Starlikeness of a generalized Bessel function

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Abstract

This paper investigates three functions $\mathtt{f}_{a,\nu}$, $\mathtt{g}_{a,\nu}$ and $\mathtt{h}_{a,\nu}$ in the class $\mathcal{A}$ consisting of analytic functions f in the unit disk satisfying $f(0) = f'(0) - 1 = 0$. Here $a \in \{1, 2, 3, \ldots\}$, and ν is real. Each function is related to the generalized Bessel function. The radius of starlikeness of positive order is obtained for each of the three functions. Further, the best range on ν is determined for a fixed a to ensure the functions $\mathtt{f}_{a,\nu}$ and $\mathtt{g}_{a,\nu}$ are starlike of positive order in the entire unit disk. When $a = 1$, the results obtained reduced to earlier known results.

1 Introduction

There is a vast literature describing the importance and applications of the Bessel function of the first kind of order p given by

$$\mathtt{J}_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$

where Γ is the familiar gamma function. Various generalizations of the Bessel function have also been studied. Perhaps a more complete generalization is that given by Baricz in [3]. In this case, the generalized Bessel function takes the form

$${}_a\mathtt{B}_{b,p,c}(x) := \sum_{k=0}^{\infty} \frac{(-c)^k}{k!\ \Gamma\left(ak+p+\frac{b+1}{2}\right)} \left(\frac{x}{2}\right)^{2k+p} \tag{1}$$

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for $a \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and $b, p, c, x \in \mathbb{R}$. It is evident that the function ${}_a\mathtt{B}_{b,p,c}$ converges absolutely at each $x \in \mathbb{R}$. This generalized Bessel function was further investigated in [1,2] for $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. It was shown in [2] that the generalized Bessel function ${}_a\mathtt{B}_{b,p,c}$ is a solution of an $(a+1)$-order differential equation

$$(D-p)\prod_{j=1}^{a}\left(D+\tfrac{2p+b+1-2j}{a}-p\right)y(x)+\frac{cx^2}{a^a 2^{1-a}}y(x)=0,$$

where the operator D is given by $D := x(d/dx)$. For $a = 1$, the differential equation reduces to

$$x^2y''(x)+bxy'(x)+(cx^2-p^2+(1-b)p)y(x)=0.$$

Thus it yields the classical Bessel differential equation for $b = c = 1$. Interesting functional inequalities for ${}_a\mathtt{B}_{b,p,-\alpha^2}$ were obtained in [2], particularly for the case $a = 2$.

In [4], Baricz *et. al* investigated geometric properties involving the Bessel function of the first kind in $\mathbb{D}$ for the following three functions :

$$\begin{aligned}
\mathtt{f}_\nu(z) &= \left(2^\nu\Gamma(\nu+1)\mathtt{J}_\nu(z)\right)^{\frac{1}{\nu}},\\
\mathtt{g}_\nu(z) &= 2^\nu\Gamma(\nu+1)z^{1-\nu}\mathtt{J}_\nu(z),\\
\mathtt{h}_\nu(z) &= 2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}\mathtt{J}_\nu(\sqrt{z}).
\end{aligned}\tag{2}$$

Each function is suitably normalized to ensure that it belongs to the class $\mathcal{A}$ consisting of analytic functions f in $\mathbb{D}$ satisfying $f(0) = f'(0) - 1 = 0$. Here the principal branch is assumed, which is positive for z positive.

An important geometric feature of a complex-valued function is starlikeness. For $0 \leq \beta < 1$, the class of starlike functions of order β, denoted by $\mathcal{S}^*(\beta)$, are functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad \text{for all} \quad z \in \mathbb{D}.$$

In the case $\beta = 0$, these functions are simply said to be starlike (with respect to the origin). Geometrically $f \in \mathcal{S}^* := \mathcal{S}^*(0)$ if the linear segment tw, $0 \leq t \leq 1$, lies completely in $f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$. A starlike function is necessarily univalent in $\mathbb{D}$.

The three functions given by (2) do not possess the property of starlikeness in the whole disk $\mathbb{D}$. Thus it is of interest to find the largest subdisk in $\mathbb{D}$ that gets mapped by these functions onto starlike domains. In general, the *radius of starlikeness of order* β for a given class $\mathcal{G}$ of $\mathcal{A}$, denoted by r^*_β, is the largest number $r_0 \in (0,1)$ such that $r^{-1}f(rz) \in \mathcal{S}^*(\beta)$ for $0 < r \leq r_0$ and for all $f \in \mathcal{G}$. Analytically,

$$r^*_\beta(\mathcal{G}) := \sup\left\{r > 0 : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}_r, f \in \mathcal{G}\right\},$$

where $\mathbb{D}_r = \{z : |z| < r\}$.

In [4], Baricz *et. al* obtained the radius of starlikeness of order β for each of the three functions $\mathtt{f}_\nu$, $\mathtt{g}_\nu$, and $\mathtt{h}_\nu$ given by (2). This extends the earlier work of Brown in [7] who obtained the radius of starlikeness (of order 0) for functions $\mathtt{f}_\nu$ and $\mathtt{g}_\nu$.

For $a \in \mathbb{N}$, we consider the following extension of the three functions in (2) involving the generalized Bessel function:

$$\begin{aligned}
\mathtt{f}_{a,\nu}(z) &:= \left(2^{a\nu-a+1}a^{-\frac{a(a\nu-a+1)}{2}}\Gamma(a\nu+1)\,{}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)\right)^{\frac{1}{a\nu-a+1}},\\
\mathtt{g}_{a,\nu}(z) &:= 2^{a\nu-a+1}a^{-\frac{a}{2}(a\nu-a+1)}\Gamma(a\nu+1)z^{a-a\nu}\,{}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z), \qquad (3)\\
\mathtt{h}_{a,\nu}(z) &:= 2^{a\nu-a+1}a^{-\frac{a}{2}(a\nu-a+1)}\Gamma(a\nu+1)z^{\frac{1}{2}(1+a-a\nu)}\,{}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}\sqrt{z}).
\end{aligned}$$

Here the function $\mathtt{f}_{a,\nu}$ is taken to be the principal branch (see section 3). Evidently for $a = 1$, these functions are those given by (2) treated by Baricz *et. al* in [4]. Denote by $r^*_\beta(f)$ to be the radius of starlikeness of order β for a given function f.

