On injectivity of the ring of real-valued continuous functions on a frame

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Abstract

We give characterizations of P-frames and extremally disconnected P-frames based on ring-theoretic features of the ring of continuous real-valued functions on a frame L, i.e. $\mathcal{R}L$. It is shown that L is a P-frame if and only if $\mathcal{R}L$ is an $\aleph_0$-self-injective ring. Consequently for pseudocompact frames if $\mathcal{R}L$ is $\aleph_0$-self-injective, then L is finite. We also prove that L is an extremally disconnected P-frame iff $\mathcal{R}L$ is a self-injective ring iff $\mathcal{R}L$ is a Baer regular ring iff $\mathcal{R}L$ is a continuous regular ring iff $\mathcal{R}L$ is a complete regular ring.

1 Introduction

We clarify from the start that, throughout, by the term "ring" we mean a commutative ring with identity. All topological spaces are completely regular and Hausdorff, and all frames are completely regular.

Recall that a P-space is a topological space in which every cozero set is closed and also a topological space X is extremally disconnected if every open set has an open closure. These notions have been extended to pointfree topology in such a way that a topological space X has one of these features if and only if the frame of its open sets, i.e. $\mathfrak{O}X$, has the corresponding property (see [1], [5]).

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Received by the editors in August 2017 - In revised form in January 2018.
Communicated by E. Colebunders.
2010 *Mathematics Subject Classification :* 06D22, 16D50, 13A30, 16D25, 54C30, 54G05.
Key words and phrases : P-frame, Extremally disconnected frame, self-injective, $\aleph_0$-self-injective, Ring of continuous real-valued functions on a frame.

By a *reduced ring* we mean a ring without nonzero nilpotent elements. In [10, 11] for a reduced ring A, some internal conditions on A that are equivalent to self-injectivity ($\aleph_0$-self-injective) of A are provided. Since $\mathcal{R}L$ is always a reduced ring, we can use these conditions to investigate the injectivity of the ring $\mathcal{R}L$. Using these conditions, Estaji and Karamzadeh [7] have shown that for a space X, the ring of real-valued continuous functions $C(X)$ is $\aleph_0$-self-injective if and only if X is a P-space. Moreover, they demonstrated that $C(X)$ is self-injective if and only if X is an extremally disconnected P-space. One of the main aims of this article is to develop these results to the more general setting of pointfree topology, that is, frames.

To prove the equivalence of $\mathcal{R}L$ is $\aleph_0$-self-injective and L is a P-frame, for an orthogonal countable set T in $\mathcal{R}L$ and $t = \bigvee_{\alpha\in T} \mathrm{coz}(\alpha)$, in Lemma 3.2, we introduce a frame map $\alpha_t \in \mathcal{R}L$ such that $\mathrm{coz}(\alpha_t) = t$, whenever L is a P-frame. Finally, using the map α_t, Lemmas 3.3 and 3.4, and [5, Proposition 3.9], it is shown that L is a P-frame iff $\mathcal{R}L$ is $\aleph_0$-self-injective, see Theorem 3.6.

In Proposition 4.3, we show that for a frame L, it is an extremally disconnected frame iff $\mathcal{R}L$ is a Baer ring or equivalently, iff $\mathcal{R}L$ is a CS-ring or equivalently, iff every nonzero ideal in $\mathcal{R}L$ is essential in a principal ideal generated by an idempotent. This proposition is proved by Dube in [6, Proposition 2.4], but here, in the proof of this proposition, a different approach is used.

To prove the equivalence of $\mathcal{R}L$ is self-injective and L is an extremally disconnected P-frame, for a set T in $\mathcal{R}L$ and $t = \bigvee_{\alpha\in T} \mathrm{coz}(\alpha)$, in Lemma 4.5, we construct a frame map $\mu_t \in \mathcal{R}L$ such that $\mathrm{coz}(\mu_t) = t^{**}$, whenever L is extremally disconnected P-frame. Using the map μ_t, Propositions 3.5, 3.6, 4.3, and 4.6, Lemmas 3.4 and 4.4, [9, Corollary 13.4], and [12, Proposition 1.7], it is proved that for a frame L, L is an extremally disconnected P-frame iff $\mathcal{R}L$ is a self-injective ring iff $\mathcal{R}L$ is a Baer regular ring iff $\mathcal{R}L$ is a continuous regular ring iff $\mathcal{R}L$ is a complete regular ring, see Theorem 4.7.

