Counting Unlabeled Bipartite Graphs Using Polya's Theorem

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Abstract

This paper solves a problem that was stated by M. A. Harrison in 1973. The problem has remained open since then, and it is concerned with counting equivalence classes of $n \times r$ binary matrices under row and column permutations. Let I and O denote two sets of vertices, where $I \cap O = \emptyset$, $|I| = n$, $|O| = r$, and $B_u(n,r)$ denote the set of unlabeled graphs whose edges connect vertices in I and O. Harrison established that the number of equivalence classes of $n \times r$ binary matrices is equal to the number of unlabeled graphs in $B_u(n,r)$. He also computed the number of such matrices (hence such graphs) for small values of n and r without providing an asymptotic formula for $|B_u(n,r)|$. Here, such an asymptotic formula is provided by proving the following two-sided inequality using Polya's Counting Theorem.

$$\frac{\binom{r+2^n-1}{r}}{n!} \leq |B_u(n,r)| \leq 2\frac{\binom{r+2^n-1}{r}}{n!}, n < r. \tag{1}$$

1 Introduction

Asymptotic counting of graphs has been an active area of research in graph theory [1–13]. Several of these efforts focused on the problem of counting unlabelled graphs with a given number of vertices and edges [3, 4, 6–8, 11]. In this

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paper, we explore asymptotic counting of bipartite graphs. Clearly, the set of bipartite graphs with n vertices in one part, and r vertices in the other part forms a subset of all graphs with $n+r$ vertices. Hence, any upper bound on the number of graphs with $n+r$ vertices and up to $(n+r)(n+r-1)/2$ edges provides an upper bound on the number of bipartite graphs with n vertices in one part and r vertices in the other part with up to nr edges. Nonetheless, the known asymptotic formulas for unlabelled graphs are valid only when the number of edges satisfies certain inequalities, see for example, (1.2) and (1.4) in [3]. Similar constraints were imposed on the number of edges in [4, 6]. Other results on the number of unlabelled graphs are related to particular types of unlabelled graphs such as regular graphs [11] or Hamiltonian graphs [9]. As we are interested in asymptotic bounds on the number of unlabeled bipartite graphs without any restrictions on n and r, and the number of edges, the known asymptotic bounds for unlabelled graphs cannot be applied to obtain such bounds. Instead, we use Polya's counting theorem to obtain our main result[1] in this paper.

The bipartite graph counting problem that is considered in this paper has been investigated in connection with the enumeration of unlabeled bipartite graphs and binary matrices [5]. Let (I, O, E) denote a graph with two disjoint sets of vertices, I, called *left vertices* and a set of vertices, O, called *right vertices*, where each edge in E connects a left vertex with a right vertex. We let $n = |I|$, $r = |O|$, and refer to such a graph as an (n,r)-bipartite graph. Let $G_1 = (I, O, E_1)$ and $G_2 = (I, O, E_2)$ be two (n,r)-bipartite graphs, and $\alpha : I \to I$ and $\beta : O \to O$ be both bijections. The pair (α, β) is an isomorphism between G_1 and G_2 provided that $((\alpha(v_1), \beta(v_2)) \in E_2$ if and only if $(v_1, v_2) \in E_1$, $\forall v_1 \in I, \forall v_2 \in O$. It is easy to establish that this mapping induces an equivalence relation, and partitions the set of 2^{nr} (n,r)-bipartite graphs into equivalence classes. This equivalence relation captures the fact that the vertices in I and O are unlabeled, and so each class of (n,r)-bipartite graphs can be represented by any one of the graphs in that class without identifying the vertices in I and O. Let $B_u(n,r)$ denote any set of (n,r)-bipartite graphs that contains exactly one such graph from each of the equivalence classes of (n,r)-bipartite graphs induced by the isomorphism we defined. It is easy to see that determining $|B_u(n,r)|$ amounts to an enumeration of non-isomorphic (n,r)-bipartite graphs that will henceforth be referred to as unlabeled (n,r)-bipartite graphs.

In [5], Harrison used Pólya's counting theorem to obtain an expression to compute the number of non-equivalent $n \times r$ binary matrices. This expression contains a nested sum, in which one sum is carried over all partitions of n while the other is carried over all partitions of r, where the argument of the nested sum involves factorial, exponentiation and greatest common divisor (gcd) computations. He further established that this formula also enumerates the number of unlabeled (n,r)-bipartite graphs, i.e., $|B_u(n,r)|$. A number of results indirectly related to Harrison's work and our result appeared in the literature [1, 2, 10]. In particular, the set $B_u(n,r)$ in our work coincides with the set of bicolored graphs described in Section 2 in [1]. Whereas [1] provides a counting polynomial for the

[1]Note that a two-sided inequality for $r < n$ directly follows from the fact that $|B_u(n,r)| = |B_u(r,n)|$.

number of bicolored graphs, we focus on the asymptotic behavior of $|B_u(n,r)|$ in this paper. Counting polynomials for other families of bipartite graphs were also reported in [2]. Likewise, [10] provides generating functions for related bipartite graph counting problems without an asymptotic analysis as provided in this paper. It should also be mentioned that Polya's theorem has been used more recently in [14–21] to count various combinatorial classes of objects. To the best of our knowledge, our work provides the first asymptotic enumeration of unlabelled bipartite graphs.

Let S_n denote the symmetric group of permutations of degree n acting on set $N = \{1,2,\cdots,n\}$. Suppose that the $n!$ permutations in S_n are indexed by $1,2,\cdots,n!$ in some arbitrary, but fixed manner. The cycle index polynomial of S_n is defined as follows([22],see p.35, (2.2.1)):

$$Z_{S_n}(x_1,x_2,\cdots,x_n) = \frac{1}{n!}\sum_{m=1}^{n!}\prod_{k=1}^{n} x_k^{p_{m,k}} \tag{2}$$

where $p_{m,k}$ denotes the number of cycles of length k in the disjoint cycle representation of the m^{th} permutation in S_n, and $\sum_{k=1}^{n} kp_{m,k} = n, \forall m = 1,2,\cdots,n!$.

