157. *On the Class of Functions with Absolutely Convergent Fourier Series.*

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1. Let

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

be the Fourier series of a periodic summable function $f(x)$ with the period 2π. As regards the absolute convergence of the series (1), Zygmund[1] has given a sufficient condition in the form that the function $f(x)$ is of limited variation and satisfies Lipschitz's condition of the positive order.

In this note, we determine the class of all the functions whose Fourier series converge absolutely.

A periodic function $f(x)$ is said to be Young's continuous function, if there exist two periodic square-summable functions $f_1(x)$, $f_2(x)$, satisfying the relation

$$f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f_1(\xi)f_2(\xi+x)d\xi,$$

here and afterwards the period being taken to be 2π. The functions of such a type were first considered by Young[2]. Now we will prove the following theorem:

The necessary and sufficient condition for the absolute convergence of a trigonometrical series in the whole interval[3], *is that the series is a Fourier series of a Young's continuous function.*

2. First we prove the necessity of the condition. Assuming the absolute convergence of the series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{1}$$

1) A. Zygmund, Remarque sur la convergence absolue des séries de Fourier, The Journal of the London Math. Soc., **3** (1928), 194-196.

2) W.H. Young, On a class of parametric integrals etc., Proc. Roy. Soc. (A), **85** (1911), 401-414.

3) N. Lusin proved that if a trigonometrical serie is absolutely convergent at a set of positive measure, it converges everywhere absolutely; see Comptes Rendus, **155** (1912), 580.

we put

$$\sqrt{|a_n|} = \alpha_n , \qquad \sqrt{|b_n|} = \beta_n ,$$
$$m_n = \text{Max}\,(\alpha_n , \beta_n)$$

and define two sequences a_0', a_1', b_1', ... and a_0'', a_1'', b_1'', ... as follows:

$$a_0' a_0'' = a_0 ,$$
$$a_n' = b_n' = m_n ,$$
$$\left.\begin{aligned} a_n'' &= \frac{1}{2m_n}(a_n + b_n) \\ b_n'' &= \frac{1}{2m_n}(a_n - b_n) \end{aligned}\right\} \quad \text{for} \quad m_n \neq 0 ,$$
$$a_n'' = b_n'' = 0 \qquad \text{for} \quad m_n = 0 ,$$
$$n = 1, 2, \ldots\ldots .$$

By Lusin's theorem[1], the series $\sum|a_n|$, $\sum|b_n|$ are convergent. We have further

$$\sum(a_n'^2 + b_n'^2) = 2\sum m_n^2 \leqq 2\sum(\alpha_n^2 + \beta_n^2) = 2\sum(|a_n| + |b_n|) ,$$
$$\sum(a_n''^2 + b_n''^2) = \sum_{m_n \neq 0}(a_n''^2 + b_n''^2) = \frac{1}{4}\sum_{m_n \neq 0}\frac{(a_n + b_n)^2 + (a_n - b_n)^2}{m_n^2}$$
$$= \frac{1}{2}\sum_{m_n \neq 0}\frac{a_n^2 + b_n^2}{m_n^2} \leqq \frac{1}{2}\sum(|a_n| + |b_n|) .$$

Thus the series $\sum(a_n'^2 + b_n'^2)$ and $\sum(a_n''^2 + b_n''^2)$ are convergent. Hence it follows from Riesz-Fischer's theorem that there exist two periodic functions $f_1(x)$ and $f_2(x)$ whose Fourier coefficients are a_0', a_1', b_1',.... and a_0'', a_1'', b_1'', ... respectively. Moreover the squares of these functions are summable.

We proceed to show that the series (1) is the Fourier's expansion of the Young's continuous function

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi) f_2(\xi + x)\, d\xi . \tag{2}$$

By the change of the order of integrations[2], we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi^2}\int_{-\pi}^{\pi} f_1(\xi)\,d\xi \int_{-\pi}^{\pi} f_2(\xi + x)\cos nx\,dx$$

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1) Loc. cit.