In this paper, we find $r^*_\beta(f_a)$ when f_a is either one of the three functions in (3). These are given in Theorem 3.7, Theorem 3.9, and Theorem 3.10 in section 3. Section 4 is devoted to finding the best range on ν corresponding to a fixed a to ensure the functions $\mathtt{f}_{a,\nu}$ and $\mathtt{g}_{a,\nu}$ are starlike of order β in the whole unit disk. These are presented in Theorem 4.1 and Theorem 4.2. When $a = 1$, the results obtained reduced to earlier known results.

2 Preliminaries

The following two results will be required. First, for $a = 1$, the generalized Bessel function (1) is simply written as $\mathtt{B}_{b,p,c} := {}_1\mathtt{B}_{b,p,c}$. Thus

$$\mathtt{B}_{b,p,c}(z) := \sum_{k=0}^{\infty}\frac{(-c)^k}{k!\,\Gamma\left(k+p+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2k+p}. \qquad (4)$$

Proposition 2.1. *[2, Proposition 2.2] Let $a \in \mathbb{N}$, and $b, p, c, \in \mathbb{R}$. Then*

$${}_a\mathtt{B}_{b,p,c}(z) = (2\pi)^{\frac{a-1}{2}}a^{-p-\frac{b}{2}}\left(\frac{z}{2}\right)^p\prod_{j=1}^{a}\left(\frac{z}{2a^{a/2}}\right)^{-\frac{p+j-1}{a}}\mathtt{B}_{\frac{b+1-a}{a},\frac{p+j-1}{a},c}\left(\frac{z}{a^{a/2}}\right),$$

where $\mathtt{B}_{b,p,c}$ is given by (4).

In [2], the generalized Bessel function was also shown to satisfy the following relations:

$$z\frac{d}{dz}{}_a\mathtt{B}_{b,p,c}(z) = p\,{}_a\mathtt{B}_{b,p,c}(z) - c\left(\frac{z}{2}\right)^{1-a}z\,{}_a\mathtt{B}_{b,p+a,c}(z),$$

and

$$z\frac{d}{dz}{}_a\mathtt{B}_{b,p,c}(z) = \frac{z}{a}{}_a\mathtt{B}_{b,p-1,c}(z) - \left(\frac{2p+b-1}{a}-p\right){}_a\mathtt{B}_{b,p,c}(z), \qquad (5)$$

which together lead to the following result.

Proposition 2.2. *[2, Proposition 2.3] Let $a \in \mathbb{N}$, $b, p, c \in \mathbb{R}$ and $z \in \mathbb{D}$. Then*

$$\frac{z}{a}{}_a\mathtt{B}_{b,p-1,c}(z) + c\left(\frac{z}{2}\right)^{1-a} z_a\mathtt{B}_{b,p+a,c}(z) = \left(\tfrac{2p+b-1}{a}\right){}_a\mathtt{B}_{b,p,c}(z).$$

3 Radius of starlikeness of generalized Bessel functions

The following preliminary result sheds insights into the zeros of the three functions given by (3).

Theorem 3.1. *Let $\nu > (a-1)/a$, $a \in \mathbb{N}$. Then all zeros of ${}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)$ are real. Further the origin is the only zero of ${}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)$ in the unit disk $\mathbb{D}$.*

Proof. Proposition 2.1 shows that

$$\begin{aligned}{}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z) = (2\pi)^{\frac{a-1}{a}} a^{-(a\nu+\frac{1}{2})} a^{\frac{a}{2}(a\nu-a+1)}\left(\frac{z}{2}\right)^{a\nu-a+1} \\ \times \prod_{j=1}^{a}\left(\frac{z}{2}\right)^{-(\nu-1)-\frac{j}{a}} \mathtt{B}_{1,(\nu-1)+j/a,1}(z).\end{aligned}$$

Since

$$\mathtt{B}_{1,(\nu-1)+j/a,1}(z) = \mathtt{J}_{(\nu-1)+j/a}(z),$$

it readily follows that

$$\begin{aligned}{}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z) = (2\pi)^{\frac{a-1}{a}} a^{\frac{1}{2}(a^2\nu-2a\nu-a^2+a-1)}\left(\frac{z}{2}\right)^{-\frac{1}{2}(a-1)} \\ \times \mathtt{J}_{(\nu-1)+1/a}(z)\mathtt{J}_{(\nu-1)+2/a}(z)\ldots\mathtt{J}_{\nu}(z).\end{aligned}$$

Now, $\nu-1+(j/a) \geq \nu-1+(1/a) > 0$, $j = 1, \ldots, a$. Further for $p > -1$, it is known [12, p. 483] that the zeros of $\mathtt{J}_p$ are all real. If $\mathtt{j}_{p,k}$ denotes the k-th positive zero of $\mathtt{J}_p$, it is also known [12, p. 508] that when p is positive, the positive zeros of $\mathtt{J}_p$ increases as p increases. Thus we infer that the zeros of ${}_a\mathtt{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)$ are all real. Since

$$\mathtt{j}_{\nu,1} > \mathtt{j}_{\nu-1,1} > \ldots > \mathtt{j}_{\nu-1+(1/a),1} > \mathtt{j}_{0,1} \approx 2.40483,$$

the only zero in $\mathbb{D}$ occurs at the origin. ∎

Theorem 3.1 shows that the function

$$\mathtt{f}_{a,\nu}(z) = z\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k\Gamma(a\nu+1)a^{ak}}{k!2^{2k}\Gamma(ak+a\nu+1)}z^{2k}\right)^{\frac{1}{a\nu-a+1}}$$

has only one zero inside $\mathbb{D}$ whenever $\nu-1+(1/a) > 0$. Thus in this instance, we may take the principal branch for $\mathtt{f}_{a,\nu} \in \mathcal{A}$. It is also readily verified that the functions $\mathtt{g}_{a,\nu}$ and

$$\begin{aligned}\mathtt{h}_{a,\nu}(z) = z - \frac{\Gamma(a\nu+1)}{1!2^2\Gamma(a+a\nu+1)}a^a z^2 + \frac{\Gamma(a\nu+1)}{2!2^4\Gamma(2a+a\nu+1)}a^{2a}z^3 + \cdots \\ + (-1)^k\frac{\Gamma(a\nu+1)}{k!2^{2k}\Gamma(ak+a\nu+1)}a^{ak}z^{k+1} + \cdots\end{aligned}$$

are both analytic and belong to the normalized class $\mathcal{A}$.

The following is another preliminary result required in the sequel.

