107. On the necessary conditions for the Fermat's last theorem.

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Concerning the Fermat's last theorem Prof. Vandiver has proved: if

$$x^p+y^p+z^p=0\,, \qquad p \nmid xyz\,,$$

then the following condition

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{\left[\frac{p}{3}\right]^2}\equiv 0 \pmod{p}\,,$$

is necessary[1].

In the present paper I will give a proof of H. Schwandt's condition[2]

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]^2}\equiv 0 \pmod{p} \qquad \text{(I)}$$

and then show that two analogous conditions

$$\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{3}\right]}\equiv 0 \pmod{p} \qquad \text{(II)}$$

and

$$\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]}\equiv 0 \pmod{p} \qquad \text{(III)}$$

are necessary.

§ 1. Proof of (I).

We put

$$a_1=\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{\left[\frac{p}{3}\right]^2}\,,$$

$$a_2=\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{\left[\frac{p}{2}\right]^2}\,,$$

1) Vandiver: Annals of Math. **26** (1924).
2) Schwandt: Jahresber. d.D.M.V. **43** (1934).

$$a_3=\begin{cases}\dfrac{1}{\left(2\left[\frac{p}{3}\right]+1\right)^2}+\dfrac{1}{\left(2\left[\frac{p}{3}\right]+2\right)^2}+\cdots\cdots+\dfrac{1}{(p-1)^2}, & \big(p\equiv 1 \pmod 3\big)\\[2ex] \dfrac{1}{\left(2\left[\frac{p}{3}\right]+2\right)^2}+\dfrac{1}{\left(2\left[\frac{p}{3}\right]+3\right)^2}+\cdots\cdots+\dfrac{1}{(p-1)^2}, & \big(p\equiv 3 \pmod 3\big)\end{cases}$$

$$a_4=\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]^2},$$

then we have by a simple calculation

$$a_4\equiv 2^2a_3-(a_2-a_1) \pmod p.$$

The congruence

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]^2}\equiv 0 \pmod p \qquad \text{(I)}$$

follows immediately from Vandiver's condition

$$a_1\equiv 0 \pmod p$$

and the congruence

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{(p-1)^2}\equiv 0 \pmod p,$$

because we have then respectively

$$a_2\equiv 0 \pmod p$$

and

$$a_3\equiv 0 \pmod p, \qquad \text{q. e. d.}$$

§ 2. Proof of (II).

Let p and r be primes and $p=nr+t$, $1\leqq t<r$, and denote the least positive solution of the congruence $hx\equiv k \pmod r$ by $\frac{k}{h} \pmod r$. Then we have the following congruence[3]:

$$\begin{aligned}\frac{r^{p-1}-1}{p}\equiv&\frac{\frac{1}{p}\pmod r}{p-1}+\frac{\frac{2}{p}\pmod r}{p-2}+\cdots\cdots+\frac{\frac{r-1}{p}\pmod r}{p-r+1}+\frac{r}{p-r}\\ &+\frac{\frac{1}{p}\pmod r}{p-r-1}+\frac{\frac{2}{p}\pmod r}{p-r-2}+\cdots\cdots+\frac{\frac{r-1}{p}\pmod r}{p-2r+1}+\frac{r}{p-2r}\\ &+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

3) Sylvester: C.R. **52** (1861) or Mathematical Papers II, pp 229–231 (cf. also the correction in p. 241).

$$+\frac{\frac{1}{p}(\bmod r)}{p-(n-1)r-1}+\frac{\frac{2}{p}(\bmod r)}{p-(n-1)r-2}+\cdots\cdots+\frac{\frac{r-1}{p}(\bmod r)}{p-nr+1}+\frac{r}{p-nr}$$

$$+\frac{\frac{1}{p}(\bmod r)}{t-1}+\cdots\cdots+\frac{\frac{t-1}{p}(\bmod r)}{1}\quad(\bmod p)\ .$$