2) This is evidently allowable.

$$=\frac{1}{\pi^2}\int_{-\pi}^{\pi} f_1(\xi)\, d\xi \int_{-\pi+\xi}^{\pi+\xi} f_2(x) \cos n(x-\xi)\, dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi)\, [a_n''\cos n\xi + b_n''\sin n\xi]\, d\xi$$

$$=a_n'a_n''+b_n'b_n'' .$$

Similarly $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = b_n'a_n''-a_n'b_n''$$

and $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = a_0'a_0''=a_0 .$$

For $m_n \neq 0$, we have

$$a_n'a_n''+b_n'b_n''=m_n(a_n''+b_n'')=a_n ,$$
$$b_n'a_n''-a_n'b_n''=m_n(a_n''-b_n'')=b_n ;$$

and for $m_n=0$, $\quad a_n=b_n=0,\ a_n'=b_n'=a_n''=b_n''=0.$

Hence,

$$\left.\begin{aligned} a_n'a_n''+b_n'b_n''&=a_n ,\\ b_n'a_n''-a_n'b_n''&=b_n .\end{aligned}\right\} \quad n=1, 2, \ldots\ldots$$

Therefore

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \begin{matrix}\cos\\ \sin\end{matrix}\, nx\, dx = \begin{matrix}a_n\\ b_n\end{matrix} , \qquad n=0, 1, 2, \ldots\ldots$$

Since the function $f(x)$ is defined by (2), the necessity of the condition is thus proved.

3. To proved the sufficiency of the given condition, let the function $f(x)$ be defined by the relation

$$f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi) f_2(\xi+x) d\xi ,$$

where $f_1(x)$, $f_2(x)$ denote two square-summable functions with the period 2π. The Fourier's constants of $f_1(x)$ are

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi) \cos n\xi\, d\xi , \qquad b_n'=\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi) \sin n\xi\, d\xi ,$$

$$n=0, 1, 2, \ldots\ldots$$

Let x be fixed and denote the Fourier's coefficients of the function $f_2(\xi+x)$ by $a_0''(x)$, $a_1''(x)$, $b_1''(x)$, ... ; thus

$$\begin{matrix} a_n''(x) \\ b_n''(x) \end{matrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi+x) \begin{matrix} \cos \\ \sin \end{matrix}\, n\xi\, d\xi\,, \qquad n=0, 1, 2, \ldots\ldots$$

Then by the Parsevals's identity we have

$$(3) \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi)f_2(\xi+x)d\xi = \frac{1}{2}a_0'a_0''(x) + \sum_{n=1}^{\infty}(a_n'a_n''(x)+b_n'b_n''(x))\,,$$

which is an absolutely convergent series, since the series

$$\sum(a_n'^2+b_n'^2) \quad \text{and} \quad \sum(a_n''^2(x)+b_n''^2(x))$$

are convergent. The series (3) is, however, nothing but the Fourier's expansion of the function $f(x)$. In fact, applying the calculations in 2,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')dx' = \frac{1}{\pi^2}\int_{-\pi}^{\pi} f_1(\xi)d\xi \cdot \int_{-\pi}^{\pi} f_2(\xi)d\xi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi)d\xi \cdot \int_{-\pi}^{\pi} f_2(\xi+x)d\xi \quad = a_0'a_0''(x)\,,$$

since $f_2(x)$ is periodic. And

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos nx'\,dx' \cdot \cos nx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\sin nx'\,dx' \cdot \sin nx$$

$$= \cos nx\left(a_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi)\cos n\xi\,d\xi + b_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi)\sin n\xi d\xi\right)$$

$$+ \sin nx\left(b_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi)\cos n\xi d\xi - a_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi)\sin n\xi\,d\xi\right)$$

$$= a_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi)\cos n(\xi-x)d\xi + b_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi)\sin n(\xi-x)d\xi$$

$$= a_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi+x)\cos n\xi d\xi + b_n' \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(\xi+x)\sin n\xi\,d\xi$$

$$= a_n'\,a_n''(x) + b_n'\,b_n''(x)\,.$$

Thus the proposition is established.