Let $S_n \times S_r$ denote the direct product of symmetric groups S_n and S_r acting on $N = \{1,2,\cdots,n\}$ and $R = \{1,2,\cdots,r\}$, respectively, where n and r are positive integers such that $n < r$. It can be inferred from Harrison ([23],Lemma 4.1 and Theorem 4.2) that the cycle index polynomial of $S_n \times S_r$ is given by [23]

$$Z_{S_n\times S_r}(x_1,x_2,\cdots,x_{nr}) = Z_{S_n}(x_1,x_2,\cdots,x_n) \boxtimes Z_{S_r}(x_1,x_2,\cdots,x_r), \tag{3}$$

where $\boxtimes$ is a particular polynomial multiplication that distributes over ordinary addition, and in which the multiplication $X_m \bigodot X_t$ of two product terms[2], $X_m = x_1^{p_{m,1}}x_2^{p_{m,2}}\cdots x_n^{p_{m,n}}$ and $X_t = x_1^{q_{t,1}}x_2^{q_{t,2}}\cdots x_r^{q_{t,r}}$ in Z_{S_n} and Z_{S_r}, respectively, is defined as[3]

$$X_m \bigodot X_t = \prod_{k=1}^{n}\prod_{j=1}^{r} x_{\text{lcm}(k,j)}^{p_{m,k}q_{t,j}\gcd(k,j)}. \tag{4}$$

Harrison further proved that [5]:

$$|B_u(n,r)| = Z_{S_n\times S_r}(\underbrace{2,2,..,2}_{nr}) \tag{5}$$

when[4] $n \neq r$.

We need one more fact that can be found in Harary ([22], p.36) in order to compute the stated lower and upper bound in (1):

$$Z_{S_r}(x_1,x_2,\ldots\ldots,x_r) = \frac{1}{r}\sum_{i=1}^{r} x_i Z_{S_{r-i}}(x_1,x_2,\ldots\ldots,x_{r-i}) \tag{6}$$

where $Z_{S_0}() = 1$.

[2]Note that we will not display the zero powers of $x_1, x_2, \cdots$ in a cycle index polynomial. We will use the same convention for all other cycle index polynomials throughout the paper.

[3]The lcm(a,b) and gcd(a,b) denote least common multiple and greatest common divisor of a and b.

[4]As noted in [5], $n = r$ case involves a different cycle index polynomial. Bounding $|B_u(n,n)|$ will be considered separately at the end of the paper.

2 An Upper Bound for $|B_u(n,r)|$

We first note that $|B_u(1,r)| = r+1 = \binom{r+2^1-1}{r}/1! \leq 2\binom{r+2^1-1}{r}/1!$. Hence the upper bound that is claimed in the abstract holds for $n=1$. Proving that it also holds for $n \geq 2$ requires a more careful analysis of the terms in

$$Z_{S_n}(x_1, x_2, \cdots, x_n) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r). \tag{7}$$

We first express $Z_{S_n}(x_1, x_2, \cdots, x_n)$ as

$$Z_{S_n}(x_1, x_2, \ldots, x_n) = Z_{S_n}[1] + Z_{S_n}[2] + \ldots + Z_{S_n}[n!], \tag{8}$$

where

$$Z_{S_n}[1] = \frac{1}{n!} x_1^n, \tag{9}$$

$$Z_{S_n}[2] = \frac{1}{n!} x_1^{n-2} x_2. \tag{10}$$

The first term is associated with the identity permutation and the second term is associated with any one of the permutations in which all but two of the elements in $N = 1, 2, \cdots, n$ are fixed to themselves. The remaining $Z_{S_n}[i] = \frac{1}{n!} \prod_{k=1}^{n} x_k^{p_{i,k}}, 3 \leq i \leq n!$ terms represent all the other product terms in the cycle index polynomial of S_n with no particular association with the permutations in S_n. Similarly, we set $Z_{S_r}(x_1, x_2, \ldots, x_r) = \frac{1}{r!} \sum_{t=1}^{r!} \prod_{j=1}^{r} x_j^{q_{t,j}}$ without identifying the actual product terms with any particular permutation in S_r.

The following equations obviously hold as the sum of the lengths of all the cycles in any cycle disjoint representation of a permutation in S_n and S_r must be n and r, respectively.

$$\sum_{k=1}^{n} k p_{i,k} = n, 1 \leq i \leq n!, \tag{11}$$

$$\sum_{j=1}^{r} j q_{t,j} = r, 1 \leq t \leq r! \tag{12}$$

Now we can proceed with the computation of the upper bound for $|B_u(n,r)|$. First, using (3) and (5) we note that

$$\begin{aligned}
|B_u(n,r)| =& Z_{S_n \times S_r}(2, 2, 2, \ldots, 2), \\
&= [Z_{S_n}(x_1, x_2, \ldots, x_n) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r)]\,(2, 2, \ldots, 2), \\
&= [(Z_{S_n}[1] + Z_{S_n}[2] + \ldots + Z_{S_n}[n!]) \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2), \\
&= [Z_{S_n}[1] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2) \\
&\quad + [Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2) \\
&\quad + \ldots + [Z_{S_n}[n!] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2).
\end{aligned} \tag{13}$$

The first term in (13) is directly computed from (9) and the following proposition.