2 Preliminaries

Here, we recall some definitions and results from the literature on frames and the pointfree version of the ring of continuous real-valued functions. For undefined terms and notations see [13] on frame-theoretic concepts, [2] on pointfree function rings, and see [8] on $C(X)$.

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee A = \bigvee\{x \wedge a : s \in A\}$$

holds for all $x \in L$ and $A \subseteq L$. The top element and the bottom element of L are denoted by $\top_L$ and $\bot_L$ respectively; dropping the subscripts if no confusion may arise. Throughout this context L will denote a frame. $\mathfrak{O}X$ is the frame of open subsets of a topological space X.

The *pseudocomplement* of an element $a \in L$ is denoted by a^* and for each $a, b \in L$ we have:

1. $a \leq a^{**}$.
2. if $a \leq b$, then $b^* \leq a^*$.
3. $(a \vee b)^* = a^* \wedge b^*$.
4. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element a in L is said to be *complemented* if $a \vee a^* = \top$.

L is said to be *regular* if $a = \bigvee\{x \in L : x \prec a\}$ for each $a \in L$, where $x \prec a$ means that $x^* \vee a = \top$. This is equivalent to saying there is an element $s \in L$, called a *separating element*, such that $x \wedge s = \perp$ and $s \vee a = \top$. It is said to be *completely regular* if, for each $a \in L$, $a = \bigvee\{x \in L : x \prec\!\!\prec a\}$, where $x \prec\!\!\prec a$ means that there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = x$, $c_1 = a$, and $c_p \prec c_q$ for $p < q$.

As described in [2], the frame of reals, denoted $\mathcal{L}(\mathbb{R})$, is the frame generated by ordered pairs (p,q) of rational numbers $p, q \in \mathbb{Q}$ subject to the relations:
(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,
(R3) $(p,q) = \bigvee\{(r,s) : p < r < s < q\}$, and
(R4) $\top = \bigvee\{(p,q) : p, q \in \mathbb{Q}\}$.

A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. A function $f : \mathcal{L}(\mathbb{R}) \to L$ which satisfies the following properties:
(R1$'$) $f((p,q) \wedge (r,s)) = f(p \vee r, q \wedge s)$,
(R2$'$) $f((p,q) \vee (r,s)) = f(p,s)$ whenever $p \leq r < q \leq s$,
(R3$'$) $f(p,q) = \bigvee\{f(r,s) : p < r < s < q\}$, and
(R4$'$) $\top = \bigvee\{f(p,q) : p, q \in \mathbb{Q}\}$
is a frame map.

Now for any frame L *the real-valued continuous functions* on L are the homomorphisms $\mathcal{L}(\mathbb{R}) \to L$. The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [2]. Further, corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, ., \wedge, \vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(u,w) : \langle r,s\rangle \diamond \langle u,w\rangle \subseteq \langle p,q\rangle\},$$

where $\langle r,s\rangle \diamond \langle u,w\rangle = \{x \diamond y : x \in \langle r,s\rangle, y \in \langle u,w\rangle\}$ and $\langle p,q\rangle = \{x \in \mathbb{Q} : p < x < q\}$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p,q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p,q) = \perp$. For any frame L, an element $\alpha \in \mathcal{R}L$ is called bounded if $\alpha(p,q) = \top$ for some $p, q \in \mathbb{Q}$, and L is called *pseudocompact* if $\mathcal{R}L = \mathcal{R}^*L$, where the subring of $\mathcal{R}L$ consisting of its bounded element is denoted by $\mathcal{R}^*L$.

Finally an important feature of $\mathcal{R}L$ is its *cozero map* $\mathrm{coz} : \mathcal{R}L \to L$ taking every $\alpha \in \mathcal{R}L$ to $\mathrm{coz}(\alpha) = \alpha((-,0) \vee (0,-))$, where

$$(0,-) = \bigvee\{(0,q)) : q \in \mathbb{Q}, q > 0\}, (-,0) = \bigvee\{(p,0)) : p \in \mathbb{Q}, p < 0\}.$$

The properties of the cozero map that we use are:

1. $\mathrm{coz}(\alpha) = \perp$ iff $\alpha = \mathbf{0}$,
2. $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)$,
3. $\alpha(p, q) = \mathrm{coz}((\alpha - \mathbf{p})^+ \wedge (\mathbf{q} - \alpha)^+)$,
4. $\mathrm{coz}(\alpha + \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$, and
5. $\alpha \in \mathcal{R}L$ is invertible iff $\mathrm{coz}(\alpha) = \top$.