Lemma 3.2. *Let $a \in \mathbb{N}$ and $\nu > -1/a$. Then*

$$\frac{z\ {}_a\mathrm{B}'_{2a-1,a\nu-a+1,1}(a^{a/2}z)}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)} = \frac{z\mathrm{J}_{\nu-1}(z)}{\mathrm{J}_\nu(z)} - (2-a)\nu + 1 - a.$$

Proof. Since

$$\mathrm{B}_{b,p,1}(z) = \left(\frac{2}{z}\right)^{\frac{b-1}{2}} \mathrm{J}_{p+\frac{b-1}{2}}(z),$$

it follows from Proposition 2.1 that

$${}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(z) =$$
$$(2\pi)^{\frac{a-1}{a}} a^{-\frac{2a\nu+1}{2}} \left(\frac{z}{2}\right)^{a\nu-a+1} \prod_{j=1}^{a} a^{\frac{a\nu-a+j}{2}} \left(\frac{2}{z}\right)^{\frac{a\nu-a+j}{a}} \mathrm{J}_{\frac{a\nu-a+j}{a}}\left(\frac{z}{a^{a/2}}\right),$$

and

$${}_a\mathrm{B}_{2a-1,a\nu-a,1}(z) =$$
$$(2\pi)^{\frac{a-1}{a}} a^{-\frac{2a\nu-1}{2}} \left(\frac{z}{2}\right)^{a\nu-a} \prod_{j=1}^{a} a^{\frac{a\nu-a+j-1}{2}} \left(\frac{2}{z}\right)^{\frac{a\nu-a+j-1}{a}} \mathrm{J}_{\frac{a\nu-a+j-1}{a}}\left(\frac{z}{a^{a/2}}\right).$$

Expanding the above products, a routine calculation shows that

$$\frac{{}_a\mathrm{B}_{2a-1,a\nu-a,1}(z)}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(z)} = a^{1-\frac{a}{2}} \frac{\mathrm{J}_{\nu-1}\left(\frac{z}{a^{a/2}}\right)}{\mathrm{J}_\nu\left(\frac{z}{a^{a/2}}\right)}.$$

With $b = 2a-1$ and $p = a\nu - a + 1$, the recurrence relation (5) gives

$$z\frac{d}{dz}{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(z) = \frac{z}{a}{}_a\mathrm{B}_{2a-1,a\nu-a,1}(z) - (\nu(2-a)+a-1)\,{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(z).$$

Replacing z by $a^{a/2}z$ leads to

$$\begin{aligned}\frac{z\ {}_a\mathrm{B}'_{2a-1,a\nu-a+1,1}(a^{a/2}z)}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)} &= \frac{a^{a/2}z}{a}\frac{{}_a\mathrm{B}_{2a-1,a\nu-a,1}(a^{a/2}z)}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)} - (2-a)\nu + 1 - a \\ &= \frac{z\mathrm{J}_{\nu-1}(z)}{\mathrm{J}_\nu(z)} - (2-a)\nu + 1 - a,\end{aligned}$$

which proves the assertion. ∎

A result on the modified Bessel function of order p given by

$$\mathrm{I}_p(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+p+1)} \left(\frac{z}{2}\right)^{2k+p}$$

is the final preliminary result required in the sequel.

Proposition 3.3. *Let $\alpha, \nu \in \mathbb{R}$ satisfy $-1 < \nu < -\alpha$. Then the equation $r\mathtt{I}'_\nu(r) + \alpha\mathtt{I}_\nu(r) = 0$ has a unique root in $(0, \infty)$.*

Proof. Consider the function

$$q(r) := \frac{r\mathtt{I}'_\nu(r)}{\mathtt{I}_\nu(r)} + \alpha.$$

It is known from [2, Theorem 3.1(c)] that $r\mathtt{I}'_\nu(r)/I_\nu(r)$ is increasing on $(0, \infty)$. Further, the asymptotic properties show that $r\mathtt{I}'_\nu(r)/\mathtt{I}_\nu(r) \to \nu$ as $r \to 0$, and $r\mathtt{I}'_\nu(r)/\mathtt{I}_\nu(r) \to \infty$ as $r \to \infty$. This implies that $q(r) \to \nu + \alpha < 0$ as $r \to 0$, and $q(r) \to \infty$ for $r \to \infty$. Thus q has exactly one zero. ∎

We also recall additional facts on the zeros of the Dini functions.

Lemma 3.4. *[12, p. 482] If $\nu > -1$ and $\alpha, \gamma \in \mathbb{R}$, then the Dini function $z \mapsto \alpha\mathtt{J}_\nu(z) + \gamma z\mathtt{J}'_\nu(z)$ has all its zeros real whenever $((\alpha/\gamma) + \nu) \geq 0$. In the case $((\alpha/\gamma) + \nu) < 0$, it also has two purely imaginary zeros.*

Lemma 3.5. *[9, Theorem 6.1] Let $\alpha \in \mathbb{R}$, $\nu > -1$ and $\nu + \alpha > 0$. Further let $x_{\nu,1}$ be the smallest positive root of $\alpha\mathtt{J}_\nu(z) + z\mathtt{J}'_\nu(z) = 0$. Then $x^2_{\nu,1} < \mathtt{j}^2_{\nu,1}$.*

Lemma 3.6. *[8, p. 78] Let $-1 < \nu < -\alpha$, and $\pm i\zeta$ be the single pair of conjugate purely imaginary zeros of the Dini function $z \mapsto \alpha\mathtt{J}_\nu(z) + z\mathtt{J}'_\nu(z)$. Then*

$$\zeta^2 < -\frac{\alpha + \nu}{2 + \alpha + \nu}\mathtt{j}^2_{\nu,1}.$$

We are now ready to present the radius of starlikeness for each function given in (3).