Proposition 1.

$$\left[\left(\frac{1}{n!}x_1^n\right) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r)\right](2,2,\ldots,2) = \frac{1}{n!}\binom{r+2^n-1}{r}.$$

Proof. $\left[\left(\frac{1}{n!}x_1^n\right) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r)\right](2,2,\ldots,2)$

$$
\begin{aligned}
&= \frac{1}{n!}\left\{\left[x_1^n \boxtimes \frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,...,2)\right\},\\
&= \frac{1}{n!}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!} x_1^n \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,...,2)\right\},\\
&= \frac{1}{n!}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(1,j)}^{nq_{t,j}\gcd(1,j)}\right](2,2,...,2)\right\},\\
&= \frac{1}{n!}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{nq_{t,j}}\right](2,2,...,2)\right\},\\
&= \frac{1}{n!}\left\{\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{nq_{t,j}}\right\},\\
&= \frac{1}{n!}\left\{\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} (2^n)^{q_{t,j}}\right\},\\
&= \frac{1}{n!}\left\{Z_{S_r}(2^n, 2^n, \ldots, 2^n)\right\}.
\end{aligned}
\tag{14}
$$

Using (6), we have

$$rZ_{S_r}(2^n, 2^n, \ldots, 2^n) = \sum_{i=1}^{r} 2^n Z_{S_{r-i}}(2^n, 2^n, \ldots, 2^n),$$

and

$$(r-1)Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n) = \sum_{i=1}^{r-1} 2^n Z_{S_{r-1-i}}(2^n, 2^n, \ldots, 2^n).$$

Subtracting the second equation from the first one gives

$$
\begin{aligned}
rZ_{S_r}(2^n, 2^n, \ldots, 2^n) - (r-1)Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n) &= 2^n Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n),\\
rZ_{S_r}(2^n, 2^n, \ldots, 2^n) &= (r+2^n-1)Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n),\\
Z_{S_r}(2^n, 2^n, \ldots, 2^n) &= (\frac{r+2^n-1}{r})Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n).
\end{aligned}
$$

Expanding the last equation inductively, we obtain

$$
\begin{aligned}
Z_{S_r}(2^n, 2^n, \ldots, 2^n) &= (\frac{r+2^n-1}{r})(\frac{r+2^n-2}{r-1})Z_{S_{r-2}}(2^n, 2^n, \ldots, 2^n),\\
Z_{S_r}(2^n, 2^n, \ldots, 2^n) &= (\frac{r+2^n-1}{r})(\frac{r+2^n-2}{r-1})(\frac{r+2^n-3}{r-2})Z_{S_{r-3}}(2^n, 2^n, \ldots, 2^n),\\
Z_{S_r}(2^n, 2^n, \ldots, 2^n) &= (\frac{r+2^n-1}{r})(\frac{r+2^n-2}{r-1})(\frac{r+2^n-3}{r-2})\ldots(\frac{2^n}{1})Z_{S_0}().
\end{aligned}
$$

Noting that $Z_{S_0}() = 1$, and combining the product terms together, we obtain

$$Z_{S_r}(2^n, 2^n, \ldots, 2^n) = \binom{r + 2^n - 1}{r}.$$

The proposition follows from the substitution of this last equation into (14). ■

Thus, it suffices to upper bound each of the remaining terms in (13) to upper bound $|B_u(n,r)|$. This will be established by proving $[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2,2,\ldots,2) \geq [Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2,2,\ldots,2), \forall i, 3 \leq i \leq n!$. We first need some preliminary facts.

Lemma 1. For all $i, 1 \leq i \leq n!$,
$[Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2,\ldots,2) = \frac{1}{n!} Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,1)}, \ldots, 2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,r)})$.

Proof.
$[Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2,2,\ldots,2)$

$$\begin{aligned}
&= \left[\frac{1}{n!}\prod_{k=1}^{n} x_k^{p_{i,k}} \boxtimes \left(\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right)\right](2,2,\ldots,2),\\
&= \left[\frac{1}{n!r!}\sum_{t=1}^{r!}\prod_{k=1}^{n} x_k^{p_{i,k}} \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,\ldots,2),\\
&= \left[\frac{1}{n!r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}\prod_{k=1}^{n} x_{\mathrm{lcm}(k,j)}^{p_{i,k}q_{t,j}\gcd(k,j)}\right](2,2,\ldots,2),\\
&= \frac{1}{n!r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}\prod_{k=1}^{n} 2^{p_{i,k}q_{t,j}\gcd(k,j)},\\
&= \frac{1}{n!}\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}(2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,j)})^{q_{t,j}}\right],\\
&= \frac{1}{n!} Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,1)}, \ldots, 2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,r)}).
\end{aligned}$$

■

Corollary 1.

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2,\ldots,2) = \frac{1}{n!} Z_{S_r}(2^{n-1}, 2^n, 2^{n-1}, 2^n, \ldots).$$

Proof. By definition, $p_{2,1} = n-2, p_{2,2} = 1, p_{2,k} = 0, 3 \leq k \leq n$. Substituting these into the last equation in Lemma 1 proves the statement. ■

Lemma 2.

$$\textstyle\sum_{k=1}^{n} p_{i,k} \leq n-1, \forall i, 2 \leq i \leq n!.$$

Proof. Recall from (11) that $\sum_{k=1}^{n} k p_{i,k} = n, \forall i, 1 \leq i \leq n!$. Hence $\sum_{k=1}^{n} p_{i,k} = n - \sum_{k=1}^{n}(k-1)p_{i,k}$, and so the maximum value of $\sum_{k=1}^{n} p_{i,k}$ occurs when $\sum_{k=1}^{n}(k-1)p_{i,k}$ is minimized. Furthermore, at least one of $p_{i,k}, \forall i, 2 \leq i \leq n!$ must be ≥ 1 for some $k \geq 2$ since none of the permutations we consider is the identity. Thus, $\sum_{k=1}^{n}(k-1)p_{i,k} \geq 1$ and the statement follows. ■

Lemma 3. If $\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,\alpha+1) = n$, then $\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,\alpha) \leq n-1$, $\forall i, 2 \leq i \leq n!$ and for any integer $\alpha \geq 2$.