A *cozero element* of L is an element of the form $\mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}L$. The *cozero part* of L, denoted by $\mathrm{Coz}\, L$, is the regular sub-σ-frame consisting of all the cozero elements of L. It is shown in [2] that a frame L is completely regular if and only if it is generated by the cozero elements.

3 P-frames

Recall that a P-frame is one in which every cozero element is complemented. This notion is the exact extension of its point-delicate namesake in that a topological space X is a P-space if and only if the frame $\mathfrak{O}X$ is a P-frame. A ring R is said to be regular (in the sense of Von Neumann) if for every $a \in R$ there is $b \in R$ with $a = a^2 b$. The following result has been proved by Dube in [5, Proposition 3.9].

Proposition 3.1. *L is a P-frame if and only if $\mathcal{R}L$ is a regular ring.*

A ring R is said to be self-injective ($\aleph_0$-self-injective) if every R-homomorphism from an ideal (a countably generated ideal) of R to R can be extended to an R-homomorphism from R to R. In this section the aim is to find a feature of a frame L that is equivalent to $\aleph_0$-self-injective of $\mathcal{R}L$. For this purpose first we recall some of these definitions and results known and are making some lemmas.

Suppose R is a commutative ring with unit. A subset S of R is said to be orthogonal provided $xy = 0$ for all $x, y \in S$ with $x \neq y$. If $S \cap T = \emptyset$ and $S \cup T$ is an orthogonal set in R, then $a \in R$ is said to separate S from T if $a \in Ann(T)$ and $s^2 a = s$, for every $s \in S$ (see [10]). In [11] it is shown that there exists an element in R which separates S from T if and only if there is an element b in R such that $b \in Ann(T)$ and $s^2 = sb$, for every $s \in S$.

The homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ given by $(p, q) \mapsto]\!]p, q[\![$ is an isomorphism, where

$$]\!]p, q[\![:= \{x \in \mathbb{R} : p < x < q\}.$$

For convenience, we put $v^0 := \tau^{-1}(\tau(v) \setminus \{0\})$, for every $v \in \mathcal{L}\mathbb{R}$.

We need the following three lemmas and one proposition which gives an algebraic characterization of P- frames. But we omit the proof of propositions for it is achieved by [10, Theorem 2.2] and [11, Proposition 1.2].

Lemma 3.2. *Let L be a P-frame. Assume that T is an orthogonal countable set in $\mathcal{R}L$ and $t = \bigvee_{\alpha \in T} \mathrm{coz}(\alpha)$. If $\alpha_t : \mathcal{L}\mathbb{R} \to L$ given by*

$$\alpha_t(v) = \begin{cases} \bigvee_{\alpha \in T} \alpha(v^0) \vee t^* & \text{if } 0 \in \tau(v) \\ \bigvee_{\alpha \in T} \alpha(v) & \text{if } 0 \notin \tau(v) \end{cases}$$

for every $v \in \mathcal{L}\mathbb{R}$, then $\alpha_t \in \mathcal{R}L$ and $\mathrm{coz}(\alpha_t) = t$.

Proof. We check the conditions (R1′)-(R4′) for α_t.

(R1′). Consider $p,q,r,s \in \mathbb{Q}$. If $0 \in \tau(p,q) \cap \tau(r,s)$, then $0 \in \tau(p \vee r, q \wedge s)$, which implies that