Theorem 3.7. *Let $0 \leq \beta < 1$, and $a \in \mathbb{N}$. If $\nu > (a-1)/a$, then $r^*_\beta(f_{a,\nu}) = \mathtt{j}^{\mathtt{a,f}}_{\nu,\beta,1}$, where $\mathtt{j}^{\mathtt{a,f}}_{\nu,\beta,1}$ is the smallest positive root of the equation*

$$ra^{a/2}\mathtt{J}'_\nu(r) - \Big((\nu-1)(1-a)a^{a/2} + \beta(a\nu - a + 1)\Big)\,\mathtt{J}_\nu(r) = 0. \tag{6}$$

If $\nu \in (-1/a, (a-1)/a)$ and

$$\frac{(a\nu - a + 1)\left(a^{a/2} - \beta\right)}{2a^{a/2} + (a\nu - a + 1)\left(a^{a/2} - \beta\right)} > -1, \tag{7}$$

then $r^*_\beta(f_{a,\nu}) = \mathtt{i}^{\mathtt{a,f}}_{\nu,\beta}$, where $\mathtt{i}^{\mathtt{a,f}}_{\nu,\beta}$ is the unique positive root of the equation

$$ra^{a/2}\mathtt{I}'_\nu(r) - \Big((\nu-1)(1-a)a^{a/2} + \beta(a\nu - a + 1)\Big)\,\mathtt{I}_\nu(r) = 0. \tag{8}$$

Proof. Differentiating logarithmically, Lemma 3.2 shows that

$$\begin{aligned}\frac{z\mathtt{f}'_{a,\nu}(z)}{\mathtt{f}_{a,\nu}(z)} &= \frac{a^{a/2}}{a\nu - a + 1}\frac{z\,{}_a\mathrm{B}'_{2a-1,a\nu-a+1,1}(a^{a/2}z)}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)} \\ &= \frac{a^{a/2}}{a\nu - a + 1}\left(\frac{z\mathtt{J}_{\nu-1}(z)}{\mathtt{J}_\nu(z)} - \nu(2-a) + 1 - a\right).\end{aligned} \tag{9}$$

Since ${}_1\mathtt{B}_{1,\nu,1}(z) = \mathtt{J}_\nu(z)$, the relation (5) leads to the well-known recurrence relation

$$z\mathtt{J}_\nu'(z) = z\mathtt{J}_{\nu-1}(z) - \nu\mathtt{J}_\nu(z),$$

and whence (9) reduces to

$$\frac{z\mathtt{f}_{a,\nu}'(z)}{\mathtt{f}_{a,\nu}(z)} = \frac{a^{a/2}}{a\nu - a + 1}\left(\frac{z\mathtt{J}_\nu'(z)}{\mathtt{J}_\nu(z)} - (\nu-1)(1-a)\right). \tag{10}$$

With $\mathtt{j}_{\nu,n}$ as the n-th positive zero of the Bessel function $\mathtt{J}_\nu$, the Bessel function $\mathtt{J}_\nu$ admits the Weierstrassian decomposition [12, p.498]

$$\mathtt{J}_\nu(z) = \frac{z^\nu}{2^\nu\Gamma(\nu+1)}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\mathtt{j}_{\nu,n}^2}\right).$$

Thus

$$\frac{z\ \mathtt{J}_\nu'(z)}{\mathtt{J}_\nu(z)} = \nu - \sum_{n=1}^{\infty}\frac{2z^2}{\mathtt{j}_{\nu,n}^2 - z^2},$$

which reduces (10) to

$$\frac{z\mathtt{f}_{a,\nu}'(z)}{\mathtt{f}_{a,\nu}(z)} = a^{a/2} - \frac{a^{a/2}}{a\nu - a + 1}\sum_{n=1}^{\infty}\frac{2z^2}{\mathtt{j}_{\nu,n}^2 - z^2}. \tag{11}$$

For $\nu > (a-1)/a$ and $|z| < \mathtt{j}_{\nu,n}$, evidently

$$\begin{aligned}\operatorname{Re}\frac{z\mathtt{f}_{a,\nu}'(z)}{\mathtt{f}_{a,\nu}(z)} &= a^{a/2} - \frac{a^{a/2}}{a\nu - a + 1}\operatorname{Re}\sum_{n=1}^{\infty}\frac{2z^2}{\mathtt{j}_{\nu,n}^2 - z^2}\\ &\geq a^{a/2} - \frac{a^{a/2}}{a\nu - a + 1}\sum_{n=1}^{\infty}\frac{2|z|^2}{\mathtt{j}_{\nu,n}^2 - |z|^2} = \frac{|z|\mathtt{f}_{a,\nu}'(|z|)}{\mathtt{f}_{a,\nu}(|z|)}.\end{aligned}$$

Equality holds for $|z| = r$, and by the minimum principle for harmonic functions,

$$\operatorname{Re}\frac{z\mathtt{f}_\nu'(z)}{\mathtt{f}_\nu(z)} \geq \beta \iff |z| \leq \mathtt{j}_{\nu,\beta,1}^{\mathtt{a,f}},$$

where $\mathtt{j}_{\nu,\beta,1}^{\mathtt{a,f}}$ is the smallest positive root of equation (6). Since

$$\nu - \left((\nu-1)(1-a) + \tfrac{\beta(a\nu - a+1)}{a^{a/2}}\right) = (a\nu - a + 1)\left(1 - \tfrac{\beta}{a^{a/2}}\right) > 0$$

for all $\nu > (a-1)/a$, we infer from Lemma 3.4 and Lemma 3.5 that $\mathtt{j}_{\nu,\beta,1}^{\mathtt{a,f}} < \mathtt{j}_{\nu,1} < \mathtt{j}_{\nu,n}$.

Consider next the case $-1/a < \nu < (a-1)/a$. It is known from [4, p. 2023] that for $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ with $\alpha \geq |z|$, then

$$\operatorname{Re}\left(\frac{z}{\alpha - z}\right) \geq -\frac{|z|}{\alpha + |z|},$$

which in turn implies that

$$\operatorname{Re}\left(\frac{z^2}{\mathtt{j}_{\nu,n}^2 - z^2}\right) \geq -\frac{|z|^2}{\mathtt{j}_{\nu,n}^2 + |z|^2}$$

whenever $|z| < \mathtt{j}_{\nu,1} < \mathtt{j}_{\nu,n}$.