Proof. If $\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,\alpha+1) = n$ as stated in the lemma, then we must have $\mathrm{gcd}(k,\alpha+1) = k$ where $p_{i,k} \geq 1$, $\forall i, 2 \leq i \leq n!$. Therefore $k \leq \alpha+1$. Now if $k = \alpha+1$, then trivially $\mathrm{gcd}(k,\alpha) < k$. On the other hand if $k < \alpha+1$, then $\alpha+1$ must be a multiple of k. Therefore, α can not be a multiple of k for any $k \geq 2$. At this point we find that $\mathrm{gcd}(k,\alpha) < k$, $\forall k, 2 \leq k \leq n$. Since as in the previous lemma, none of the permutations we consider is the identity, at least one of $p_{i,k}, \forall i, 2 \leq i \leq n!$ must be ≥ 1 for some $k \geq 2$ and so we conclude that $\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,\alpha) \leq n-1$. ∎

Lemma 4. $Z_{S_r}(2^{n-1},2^n,\ldots) \geq Z_{S_{r-1}}(2^{n-1},2^n,\ldots)$, for $2 \leq n$.

Proof. Using (6), we get

$$rZ_{S_r}(2^{n-1},2^n,\ldots) = \sum_{\textbf{odd } i}^{r-\beta_1} 2^{n-1}Z_{S_{r-i}}(2^{n-1},2^n,\ldots) + \sum_{\textbf{even } i}^{r-\beta_2} 2^{n}Z_{S_{r-i}}(2^{n-1},2^n,\ldots), \tag{15}$$

where $\beta_1 = 1, \beta_2 = 0$ if r is even and $\beta_1 = 0, \beta_2 = 1$ if r is odd. Similarly, for $r-1$,

$$(r-1)Z_{S_{r-1}}(2^{n-1},2^n,\ldots) = \sum_{\textbf{odd } i}^{r-1-\beta_2} 2^{n-1}Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots) + \sum_{\textbf{even } i}^{r-1-\beta_1} 2^{n}Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots). \tag{16}$$

Subtracting 16 from 15 gives

$$\begin{aligned} &rZ_{S_r}(2^{n-1},2^n,\ldots) - (r-1)Z_{S_{r-1}}(2^{n-1},2^n,\ldots) \\ &\quad= \sum_{\textbf{even } i}^{r-\beta_2} 2^{n}Z_{S_{r-i}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{odd } i}^{r-1-\beta_2} 2^{n-1}Z_{S_{r-1-i}}(2^{n-1},2^n\ldots) \\ &\quad\quad+ \sum_{\textbf{odd } i}^{r-\beta_1} 2^{n-1}Z_{S_{r-i}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{even } i}^{r-1-\beta_1} 2^{n}Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots), \end{aligned} \tag{17}$$

$$\begin{aligned} &rZ_{S_r}(2^{n-1},2^n,\ldots) - (r-1)Z_{S_{r-1}}(2^{n-1},2^n,\ldots) \\ &\quad= \sum_{\textbf{even } i}^{r-\beta_2} 2^{n-1}Z_{S_{r-i}}(2^{n-1},2^n,\ldots) + 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots) - \\ &\quad\quad\quad \sum_{\textbf{even } i}^{r-1-\beta_1} 2^{n-1}Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots), \end{aligned}$$

$$\begin{aligned} &rZ_{S_r}(2^{n-1},2^n,\ldots) = (r-1+2^{n-1})Z_{S_{r-1}}(2^{n-1},2^n\ldots) \\ &\quad+ 2^{n-1}\left(\sum_{\textbf{even } i}^{r-\beta_2} Z_{S_{r-i}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{even } i}^{r-1-\beta_1} Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots)\right). \end{aligned} \tag{18}$$

We now prove the lemma by induction on r.

Basis $r = 1$. By (6), $Z_{S_1}(2^{n-1}) = 2^{n-1}Z_{S_0}() = 2^{n-1}$. So we have $Z_{S_1}(2^{n-1}) = 2^{n-1} \geq Z_{S_0}() = 1$ for $2 \leq n$.

Induction Step. Suppose that the lemma holds from 1 to $r-1$. That is, $Z_{S_{r-i}} - Z_{S_{r-i-1}} \geq 0, 1 \leq i \leq r-1$. Now recall that if r is even then $\beta_1 = 1$, and $\beta_2 = 0$, and hence the difference of the two sums in (18) becomes $(Z_{S_{r-2}} - Z_{S_{r-3}}) + (Z_{S_{r-4}} - Z_{S_{r-5}}) \ldots + (Z_{S_2} - Z_{S_1}) + Z_{S_0}$, which is clearly ≥ 0 by the induction hypothesis. Therefore,

$$rZ_{S_r}(2^{n-1}, 2^n, \ldots) \geq (r - 1 + 2^{n-1}) Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots), \tag{19}$$

$$Z_{S_r}(2^{n-1}, 2^n, \ldots) \geq Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots), n \geq 2. \tag{20}$$

On the other hand, if r is odd then $\beta_1 = 0$, and $\beta_2 = 1$, and hence the difference of the two sums in the same equation becomes $(Z_{S_{r-2}} - Z_{S_{r-3}}) + (Z_{S_{r-4}} - Z_{S_{r-5}}) \ldots + (Z_{S_2} - Z_{S_1}) + (Z_{S_1} - Z_{S_0})$, which is again ≥ 0, and the statement follows in this case as well. ■

We now are ready to prove that
$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, \ldots, 2) \geq [Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, \ldots, 2)$, $\forall i, 2 \leq i \leq n!$.

