28. *Note on Banach Spaces (II): An Ergodic Theorem for Abelian Semi-Groups.*

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G. Birkhoff and L. Alaoglu[1] defined that a Banach space E is *ergodic under a semi-group G of linear operators* if and only if for any element x of E, convex closure $K(x)$ of its transforms xT^{α} for all $\alpha \in G$ contains one and only one fix point.

Under this definition we prove the following

Theorem. Every reflexive Banach space is ergodic under a uniformly bounded abelian semi-groups of linear operators.

This was proved by G. Birkhoff and L. Alaoglu with the use of Banach mean. Our proof given here is quitely different from that of them and is simpler. Used lemma is the ergodic theorem of K. Yosida[2] only.

Let G^* be the convex closure of G and x be an arbitrary but fixed point in E. We introduce an ordering in G^* or in $K(x)$ by $\alpha < \beta$ means $xT^{\alpha}T^{\beta} = xT^{\beta}$. Then the ordering is evidently transitive and asymmetric, but not reflexive in general.

Since the definition of the ordering shows that $\alpha < \beta$ if and only if xT^{β} of $K(x)$ is invariant under T^{α}, we can find a successor of any xT^{α}, by the help of the Yosida's theorem, as the limit of the arithmetic means of its transforms $xT^{n\alpha}$. And moreover we have if $\alpha < \gamma$ and $\beta < \delta$ then $\alpha, \beta < \gamma + \delta$ — or in another words — our ordering has the Moore-Smith property.

Let now A_{α} be the subset of $K(x)$ invariant by T^{α}, then A_{α} is weakly closed and non-void. As the above considered, A_{α}'s have the finite intersection property, and E is locally weakly bicompact, hence the intersection A of all A_{α} is not empty. Evidently, all points of A is invariant under G.

On the other hand, if G is abelian, then $K(x)$ contains at most one fix point[3]. Therefore A contains exactly one point, that is, x is ergodic. But x is arbitrary, thus the theorem is proved.

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1) L. Alaoglu and G. Birkhoff, Ann. of Math., **41** (1940), 293-309.

2) K. Yosida, Proc. **14** (1938), 292-294.

3) Loc. cit., 1) Lemma 2 of Theorem 5.