The expression (11) yields

$$\operatorname{Re}\frac{z\mathtt{f}'_{a,\nu}(z)}{\mathtt{f}_{a,\nu}(z)} \geq a^{a/2} + \frac{a^{a/2}}{a\nu - a + 1}\sum_{n=1}^{\infty}\frac{2|z|^2}{\mathtt{j}_{\nu,n}^2 + |z|^2} = \frac{i|z|\mathtt{f}'_{a,\nu}(i|z|)}{\mathtt{f}_{a,\nu}(i|z|)}.$$

Equality holds for $|z| = i|z| = ir$. Hence

$$\operatorname{Re}\frac{z\mathtt{f}'_{a,\nu}(z)}{\mathtt{f}_{a,\nu}(z)} \geq \beta$$

if $|z| \leq \mathtt{i}_{\nu,\beta}^{a,f}$, where $\mathtt{i}_{\nu,\beta}^{a,f}$ is a root of $i|z|\mathtt{f}'_{a,\nu}(i|z|) = \beta\mathtt{f}_{a,\nu}(i|z|)$, that is, $\mathtt{i}_{\nu,\beta}^{a,f}$ is a root of

$$\frac{a^{a/2}}{a\nu - a + 1}\left(\frac{i|z|\mathtt{J}'_\nu(i|z|)}{\mathtt{J}_\nu(i|z|)} - (\nu - 1)(1 - a)\right) = \beta.$$

Since $\mathtt{I}_\nu(z) = i^{-\nu}\mathtt{J}_\nu(iz)$, the above equation is equivalent to (8). It also follows from Proposition 3.3 that the root $\mathtt{i}_{\nu,\beta}^{a,f}$ is unique. Finally, that $\mathtt{i}_{\nu,\beta}^{a,f} < \mathtt{j}_{\nu,n}$ is a consequence of Lemma 3.6 and assumption (7). Indeed,

$$\left(\mathtt{i}_{\nu,\beta}^{a,f}\right)^2 < -\frac{(a\nu - a + 1)\left(a^{a/2} - \beta\right)}{2a^{a/2} + (a\nu - a + 1)\left(a^{a/2} - \beta\right)}\mathtt{j}_{\nu,1}^2 < \mathtt{j}_{\nu,1}^2 < \mathtt{j}_{\nu,n}^2,$$

which completes the proof. ∎

Interestingly, Theorem 3.7 reduces to earlier known result for $a = 1$.

Corollary 3.8. *[4, Theorem 1(a)] Let $0 \leq \beta < 1$. If $\nu > 0$, then $r^*_\beta(f_{1,\nu})$ is the smallest positive root $\mathtt{j}_{\nu,\beta,1}^{1,\mathtt{f}}$ of the equation*

$$r\mathtt{J}'_\nu(r) - \beta\nu\mathtt{J}_\nu(r) = 0.$$

In the case $\nu \in (-1, 0)$, then $r^*_\beta(f_{1,\nu})$ is the unique positive root $\mathtt{i}_{\nu,\beta}^{1,\mathtt{f}}$ of the equation

$$r\mathtt{I}'_\nu(r) - \beta\nu\mathtt{I}_\nu(r) = 0.$$

The next two results find the radius of starlikeness of order β for the functions $\mathtt{g}_{a,\nu}$ and $\mathtt{h}_{a,\nu}$ given in (3).

Theorem 3.9. *Let $\beta \in [0, 1)$, $a \in \mathbb{N}$, and $\nu > -1/a$. If $a(\nu - 1)(a^{a/2} - 1) + a^{a/2} - \beta \geq 0$, then $r^*_\beta(\mathtt{g}_{a,\nu}) = \mathtt{j}_{\nu,\beta,1}^{\mathtt{a,g}}$, where $\mathtt{j}_{\nu,\beta,1}^{\mathtt{a,g}}$ is the smallest positive root of the equation*

$$ra^{a/2}\mathtt{J}'_\nu(r) - \left((\nu - 1)(1 - a)a^{a/2} - a(1 - \nu) + \beta\right)\mathtt{J}_\nu(r) = 0. \tag{12}$$

Proof. It follows from (3) that

$$\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)} = a(1-\nu) + a^{a/2}\frac{z\ {}_a\mathrm{B}'_{2a-1,a\nu-a+1,1}(a^{a/2}z)}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}z)}.$$

As in the proof of Theorem 3.7 (see (9) and (10)), it is readily shown that

$$\begin{aligned}\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)} &= a(1-\nu) + a^{a/2}\left(\frac{z\mathrm{J}_{\nu-1}(z)}{\mathrm{J}_{\nu}(z)} - \nu(2-a) + 1 - a\right)\\ &= a(1-\nu) + a^{a/2}\left(\frac{z\mathrm{J}'_{\nu}(z)}{\mathrm{J}_{\nu}(z)} - (\nu-1)(1-a)\right)\\ &= a(1-\nu) + a^{a/2}\left[a\nu + 1 - a - \sum_{n=1}^{\infty}\frac{2z^2}{\mathrm{j}_{\nu,n}^2 - z^2}\right]. \qquad (13)\end{aligned}$$

This implies that

$$\operatorname{Re}\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)} \geq a(1-\nu) + a^{a/2}\left[a\nu + 1 - a - \sum_{n=1}^{\infty}\frac{2|z|^2}{\mathrm{j}_{\nu,n}^2 - |z|^2}\right] = \frac{|z|\mathrm{g}'_{a,\nu}(|z|)}{\mathrm{g}_{a,\nu}(|z|)}$$

provided $|z| < \mathrm{j}_{\nu,n}$. Equality holds at $|z| = r$. The minimum principle for harmonic functions leads to

$$\operatorname{Re}\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)} \geq \beta \iff |z| \leq r^*_\beta(\mathrm{g}_{a,\nu}).$$

The exact value of $r^*_\beta(\mathrm{g}_{a,\nu})$ is obtained from the equation $r\mathrm{g}'_{a,\nu}(r) = \beta\mathrm{g}_{a,\nu}(r)$. From (13), this is equivalent to determining the root of (12).

If $a(\nu-1)(a^{a/2}-1) + a^{a/2} - \beta \geq 0$, then Lemma 3.4 shows that all roots of (12) are real. In this case, $r^*_\beta(\mathrm{g}_{a,\nu})$ is its smallest positive root $\mathrm{j}^{a,g}_{\nu,\beta,1}$. Finally, Lemma 3.5 shows that $\mathrm{j}^{a,g}_{\nu,\beta,1} < \mathrm{j}_{\nu,1}$, and whence $|z| < r^*_\beta(\mathrm{g}_{a,\nu}) < \mathrm{j}_{\nu,1}$. ∎

Theorem 3.10. *Let $\beta \in [0,1)$, $a \in \mathbb{N}$, and $\nu > -1/a$. If $(a^{a/2}-1)(1-a+a\nu) + 2(1-\beta) > 0$, then $r^*_\beta(\mathrm{h}_{a,\nu}) = \mathrm{j}^{\mathrm{a,h}}_{\nu,\beta,1}$, where $\mathrm{j}^{\mathrm{a,h}}_{\nu,\beta,1}$ is the smallest positive root of the equation*

$$a^{a/2}r\mathrm{J}'_\nu(r) + \Big((a^{a/2}-1)(1-a+a\nu) - a^{a/2}\nu + 2(1-\beta)\Big)\mathrm{J}_\nu(r) = 0. \qquad (14)$$