Theorem 1.

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, 2, \ldots, 2) \geq [Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, 2, \ldots, 2) \tag{21}$$

$\forall i, 2 \leq i \leq n!$ and $\forall n, n < r$.

Proof. Using Lemma 1 and Corollary 1 it suffices to show that

$$Z_{S_r}(2^{n-1}, 2^n, \ldots) \geq Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots, 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)}). \tag{22}$$

We prove the statement by induction on r.

Basis: $(r = 1)$. By (6), $Z_{S_1}(2^{n-1}) = 2^{n-1} Z_{S_0}() = 2^{n-1}$. Similarly, by (6), $Z_{S_1}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}) = 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)} Z_{S_0}() = 2^{\sum_{k=1}^{n} p_{i,k}}$. Given that $\sum_{k=1}^{n} p_{i,k} \leq n-1$ by Lemma 2, we have $2^{\sum_{k=1}^{n} p_{i,k}} \leq 2^{n-1}$, and hence the statement holds in this case.

Induction Step: First, by (6),

$$Z_{S_r}(2^{n-1}, 2^n, \ldots) = \frac{1}{r} \begin{bmatrix} 2^{n-1} Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) \\ +2^n Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) \\ +2^{n-1} Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) \\ \vdots \\ +2^{\beta} Z_{S_0}() \end{bmatrix}, \tag{23}$$

where $\beta = n$ if r is even and $\beta = n-1$ if r is odd. Similarly,

$$Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots, 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)}) =$$

$$\frac{1}{r} \begin{bmatrix} 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)} Z_{S_{r-1}}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots) \\ +2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,2)} Z_{S_{r-2}}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots) \\ +2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,3)} Z_{S_{r-3}}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots) \\ \vdots \\ +2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)} Z_{S_0}() \end{bmatrix} \tag{24}$$

Subtracting (24) from (23), we have

$$Z_{S_r}(2^{n-1},2^n,\dots) - Z_{S_r}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},\dots,2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)})$$
$$= \frac{1}{r}\begin{bmatrix} 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\dots) \\ +2^nZ_{S_{r-2}}(2^{n-1},2^n,\dots) \\ +2^{n-1}Z_{S_{r-3}}(2^{n-1},2^n,\dots) \\ \vdots \\ +2^{\beta}Z_{S_0}() \end{bmatrix}$$
$$-\frac{1}{r}\begin{bmatrix} 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)},\dots) \\ +2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)},\dots) \\ +2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)},\dots) \\ \vdots \\ +2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \end{bmatrix} \tag{25}$$

Thus, it suffices to show that the right hand side of the above equation is ≥ 0, or

$$\begin{array}{c} 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)},\dots) \\ +2^nZ_{S_{r-2}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)},\dots \\ +2^{n-1}Z_{S_{r-3}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)},2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)},\dots) \\ \vdots \\ +2^{\beta}Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \geq 0. \end{array} \tag{26}$$

Now by induction hypothesis, (22) holds for $1,2,\cdots,r-1$. Thus, (26) can be replaced by

$$\begin{array}{c} 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{n-1},2^n,\dots) \\ +2^nZ_{S_{r-2}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{n-1},2^n,\dots) \\ +2^{n-1}Z_{S_{r-3}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{n-1},2^n,\dots) \\ \vdots \\ +2^{\beta}Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \geq 0. \end{array} \tag{27}$$

Moreover, invoking Lemma 2 gives

$$\begin{aligned} &2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{n-1},2^n\dots) \\ &\qquad \geq 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\dots) - 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\dots) = 0. \end{aligned} \tag{28}$$

Hence the difference in the first line in (27) ≥ 0, and therefore it is sufficient to show that

$$\begin{array}{c} 2^nZ_{S_{r-2}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{n-1},2^n,\dots) \\ +2^{n-1}Z_{S_{r-3}}(2^{n-1},2^n,\dots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{n-1},2^n,\dots) \\ \vdots \\ +2^{\beta}Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \geq 0. \end{array} \tag{29}$$

To prove this inequality, we will combine four terms in pairs of consecutive lines for the remaining $r-1$ lines by considering two cases. If r is odd then $\beta = n-1$ and no extra line remains in this pairing. Thus, for all even $\alpha, 2 \leq \alpha \leq r-1$, it suffices to prove

$$\begin{aligned} 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,\alpha)} Z_{S_{r-\alpha}}(2^{n-1}, 2^n \ldots), \\ +2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,\alpha+1)} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n \ldots) \geq 0. \end{aligned} \tag{30}$$

or,

$$\begin{aligned} 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}k} Z_{S_{r-\alpha}}(2^{n-1}, 2^n \ldots) \\ +2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,\alpha+1)} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n \ldots) \geq 0. \end{aligned}$$

Now if $\sum_{k=1}^n p_{i,k}\gcd(k, \alpha+1) \leq n-1$, then

$$\begin{aligned} 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}k} Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \\ +2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \geq \\ 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \\ +2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) = 0. \end{aligned}$$

On the other hand, if $\sum_{k=1}^n p_{i,k}\gcd(k, \alpha+1) = n$, then we prove (30) by noting that $\sum_{k=1}^n p_{i,k}\gcd(k, \alpha) \leq n-1$ by Lemma 3. Thus,

$$\begin{aligned} & 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \\ & + 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^n Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \\ & = 2^{n-1} Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \\ & 2^{n-1} \left[Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \right] \end{aligned}$$

Now by Lemma 4, $Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \geq Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots)$ and the statement is proved for odd $r, n < r$. For even r, the last line in (29) is left out in the pairing of consecutive lines and $\beta = n$. In this case we have $2^n Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)} Z_{S_0}() \geq 2^n Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}k} Z_{S_0}() = 2^n Z_{S_0}() - 2^n Z_{S_0}() = 0$ and the statement follows. ∎

Theorem 2.