We define a, a_1, a_2 and a_3 respectively by

1) for $p\equiv 1 \pmod 3$

$$a=\frac{3^{p-1}-1}{p}\equiv\frac{1}{p-1}+\frac{2}{p-2}+\frac{3}{p-3}+\frac{1}{p-4}+\frac{2}{p-5}+\frac{3}{p-6}$$
$$+\cdots\cdots+\frac{1}{3}+\frac{2}{2}+\frac{3}{1}\quad(\bmod p)\,,$$

$$a_1=\frac{1}{p-1}+\frac{1}{p-4}+\cdots\cdots+\frac{1}{3}\,,$$

$$a_2=\frac{1}{p-2}+\frac{1}{p-5}+\cdots\cdots+\frac{1}{2}\,,$$

$$a_3=\frac{1}{p-3}+\frac{1}{p-6}+\cdots\cdots+\frac{1}{1}\,.$$

2) for $p\equiv 2 \pmod 3$

$$a=\frac{3^{p-1}-1}{p}\equiv\frac{2}{p-1}+\frac{1}{p-2}+\frac{3}{p-3}+\frac{2}{p-4}+\frac{1}{p-5}+\frac{3}{p-6}$$
$$+\cdots\cdots+\frac{1}{3}+\frac{3}{2}+\frac{2}{1}\quad(\bmod p)\,,$$

$$a_1=\frac{1}{p-2}+\frac{1}{p-5}+\cdots\cdots+\frac{1}{3}\,,$$

$$a_2=\frac{1}{p-1}+\frac{1}{p-4}+\cdots\cdots+\frac{1}{1}\,,$$

$$a_3=\frac{1}{p-3}+\frac{1}{p-6}+\cdots\cdots+\frac{1}{2}\,,$$

then we have in both cases

$$a_1+2a_2+3a_3\equiv a\quad(\bmod p)\,,$$
$$a_1+a_2+a_3\equiv 0\quad(\bmod p)\,,$$
$$a_1+a_3\equiv 0\quad(\bmod p)\,,$$
$$a_2\equiv 0\quad(\bmod p)\,.$$

From these it follows

$$a\equiv-2a_1\equiv-\frac{2}{3}\left\{\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{3}\right]}\right\}\quad(\bmod p)\,.$$

By Mirimanoff's condition $3^{p-1}\equiv 1 \pmod{p^2}$ we have

$$a\equiv 0 \pmod{p},$$

i.e.

$$\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{3}\right]}\equiv 0 \pmod{p}.$$

Similarly we have also

$$\frac{2^{p-1}-1}{p}\equiv-\frac{1}{2}\left\{\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{2}\right]}\right\} \pmod{p},$$

and in connection with Wieferich's condition $2^{p-1}\equiv 1 \pmod{p^2}$,

$$\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{2}\right]}\equiv 0 \pmod{p}.$$

§ 3. Proof of (III).

Now I will show the following condition

$$\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]}\equiv 0 \pmod{p}$$

by the similar method as in § 1. Put

$$a_1=\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{3}\right]},$$

$$a_2=\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{2}\right]},$$

$$a_3=\begin{cases}\dfrac{1}{2\left[\frac{p}{3}\right]+1}+\dfrac{1}{2\left[\frac{p}{3}\right]+2}+\cdots\cdots+\dfrac{1}{p-1}, & \left(p\equiv 1 \pmod{3}\right)\\[2ex] \dfrac{1}{2\left[\frac{p}{3}\right]+2}+\dfrac{1}{2\left[\frac{p}{3}\right]+3}+\cdots\cdots+\dfrac{1}{p-1}, & \left(p\equiv 2 \pmod{3}\right)\end{cases}$$

$$a_4=\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]}.$$

Then we have

$$a_4\equiv-\{2a_3-(a_2-a_1)\} \pmod{p}.$$

Now we will prove

$$\frac{1}{1}+\frac{1}{2}+\cdots\cdots+\frac{1}{\left[\frac{p}{6}\right]}\equiv 0 \pmod{p}. \tag{III}$$

Proof: We have already

$$a_1\equiv 0 \pmod{p},$$

$$a_2\equiv 0 \pmod{p},$$

and from

$$a_1+a_3\equiv 0 \pmod{p},$$

we have

$$a_3\equiv 0 \pmod{p};$$

thus the required congruence (III) is proved.