Proof. It follows from (3) that

$$\frac{\mathrm{h}'_{a,\nu}(z)}{\mathrm{h}_{a,\nu}(z)} = \frac{1+a-a\nu}{2z} + \frac{a^{a/2}}{2\sqrt{z}}\frac{{}_a\mathrm{B}'_{2a-1,a\nu-a+1,1}(a^{a/2}\sqrt{z})}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}\sqrt{z})},$$

and thus Lemma 3.2 yields

$$\begin{aligned}\frac{z\mathrm{h}'_{a,\nu}(z)}{\mathrm{h}_{a,\nu}(z)} &= \frac{1+a-a\nu}{2} + \frac{a^{a/2}\sqrt{z}}{2}\frac{{}_a\mathrm{B}'_{2a-1,a\nu-a+1,1}(a^{a/2}\sqrt{z})}{{}_a\mathrm{B}_{2a-1,a\nu-a+1,1}(a^{a/2}\sqrt{z})}\\ &= \frac{1+a-a\nu}{2} + \frac{a^{a/2}}{2}\left(\frac{\sqrt{z}\mathrm{J}'_\nu(\sqrt{z})}{\mathrm{J}_\nu(\sqrt{z})} - (\nu-1)(1-a)\right)\\ &= 1 - \frac{(a\nu+1-a)(1-a^{a/2})}{2} - a^{a/2}\sum_{n=1}^{\infty}\frac{z}{\mathrm{j}_{\nu,n}^2 - z}.\end{aligned}$$

Proceeding similarly as in the proof of Theorem 3.7, it is readily shown that

$$\operatorname{Re}\frac{z\mathtt{h}'_{a,\nu}(z)}{\mathtt{h}_{a,\nu}(z)} \geq 1 - \frac{(a\nu+1-a)(1-a^{a/2})}{2} - a^{a/2}\sum_{n=1}^{\infty}\frac{|z|}{\mathtt{j}^2_{\nu,n}-|z|} = \frac{|z|\mathtt{h}'_{a,\nu}(|z|)}{\mathtt{h}_{a,\nu}(|z|)} = \beta$$

if and only if $|z| \leq r^*(\mathtt{h}_{\nu,\beta}) < \mathtt{j}_{\nu,n}$. Here $r^*(\mathtt{h}_{\nu,\beta})$ is the smallest root of the equation $r\mathtt{h}'_{a,\nu}(r)/\mathtt{h}_{a,\nu}(r) = \beta$, that is, a root of

$$\frac{1+a-a\nu}{2} + \frac{a^{a/2}}{2}\left(\frac{\sqrt{r}\mathtt{J}'_\nu(\sqrt{r})}{\mathtt{J}_\nu(\sqrt{r})} - (\nu-1)(1-a)\right) = \beta,$$

or equivalently, of the equation

$$a^{a/2}r\mathtt{J}'_\nu(r) + \left((a^{a/2}-1)(1-a+a\nu) - a^{a/2}\nu + 2(1-\beta)\right)\mathtt{J}_\nu(r) = 0.$$

Thus by Lemma 3.4, $r^*(\mathtt{h}_{\nu,\beta})$ is the smallest positive root $\mathtt{j}^{a,h}_{\nu,\beta,1}$ of (14) when $(a^{a/2}-1)(1-a+a\nu) + 2(1-\beta) > 0$. ∎

Remark 1. In the case $a = 1$, the condition $a(\nu-1)\left(a^{a/2}-1\right) + a^{a/2} - \beta = 1-\beta > 0$ and $\left(a^{a/2}-1\right)(1-a+a\nu) + 2(1-\beta) = 2(1-\beta) > 0$ both hold trivially for all $\beta \in [0,1)$. Both theorems therefore coincide with the earlier results in [4].

Further, it is of interest to determine the radius of starlikeness $r^*_\beta(\mathtt{g}_{a,\nu})$ in Theorem 3.9 in the event that $a(\nu-1)(a^{a/2}-1) + a^{a/2} - \beta < 0$, as well as that of $r^*_\beta(\mathtt{h}_{a,\nu})$ in Theorem 3.10 when $(a^{a/2}-1)(1-a+a\nu) + 2(1-\beta) < 0$.

4 Starlikeness of the generalized Bessel function

In this final section, the best range on ν is obtained for a fixed $a \in \mathbb{N}$ to ensure the functions $\mathtt{f}_{a,\nu}$ and $\mathtt{g}_{a,\nu}$ given by (3) are starlike of order β in $\mathbb{D}$.

Theorem 4.1. *For a fixed $a \in \mathbb{N}$, the function $\mathtt{f}_{a,\nu}$ given by* (3) *is starlike of order $\beta \in [0,1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_f(a,\beta)$, where $\nu_f(a,\beta)$ is the unique root of*

$$(a\nu - a + 1)(a^{a/2} - \beta)\mathtt{J}_\nu(1) = a^{a/2}\mathtt{J}_{\nu+1}(1)$$

in $((a-1)/a, \infty)$.

Proof. For $\nu > (a-1)/a$ and $|z| = r \in [0,1)$, it follows from (11) that

$$\operatorname{Re}\left(\frac{z\mathtt{f}'_{a,\nu}(z)}{\mathtt{f}_{a,\nu}(z)}\right) \geq \frac{r\mathtt{f}'_{a,\nu}(r)}{\mathtt{f}_{a,\nu}(r)} = a^{a/2} - \frac{a^{a/2}}{a\nu-a+1}\sum_{n=1}^{\infty}\frac{2r^2}{\mathtt{j}^2_{\nu,n}-r^2}.$$

The above inequality holds since $r < 1$ and it is known ([12, p. 508], [11, p. 236]) that the function $\nu \mapsto \mathtt{j}_{\nu,n}$ is increasing on $(0,\infty)$ for each fixed $n \in \mathbb{N}$, and whence $\mathtt{j}_{\nu,1} \geq \mathtt{j}_{(a-1)/a,1} \geq \mathtt{j}_{0,1} \approx 2.40483\ldots$.