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2) \leq \frac{\binom{r+2^n-1}{r}}{n!(n!-1)}. \tag{31}$$

where $2 \leq n < r$.

Proof. By Corollary 1

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2) = \frac{1}{n!} Z_{S_r}(2^{n-1}, 2^n, \ldots). \tag{32}$$

Thus, to prove the theorem, it is sufficient to show

$$\frac{1}{n!} Z_{S_r}(2^{n-1}, 2^n, 2^{n-1}, 2^n, \ldots) \leq \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} \tag{33}$$

where $2 \leq n < r$.

Now, using (6), we get

$$rZ_{S_r}(2^{n-1},2^n,\ldots) = \sum_{\textbf{odd } i}^{r-\beta_1} 2^{n-1} Z_{S_{r-i}}(2^{n-1},2^n,\ldots) + \sum_{\textbf{even } i}^{r-\beta_2} 2^n Z_{S_{r-i}}(2^{n-1},2^n,\ldots) \tag{34}$$

where $\beta_1 = 1, \beta_2 = 0$ if r is even and $\beta_1 = 0, \beta_2 = 1$ if r is odd. Similarly, for $r-2$,

$$(r-2)Z_{S_{r-2}}(2^{n-1},2^n,\ldots) = \sum_{\textbf{odd } i}^{r-2-\beta_1} 2^{n-1} Z_{S_{r-2-i}}(2^{n-1},2^n,\ldots) + \sum_{\textbf{even } i}^{r-2-\beta_2} 2^n Z_{S_{r-2-i}}(2^{n-1},2^n,\ldots). \tag{35}$$

Subtracting (35) from (34) gives

$$\begin{aligned}
&rZ_{S_r}(2^{n-1},2^n,\ldots) - (r-2)Z_{S_{r-2}}(2^{n-1},2^n,\ldots) \\
&= 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots) + 2^n Z_{S_{r-2}}(2^{n-1},2^n,\ldots), \\
&rZ_{S_r}(2^{n-1},2^n,\ldots) = 2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots) + (r-2+2^n)Z_{S_{r-2}}(2^{n-1},2^n,\ldots), \\
&Z_{S_r}(2^{n-1},2^n,\ldots) = \frac{1}{r}\left[2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots) + (r-2+2^n)Z_{S_{r-2}}(2^{n-1},2^n,\ldots)\right].
\end{aligned} \tag{36}$$

We will use induction on r and the recurrence given in (36) to prove this inequality.

Basis. Case $r = 3$: Recall that

$$Z_{S_n}[2] = \frac{1}{n!}x_1^{n-2}x_2,$$
$$Z_{S_3}(x_1,x_2,x_3) = \frac{1}{3!}(x_1^3 + 3x_1x_2 + 2x_3).$$

Thus,

$$\begin{aligned}
&\left[Z_{S_n}[2] \boxtimes Z_{S_3}(x_1,x_2,x_3)\right](2,2,\ldots,2) \\
&= \left[\frac{1}{n!}(x_1^{n-2}x_2) \boxtimes \frac{1}{3!}(x_1^3 + 3x_1x_2 + 2x_3)\right](2,2,\ldots,2), \\
&= \frac{1}{3!n!}\left[(x_1^{n-2}x_2) \odot x_1^3 + (x_1^{n-2}x_2) \odot (3x_1x_2) + (x_1^{n-2}x_2) \odot 2x_3\right](2,2,\ldots,2), \\
&= \frac{1}{3!n!}\left[x_1^{3(n-2)}x_2^3 + 3x_1^{n-2}x_2x_2^{n-2}x_2^2 + 2x_3^{n-2}x_6\right](2,2,\ldots,2), \\
&= \frac{1}{3!n!}\left[2^{3n-3} + 3\times 2^{2n-1} + 2^n\right] \le \frac{\binom{r+2^n-1}{r}}{n!(n!-1)}.
\end{aligned}$$

for $n = 2$ and $r = 3$.

Case $r = 4$. In this case we have

$$
\begin{aligned}
&\left[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)\right](2, \ldots, 2) \\
&= \left[\frac{1}{n!}(x_1^{n-2}x_2) \boxtimes \frac{1}{4!}(x_1^4 + 6x_1^2x_2 + 3x_2^2 + 8x_1x_3 + 6x_4)\right](2, \ldots, 2), \\
&= \frac{1}{4!n!}\Big[(x_1^{n-2}x_2) \odot x_1^4 + (x_1^{n-2}x_2) \odot (6x_1^2x_2) + (x_1^{n-2}x_2) \odot 3x_2^2 \\
&\quad + (x_1^{n-2}x_2) \odot (8x_1x_3) + (x_1^{n-2}x_2) \odot 6x_4\Big](2, \ldots, 2), \\
&= \frac{1}{4!n!}\left[x_1^{4(n-2)}x_2^4 + 6x_1^{2(n-2)}x_2^{n-2}x_2^2x_2^2 + 3x_1^{2(n-2)}x_2^4 + 8x_1^{n-2}x_3^{n-2}x_2x_6 + 6x_4^{n-2}x_4^2\right] \\
&\qquad\qquad (2, \ldots, 2), \\
&= \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 3 \times 2^{2n} + 8 \times 2^{2n-2} + 6 \times 2^n\right], \\
&= \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 5 \times 2^{2n} + 6 \times 2^n\right].
\end{aligned}
$$

Now, given that $r = 4$, the only possible values of n are 2 and 3. If $n = 2$ then:

$$
\begin{aligned}
&\left[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)\right](2, 2, \ldots, 2) \\
&\qquad = \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 5 \times 2^{2n} + 6 \times 2^n\right], \\
&\qquad = \frac{1}{4!2!}\left[2^4 + 6 \times 2^4 + 5 \times 2^4 + 6 \times 2^2\right], \\
&\qquad = \frac{16 + 96 + 80 + 24}{4!2!} = 4.5, \\
&\qquad \le \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} = \frac{\binom{7}{4}}{2!(2!-1)} = \frac{35}{2} = 17.5.
\end{aligned}
$$

