48. *Relation between the Measures* $\wedge_a(X)$ *and* $m^*(X)$.

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A function of set $\Gamma(X)$, which is defined and non-negative for all sets of a metrical space, will be called an outer measure in the sense of Carathéodory or simply Carathéodory measure, if it subject the following conditions:

(C–I) $\Gamma(X) \leqq \Gamma(Y)$ whenever a set Y includes a set X,

(C–II) $\Gamma(\sum_{n=1}^{\infty} X_n) \leqq \sum_{n=1}^{\infty} \Gamma(X_n)$ for every sequence $\{X_n\}_{n=1,2,\dots}$ of sets,

(C–III) $\Gamma(X+Y)=\Gamma(X)+\Gamma(Y)$ whenever the distance $\rho(X, Y)>0$.

Now in any separable metrical space, we can define, as the following manner, a function of set which is one of Carathéodory measures.

Let a be an arbitrary positive number. Given a set in the space, we shall denote, for each positive number ε, by $\wedge_a^{(\varepsilon)}(X)$ the greatest lower bound of the sums $\sum_{n=1}^{\infty}[d(X_n)]^a$, for which $(X_n)_{n=1,2,\dots}$ is an arbitrary partition of the set X into a sequence of sets whose diameters $d(X_n)$ are less than ε.

When ε tends to 0, the number $\wedge_a^{(\varepsilon)}(X)$ tends, in a monotone non-decreasing manner, to a unique limit (finite or infinite), which we shall denote by $\wedge_a(X)$. The function $\wedge_a(X)$ of set thus defined is an outer measure in the sense of Carathéodory.

For, when $\varepsilon>0$, we clearly have,

(I) $\wedge_a^{(\varepsilon)}(X) \leqq \wedge_a^{(\varepsilon)}(Y)$ whenever $X \subset Y$,

(II) $\wedge_a^{(\varepsilon)}(\sum_{n=1}^{\infty} X_n) \leqq \sum_{n=1}^{\infty} \wedge_a^{(\varepsilon)}(X_n)$ for any sequence of sets,

(III) $\wedge_a^{(\varepsilon)}(X+Y)=\wedge_a^{(\varepsilon)}(X)+\wedge_a^{(\varepsilon)}(Y)$ if $\rho(X, Y)>\varepsilon$.

Making here $\varepsilon \to 0$, (I), (II) and (III) become respectively the three conditions (C–I), (C–II) and (C–III). (vide; Saks: Theory of the integral, § 8, Chap.-II.)

In particular, we shall study the relation of $\wedge_2(X)$ thus defined and Lebesgue outer measure $m^*(X)$ in the two-dimensional Euclidean space.

First, we shall show two necessary lemmas.

Lemma I. Among the class of all sets whose diameters do not exceed a given constant, the set having the largest Lebesgue outer measure is the circle. In the other words $d(E)^2\pi/4 \geqq m^*(E)$ for each bounded set E. (vide; Bonnesen-Fenchel; Theorie d. Konvexen Körper, S. 76 u. 107.)

Lemma II. Let $\{I_n\}_{n=1,2,\dots}$ be an arbitrary sequence of the closed circles which covers a set X. Then the greatest lower bound of the sums $\sum_{n=1}^{\infty} m^*(I_n)$ coincides with Lebesgue outer measure of X.

Further we may restrict ourselves to each diameter of I_n less than a given positive number ε. (vide: Hausdorff, Math. Ann. **79**, 163.)

We are now able to prove

Theorem. $\Lambda_2(X)=4m^*(X)/\pi$ for each set X in the two-dimensional Euclidean space.

Proof. For brevity, we shall denote $\Lambda_2(X)$ and $\Lambda_2^{(\varepsilon)}(X)$ by $\Lambda(X)$ and $\Lambda^{(\varepsilon)}(X)$ respectively. For an arbitrary positive number γ, by the definition, there exists a sequence $\{X_n\}_{n=1,2,\dots}$ of partition of X subjecting to the conditions,

$$d(X_n)<\varepsilon \quad (n=1,2,\dots) \quad \text{and} \quad \Lambda^{(\varepsilon)}(X)\geqq\sum_{n=1}^{\infty}d(X_n)^2-\gamma \tag{1}$$

Now by Lemma I

$$d(X_n)^2\geqq 4m^*(X_n)/\pi \tag{2}$$

and it is very evident that

$$\sum_{n=1}^{\infty}m^*(X_n)\geqq m^*(\sum_{n=1}^{\infty}X_n)=m^*(X). \tag{3}$$

It follows from (1), (2) and (3) that

$$\Lambda^{(\varepsilon)}(X)\geqq\sum_{n=1}^{\infty}d(X_n)^2-\gamma\geqq 4\sum_{n=1}^{\infty}m^*(X_n)/\pi-\gamma\geqq 4m^*(X)/\pi-\gamma.$$

Since γ is arbitrary, this shows clearly

$$\Lambda^{(\varepsilon)}(X)\geqq 4m^*(X)/\pi. \tag{4}$$

To prove the inverse inequality, let γ' be an arbitrary positive number. Then by virtue of Lemma II, there exists a sequence of closed circles $\{I_n\}_{n=1,2,\dots}$ having the properties:

$$\sum_{n=1}^{\infty}I_n\supset X,\quad d(I_n)<\varepsilon \quad \text{for} \quad n=1,2,\dots$$

$$m^*(X)\geqq\sum_{n=1}^{\infty}m^*(I_n)-\gamma'=\sum_{n=1}^{\infty}\pi d(I_n)^2/4-\gamma'=\pi\sum_{n=1}^{\infty}d(I_n)^2/4-\gamma'. \tag{5}$$

Here, when we write $X_1=X\cdot I_1$ and $X_n=X(I_n-\sum_{i=1}^{n-1}I_i)$ for $n=2,3,\dots$, $\{X_n\}_{n=1,2,\dots}$ is evidently one of the partitions of X and each X_n has the diameter less than ε. Therefore, the following inequality holds at once.

$$\sum_{n=1}^{\infty}d(I_n)^2\geqq\sum_{n=1}^{\infty}d(X_n)^2\geqq\Lambda^{(\varepsilon)}(X). \tag{6}$$

From (5) and (6) we get

$$m^*(X)\geqq\pi\Lambda^{(\varepsilon)}(X)/4-\gamma'.$$

Since γ' is arbitrary, this inequality shows

$$m^*(X)\geqq\pi\Lambda^{(\varepsilon)}(X)/4. \tag{7}$$

By virtue of (4) and (7), we finally get

$$m^*(X)=\pi\Lambda^{(\varepsilon)}(X)/4$$

Accordingly, $\Lambda(X)=\mathrm{Lim}_{\varepsilon\to 0}\Lambda^{(\varepsilon)}(X)=4m^*(X)/\pi$, which completes the proof.

In the preceding proof, we find that $\Lambda^{(\epsilon)}(X)$ is independent of ϵ, and is always equal to $\Lambda(X)$.

In the general n-dimensional Euclidean space, we can get, by the similar manner, the following relations:

$$\Lambda_n(X)=\frac{2\cdot 4\cdot\cdots\cdot n\cdot 2^{n/2}}{\pi^{n/2}}m^*(X) \qquad \text{for even } n,$$

$$\Lambda_n(X)=\frac{1\cdot 3\cdot 5\cdot\cdots\cdot n\cdot 2^{(n-1)/2}}{\pi^{(n-1)/2}}m^*(X) \qquad \text{for odd } n.$$