A computation yields

$$\frac{d}{dr}\left(\frac{r\mathtt{f}'_{a,\nu}(r)}{\mathtt{f}_{a,\nu}(r)}\right) = -\frac{2a^{a/2}}{a\nu-a+1}\sum_{n=1}^{\infty}\frac{2r\mathtt{j}^2_{\nu,n}}{\left(\mathtt{j}^2_{\nu,n}-r^2\right)^2} \leq 0.$$

Hence

$$\operatorname{Re}\left(\frac{z\mathtt{f}'_{a,\nu}(z)}{\mathtt{f}_{a,\nu}(z)}\right) \geq a^{a/2} - \frac{a^{a/2}}{a\nu - a + 1}\sum_{n=1}^{\infty}\frac{2}{\mathtt{j}_{\nu,n}^2 - 1} = \frac{\mathtt{f}'_{a,\nu}(1)}{\mathtt{f}_{a,\nu}(1)}.$$

The monotonicity property of $\nu \mapsto \mathtt{j}_{\nu,n}$ leads to

$$\frac{\mathtt{f}'_{a,\mu}(1)}{\mathtt{f}_{a,\mu}(1)} = a^{a/2} - \frac{a^{a/2}}{a\mu - a + 1}\sum_{n=1}^{\infty}\frac{2}{\mathtt{j}_{\mu,n}^2 - 1} \geq a^{a/2} - \frac{a^{a/2}}{a\nu - a + 1}\sum_{n=1}^{\infty}\frac{2}{\mathtt{j}_{\nu,n}^2 - 1} = \frac{\mathtt{f}'_{a,\nu}(1)}{\mathtt{f}_{a,\nu}(1)},$$

$\mu \geq \nu > -1$. Since $\nu \mapsto \mathtt{f}'_{a,\nu}(1)/\mathtt{f}_{a,\nu}(1)$ is increasing in $((a-1)/a, \infty)$, and from consideration of the asymptotic behavior of $\mathtt{f}'_{a,\nu}(1)/\mathtt{f}_{a,\nu}(1)$, evidently $\mathtt{f}'_{a,\nu}(1)/\mathtt{f}_{a,\nu}(1) \geq \beta$ if and only if $\nu \geq \nu_f(a,\beta)$, where $\nu_f(a,\beta)$ is the unique root of the equation $\mathtt{f}'_{a,\nu}(1) = \beta\mathtt{f}_{a,\nu}(1)$. From (9), the latter equation is equivalent to

$$a^{a/2}\mathtt{J}_{\nu-1}(1) = \left(a^{a/2}(\nu(2-a) + a - 1) + \beta(a\nu - a + 1)\right)\mathtt{J}_\nu(1).$$

The recurrence relation in Proposition 2.2 now shows that $\nu_f(a,\beta)$ is a unique root of $(a\nu - a + 1)(a^{a/2} - \beta)\mathtt{J}_\nu(1) = a^{a/2}\mathtt{J}_{\nu+1}(1)$. Since all inequalities are sharp, it follows that the value $\nu_f(a,\beta)$ is best. ∎

Remark 2. With regards to Theorem 4.1, we tabulate the best value ν for a fixed β and a for which $\mathtt{f}_{a,\nu}$ is starlike of order β. These values are given in Table 1.

	$\beta = 0$	$\beta = 0.5$	$\beta = 0.95$
$a = 1$	$\nu = 0.39001$	$\nu = 0.645715$	$\nu = 2.72421$
$a = 2$	$\nu = 0.659908$	$\nu = 0.706779$	$\nu = 0.781815$
$a = 3$	$\nu = 0.766251$	$\nu = 0.776181$	$\nu = 0.786989$

Table 1 Values of ν for $\mathtt{f}_{a,\nu}$ to be starlike

Using (6), we tabulate the radius of starlikeness for $\mathtt{f}_{a,\nu}$ in Theorem 3.7 for a fixed $\nu = 0.7$, $a = 1, 2, 3$, and respectively $\beta = 0$, $\beta = 0.5$, and $\beta = 0.95$. These are given in Table 2. Here the value of $\mathtt{j}_{\nu,1}$ at $\nu = 0.7$ is $\mathtt{j}_{0.7,1} = 3.42189$. With reference to Table 1, we expect the radius of starlikeness to be less than 1 whenever $\nu = 0.7$ is less than the given values of ν in Table 1.

	$\beta = 0$	$\beta = 0.5$	$\beta = 0.95$
$a = 1$	$r_0^*(f_{1,0.7}) = 1.44678$	$r_{1/2}^*(f_{1,0.7}) = 1.05621$	$r_{0.95}^*(f_{1,0.7}) = 0.343848$
$a = 2$	$r_0^*(f_{2,0.7}) = 1.12397$	$r_{1/2}^*(f_{2,0.7}) = 0.982365$	$r_{0.95}^*(f_{2,0.7}) = 0.828745$
$a = 3$	$r_0^*(f_{3,0.7}) = 0.577726$	$r_{1/2}^*(f_{3,0.7}) = 0.549716$	$r_{0.95}^*(f_{3,0.7}) = 0.523133$

Table 2 Radius of starlikeness for $\mathtt{f}_{a,\nu}$ when $\nu = 0.7$

Letting

$$F(r) := \frac{r\mathtt{J}'_\nu(r)}{\mathtt{J}_\nu(r)},$$

then (6) takes the form $F(r) = -\alpha$, where

$$\alpha := \alpha(a, \beta, \nu) = -\left((\nu - 1)(1 - a) + \frac{\beta(a\nu - a + 1)}{a^{a/2}}\right).$$

For $\nu > 0$, it is known [6] that F(r) is strictly decreasing on $(0, \infty)$ except at the zeros of $\mathrm{J}_\nu(r)$. Differentiating with respect to β, it is clear that α is decreasing with respect to β so long as $a\nu - a + 1 > 0$, and thus r^*_β is decreasing. Further, for a fixed $\nu < 1$ and $\beta = 0$, then α is monotonically decreasing with respect to a, that is, r^*_0 is decreasing as a function of a. However, for β near 1, then α is no longer monotonic. For instance, choosing $\nu = 0.7$ and $\beta = 0.95$, Table 2 illustrates the fact that $r^*_{0.95}$ is not monotonic with respect to the parameter a.

Theorem 4.2. *Let $a \in \mathbb{N}$, $\nu > -1/a$, and $\mathrm{j}_{\nu,1}$ be the first positive zero of J_ν. Then the function $\mathrm{g}_{a,\nu}$ given by* (3) *is starlike of order $\beta \in [0, 1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_g(a, \beta)$, where $\nu_g(a, \beta)$ is the unique root in $(\max\{\tilde{\nu}, -1/a\}, \infty)$ of*

$$\left(a(\nu - 1)(a^{a/2} - 1) + a^{a/2} - \beta\right)\mathrm{J}_\nu(1) = a^{a/2}\mathrm{J}_{\nu+1}(1),$$

and $\tilde{\nu} \simeq -0.7745\ldots$ is the unique root of $\mathrm{j}_{\nu,1} = 1$.