On the other hand, if $n = 3$ then:

$$
\begin{aligned}
&\left[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)\right](2, 2, \ldots, 2) \\
&\qquad = \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 5 \times 2^{2n} + 6 \times 2^n\right], \\
&\qquad = \frac{1}{4!3!}\left[2^8 + 6 \times 2^7 + 5 \times 2^6 + 6 \times 2^3\right], \\
&\qquad = \frac{256 + 768 + 320 + 48}{4!3!} = \frac{29}{3}, \\
&\qquad \le \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} = \frac{\binom{11}{4}}{3!(3!-1)} = \frac{330}{30} = 11.
\end{aligned}
$$

Induction Step: Suppose that (33) holds for all values from 3 to $r - 1$. Using the recurrence given in (36) and the induction hypothesis for $r - 1$ and $r - 2$ we get:

$$
\begin{aligned}
\frac{1}{n!}Z_{S_r}(2^{n-1},2^n,\ldots) &= \frac{1}{n!r}\left[2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots)+(r-2+2^n)Z_{S_{r-2}}(2^{n-1},2^n,\ldots)\right],\\
&= \frac{2^{n-1}}{n!r}Z_{S_{r-1}}(2^{n-1},2^n,\ldots)+\frac{r-2+2^n}{n!r}Z_{S_{r-2}}(2^{n-1},2^n,\ldots),\\
&\le \frac{2^{n-1}}{r}\frac{\binom{r+2^n-2}{r-1}}{n!(n!-1)}+\frac{r-2+2^n}{r}\frac{\binom{r+2^n-3}{r-2}}{n!(n!-1)},\\
&\le \frac{2^{n-1}}{n!(n!-1)r}\frac{(r+2^n-2)!}{(r-1)!(2^n-1)!}+\frac{r-2+2^n}{n!(n!-1)r}\frac{(r+2^n-3)!}{(r-2)!(2^n-1)!},
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{2^{n-1}}{n!(n!-1)r}\frac{(r+2^n-2)!}{(r-1)!(2^n-1)!}+\frac{(r-1)(r+2^n-2)!}{n!(n!-1)r!(2^n-1)!},\\
&\le \frac{(r+2^n-2)!(r+2^{n-1}-1)}{n!(n!-1)r!(2^n-1)!}\le\frac{(r+2^n-2)!(r+2^n-1)}{n!(n!-1)r!(2^n-1)!},\\
&\le \frac{(r+2^n-1)!}{n!(n!-1)r!(2^n-1)!}=\frac{1}{n!(n!-1)}\binom{r+2^n-1}{r},\\
&\le \frac{1}{n!(n!-1)}\binom{r+2^n-1}{r}.
\end{aligned}
$$

This completes the proof. ∎

Combining Theorems 1 and 2 concludes the upper bound calculation.

Theorem 3. $|B_u(n,r)| \le \frac{2\binom{r+2^n-1}{r}}{n!}$.

Proof. By (13),

$$
\begin{aligned}
|B_u(n,r)| = {}& [(Z_{S_n}[1])\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,2,\ldots,2)\\
&+[(Z_{S_n}[2])\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,2,\ldots,2)\\
&+\ldots+[(Z_{S_n}[n!])\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,2,\ldots,2),
\end{aligned}
$$

and by Theorem 1,

$$
\begin{aligned}
|B_u(n,r)| \le {}& [(Z_{S_n}[1])\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,2,\ldots,2)+\\
&[(Z_{S_n}[2])\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,2,\ldots,2)\\
&+\ldots+[(Z_{S_n}[2])\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,2,\ldots,2).
\end{aligned}
$$

Finally by Theorem 2,

$$|B_u(n,r)| \le \frac{\binom{r+2^n-1}{r}}{n!}+(n!-1)\frac{\binom{r+2^n-1}{r}}{n!(n!-1)}=\frac{2\binom{r+2^n-1}{r}}{n!}.$$ ∎

3 The Lower Bound for $|B_u(n,r)|$

Using (9) and (13) gives

$$|B_u(n,r)| \ge \left[\left(\frac{1}{n!}x_1^n\right)\boxtimes Z_{S_r}(x_1,x_2,\cdots,x_r)\right](2,2,\ldots,2). \tag{37}$$

Thus the following lower bound trivially follows from Proposition 1 and Eqn. (37):

$$|B_u(n,r)| \ge \frac{1}{n!}Z_{S_r}(2^n,2^n,\ldots,2^n)\ge\frac{\binom{r+2^n-1}{r}}{n!}. \quad ∎ \tag{38}$$

4 Remarks

1. It should be mentioned that, if $r < n$, the obtained lower and upper bounds together with the relation $|B_u(n,r)| = |B_u(r,n)|$ give
$$\frac{\binom{n+2^r-1}{n}}{r!} \leq |B_u(n,r)| \leq 2\frac{\binom{n+2^r-1}{n}}{r!}, r < n.$$
Furthermore, if $r = n$, using the cycle index representation of bi-colored graphs provided in Section 3 in [1] and Theorem 3 gives
$$|B_u(n,n)| \geq \frac{\binom{n+2^n-1}{n}}{2n!}.$$
The Z' term in the cycle index representation of bi-colored graphs in [1] prevents us from deriving an upper bound for $|B_u(n,n)|$ that is a constant multiple of the lower bound in this case. On the other hand, an obvious upper bound for $|B_u(n,n)|$ can be derived by setting $r = n+1$ in the inequality in Theorem 3.

2. It may be worthwhile to mention that the proof of the upper bound on $|B_u(n,r)|$ hinges on identifying the largest term in the cycle index polynomial of $|B_u(n,r)|$, i.e., $[\left(\frac{1}{n!}x_1^n\right) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r)](2,2,\ldots,2)$, and the second largest term, $[\left(\frac{1}{n!}x_1^{n-2}x_2\right) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r)](2,2,\ldots,2)$. Identifying the second largest term turned out to be the most complex step in our proof. Establishing an upper bound (Theorem 2) leads to the upper bound for $|B_u|(n,r)|$. Improving this upper bound significantly will likely require identifying the third and possibly fourth largest term, and obtaining upper bounds for each.

3. The novelty of our proof rests in combining Harrison's cycle index representation of the number of unlabelled bipartite graphs in $B_u(n,r)$ with Harary's recursive formula (Eqn. 6). More specifically, we transform the terms in Harrison's cycle index representation into a form that allows us to use Harary's recurrence to carry out the computations in our theorems.

4. It is noted that when $n = 2$ or $n = 3$, exact values of $|B(n,r)|$ were reported in [13], but computing the exact values of $|B_u(n,r)|$ for $n \geq 4$ remains open.

5. It is also noted that one can provide a simple combinatorial proof of the lower bound on $|B_u(n,r)|$, by counting r-selections of all subsets of n left vertices with repetition, and dividing it by $n!$. However, a similar combinatorial proof for the upper bound does not appear to be within our reach.

Appendix:

Table 1 lists $\ln|B_u(n,r)|$ along with the natural logarithms of lower and upper bounds for $1 < n < r < 15$.

n *r*	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1.09861 1.09861 1.79176	1.38629 1.38629 2.07944	1.60944 1.60944 2.30259	1.79176 1.79176 2.48491	1.94591 1.94591 2.63906	2.07944 2.07944 2.77259	2.19722 2.19722 2.89037	2.30259 2.30259 2.99573	2.3979 2.3979 3.09104	2.48491 2.48491 3.17805	2.56495 2.56495 3.2581	2.63906 2.63906 3.3322	2.70805 2.70805 3.4012	2.77259 2.77259 3.46574
2		2.30259 2.56495 2.99573	2.83321 3.09104 3.55535	3.3322 3.52636 4.02535	3.73767 3.91202 4.43082	4.09434 4.2485 4.78749	4.40672 4.55388 5.10595	4.70048 4.82831 5.39363	4.96284 5.0814 5.65599	5.20401 5.31321 5.89715	5.42495 5.52943 6.1203	5.63479 5.7301 6.32794	5.82895 5.91889 6.52209	6.01127 6.09582 6.70441
3			4.00733 4.46591 4.70048	4.8828 5.24702 5.57595	5.65599 5.95584 6.34914	6.34914 6.59851 7.04229	6.97728 7.18841 7.67089	7.55276 7.73368 8.24617	8.08364 8.24012 8.77678	8.57622 8.71276 9.26936	9.03575 9.1562 9.7289	9.46653 9.57345 10.1597	9.872 9.96754 10.5651	10.255 10.3409 10.9481
4				6.4708 6.9594 7.16395	7.72356 8.08641 8.41671	8.86869 9.14238 9.56184	9.92471 10.1349 10.6179	10.9056 11.0692 11.5987	11.8219 11.9512 12.515	12.6821 12.7855 13.3752	13.493 13.5767 14.1861	14.2603 14.3287 14.9534	14.9885 15.045 15.6816	15.6816 15.7287 16.3748
5					9.87164 10.2603 10.5648	11.5633 11.826 12.2565	13.1474 13.3276 13.8406	14.6391 14.7645 15.3322	16.0501 16.1388 16.7432	17.3899 17.4535 18.083	18.6662 18.7124 19.3593	19.8854 19.9195 20.5785	21.053 21.0784 21.7461	22.1736 22.1927 22.8667
6						14.3253 14.5771 15.0185	16.5086 16.6637 17.2017	18.588 18.6849 19.2811	20.5759 20.6372 21.269	22.482 22.5215 23.1752	24.3146 24.3403 25.0078	26.0804 26.0974 26.7736	27.7852 27.7965 28.4783	29.4338 29.4415 30.127
7							19.9011 20.0463 20.5942	22.6165 22.6996 23.3097	25.2339 25.282 25.927	27.7633 27.7915 28.4564	30.2128 30.2295 30.906	32.5895 32.5995 33.2827	34.8992 34.9053 35.5924	37.147 37.1507 37.8401
8								26.6393 26.7201 27.3324	29.9164 29.9604 30.6096	33.102 33.1261 33.7952	36.2043 36.2177 36.8975	39.2304 39.2378 39.9235	42.186 42.1902 42.8792	45.0764 45.0788 45.7696
9									34.5644 34.6096 35.2575	38.4241 38.4479 39.1173	42.1988 42.2114 42.892	45.8953 45.902 46.5885	49.5197 49.5233 50.2128	53.0769 53.0789 53.7701
10										43.693 43.7187 44.3861	48.1502 48.1635 48.8434	52.5284 52.5353 53.2216	56.8335 56.837 57.5266	61.0705 61.0723 61.7636
11											54.0381 54.0528 54.7312	59.1036 59.1111 59.7967	64.0955 64.0993 64.7886	69.0189 69.0208 69.712
12												65.6106 65.6191 66.3038	71.2925 71.2968 71.9856	76.9056 76.9078 77.5988
13													78.4205 78.4254 79.1137	84.7251 84.7275 85.4182
14														92.4768 92.4797 93.17

Table 1: Exact values of $\ln|B_u(n,r)|, 1 \leq n < r \leq 15$, and natural logarithms of lower and upper bounds.

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