Proof. It follows from (13) that

$$\begin{aligned}\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)} &= a(1 - \nu) + a^{a/2}\left(\frac{z\mathrm{J}'_\nu(z)}{\mathrm{J}_\nu(z)} - (\nu - 1)(1 - a)\right)\\ &= a(1 - \nu) + a^{a/2}\left(a\nu - a + 1 - \sum_{n=1}^{\infty}\frac{2z^2}{\mathrm{j}^2_{\nu,n} - z^2}\right),\end{aligned}$$

and

$$\mathrm{Re}\left(\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)}\right) \geq \frac{r\mathrm{g}'_{a,\nu}(r)}{\mathrm{g}_{a,\nu}(r)} = a(1 - \nu) + a^{a/2}\left(a\nu - a + 1 - \sum_{n=1}^{\infty}\frac{2r^2}{\mathrm{j}^2_{\nu,n} - r^2}\right)$$

for $|z| < \mathrm{j}_{\nu,1}$. The function $r \mapsto r\mathrm{g}'_{a,\nu}(r)/\mathrm{g}_{a,\nu}(r)$ is decreasing on $[0, 1)$. Since $\mathrm{j}_{\nu,1} > 1$ for $\nu > \max\{\tilde{\nu}, -1/a\}$, it follows that $\mathrm{j}_{\nu,n} > 1$ for each n, and consequently

$$\begin{aligned}\mathrm{Re}\left(\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)}\right) \geq \frac{r\mathrm{g}'_{a,\nu}(r)}{\mathrm{g}_{a,\nu}(r)} > \frac{\mathrm{g}'_{a,\nu}(1)}{\mathrm{g}_{a,\nu}(1)} = \\ a(1 - \nu) + a^{a/2}\left(a\nu - a + 1 - \sum_{n=1}^{\infty}\frac{2}{\mathrm{j}^2_{\nu,n} - 1}\right).\end{aligned}$$

Further, for $\mu \geq \nu > \max\{\tilde{\nu}, -1/a\}$,

$$\begin{aligned}\frac{\mathrm{g}'_{a,\mu}(1)}{\mathrm{g}_{a,\mu}(1)} &= a\left(a^{a/2} - 1\right)(\mu - 1) + a^{a/2} - \sum_{n=1}^{\infty}\frac{2a^{a/2}}{\mathrm{j}^2_{\mu,n} - 1}\\ &\geq a\left(a^{a/2} - 1\right)(\nu - 1) + a^{a/2} - \sum_{n=1}^{\infty}\frac{2a^{a/2}}{\mathrm{j}^2_{\nu,n} - 1} = \frac{\mathrm{g}'_{a,\nu}(1)}{\mathrm{g}_{a,\nu}(1)}.\end{aligned}$$

This implies that $\nu \mapsto \mathrm{g}'_{a,\nu}(1)/\mathrm{g}_{a,\nu}(1)$ is increasing on $(\max\{\tilde{\nu}, -1/a\}, \infty)$.

Thus

$$\operatorname{Re}\left(\frac{z\mathrm{g}'_{a,\nu}(z)}{\mathrm{g}_{a,\nu}(z)}\right) > \frac{\mathrm{g}'_{a,\nu}(1)}{\mathrm{g}_{a,\nu}(1)} \geq \beta$$

if and only if $\nu \geq \nu_g(a,\beta)$, where $\nu_g(a,\beta)$ is the unique root of $\mathrm{g}'_{a,\nu}(1) = \beta \mathrm{g}_{a,\nu}(1)$, or equivalently,

$$a(1-\nu) + a^{a/2}\left(\frac{\mathrm{J}'_\nu(1)}{\mathrm{J}_\nu(1)} - (\nu-1)(1-a)\right) = \beta.$$

Finally, Proposition 2.2 implies that $\nu_g(a,\beta)$ is a unique root of

$$\left(a(\nu-1)(a^{a/2}-1) + a^{a/2} - \beta\right)\mathrm{J}_\nu(1) = a^{a/2}\mathrm{J}_{\nu+1}(1). \qquad \blacksquare$$

Remark 3. The best value ν obtained from Theorem 4.2 for a fixed β and a for which $\mathrm{g}_{a,\nu}$ is starlike of order β is given in Table 3.

	$\beta = 0$	$\beta = 0.5$	$\beta = 0.95$
$a = 1$	$\nu = -0.340092$	$\nu = 0.122499$	$\nu = 9.02272$
$a = 2$	$\nu = 0.39002$	$\nu = 0.586273$	$\nu = 0.772587$
$a = 3$	$\nu = 0.714616$	$\nu = 0.751407$	$\nu = 0.784626$

Table 3 Values of ν for $\mathrm{g}_{a,\nu}$ to be starlike

The radius of starlikeness for $\mathrm{g}_{a,\nu}$ drawn from Theorem 3.9 is tabulated in Table 4 for a fixed $\nu = 0.7$, $a = 1, 2, 3$, and respectively $\beta = 0$, $\beta = 0.5$ and $\beta = 0.95$. Here the radius of starlikeness is expectedly less than 1 whenever $\nu = 0.7$ is less than the given values of ν in Table 3. A similar situation occurs as for the function $\mathtt{f}_{a,\nu}$ with regard to the monotonicity of the radius of starlikeness with respect to either parameter β or a.

	$\beta = 0$	$\beta = 0.5$	$\beta = 0.95$
$a = 1$	$r_0^*(g_{1,0.7}) = 1.68326$	$r_{1/2}^*(g_{1,0.7}) = 1.24519$	$r_{0.95}^*(g_{1,0.7}) = 0.410407$
$a = 2$	$r_0^*(g_{2,0.7}) = 1.44678$	$r_{1/2}^*(g_{2,0.7}) = 1.1867$	$r_{0.95}^*(g_{2,0.7}) = 0.856647$
$a = 3$	$r_0^*(g_{3,0.7}) = 0.939782$	$r_{1/2}^*(g_{3,0.7}) = 0.763126$	$r_{0.95}^*(g_{3,0.7}) = 0.549716$

Table 4 The radius of starlikeness for $\mathrm{g}_{a,\nu}$ when $\nu = 0.7$

Remark 4. For $a = 1$, Theorem 4.1 and Theorem 4.2, respectively reduces to Theorem 1 and Theorem 2 in [5].

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