$$\begin{aligned}
\alpha_t(p,q) \wedge \alpha_t(r,s) &= [\textstyle\bigvee_{\alpha \in T} \alpha((p,q)^0) \vee t^*] \wedge \\
&\quad [\textstyle\bigvee_{\alpha \in T} \alpha((r,s)^0) \vee t^*] \\
&= \textstyle\bigvee_{\alpha,\alpha' \in T} (\alpha((p,q)^0) \wedge \alpha'((r,s)^0) \vee t^* \\
&= \textstyle\bigvee_{\alpha \in T} (\alpha((p,q)^0) \wedge \alpha((r,s)^0) \vee t^* \\
&= \textstyle\bigvee_{\alpha \in T} \alpha((p \vee r, q \wedge s)^0) \vee t^* \\
&= \alpha_t(p \vee r, q \wedge s),
\end{aligned}$$

because

$$\alpha((p,q)^0) \wedge \alpha'((r,s)^0) \leq \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\alpha') = \mathrm{coz}(\alpha\alpha') = \mathrm{coz}(\mathbf{0}) = \perp$$

for every $\alpha, \alpha' \in S$ with $\alpha \neq \alpha'$.

If $0 \notin \tau(p,q) \cup \tau(r,s)$, then $0 \notin \tau(p \vee r, q \wedge s)$, which implies that

$$\begin{aligned}
\alpha_t(p,q) \wedge \alpha_t(r,s) &= \textstyle\bigvee_{\alpha \in T} \alpha(p,q) \wedge \bigvee_{\alpha \in T} \alpha(r,s) \\
&= \textstyle\bigvee_{\alpha,\alpha' \in T} \alpha(p,q) \wedge \alpha'(r,s) \\
&= \textstyle\bigvee_{\alpha \in T} \alpha(p,q) \wedge \alpha(r,s) \\
&= \textstyle\bigvee_{\alpha \in T} \alpha(p \vee r, q \wedge s) \\
&= \alpha_t(p \vee r, q \wedge s),
\end{aligned}$$

because

$$\alpha(p,q) \wedge \alpha'(r,s) \leq \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\alpha') = \mathrm{coz}(\alpha\alpha') = \mathrm{coz}(\mathbf{0}) = \perp$$

for every $\alpha, \alpha' \in T$ with $\alpha \neq \alpha'$.

If $0 \in \tau(p,q) \setminus \tau(r,s)$, then $0 \notin \tau(p \vee r, q \wedge s)$, which implies that

$$\begin{aligned}
\alpha_t(p,q) \wedge \alpha_t(r,s) &= [\textstyle\bigvee_{\alpha \in T} \alpha((p,q)^0) \vee t^*] \wedge \bigvee_{\alpha \in T} \alpha(r,s) \\
&= [\textstyle\bigvee_{\alpha,\alpha' \in T} \alpha((p,q)^0) \wedge \alpha'(r,s)] \vee [t^* \wedge \bigvee_{\alpha \in T} \alpha(r,s)] \\
&= [\textstyle\bigvee_{\alpha \in T} \alpha((p,q)^0) \wedge \alpha(r,s)] \vee \perp \\
&= \alpha_t(p \vee r, q \wedge s),
\end{aligned}$$

because $t^* \wedge \bigvee_{\alpha \in T} \alpha(r,s) \leq t^* \wedge t = \perp$.

(R2′). Let $p,q,r,s \in \mathbb{Q}$ with $p \leq r < q \leq s$. If $0 \in \tau(r,q)$, then $0 \in \tau(p,q)$ and $0 \in \tau(r,s)$, which implies that

$$\begin{aligned}
\alpha_t(p,q) \vee \alpha_t(r,s) &= [\textstyle\bigvee_{\alpha \in T} \alpha((p,q)^0) \vee t^*] \vee [\bigvee_{\alpha \in T} \alpha((r,s)^0) \vee t^*] \\
&= \textstyle\bigvee_{\alpha \in T} (\alpha(((p,q)^0) \vee \alpha((r,s)^0) \vee t^* \\
&= \textstyle\bigvee_{\alpha \in T} \alpha((p,s)^0) \vee t^* \\
&= \alpha_t(p,s).
\end{aligned}$$

If $p < 0 \leq r$, then $0 \in \tau(p,q)$ and $0 \notin \tau(r,s)$, which implies that

$$\begin{aligned}
\alpha_t(p,q) \vee \alpha_t(r,s) &= [\textstyle\bigvee_{\alpha \in T} \alpha((p,q)^0) \vee t^*] \vee \bigvee_{\alpha \in T} \alpha(r,s) \\
&= [\textstyle\bigvee_{\alpha \in T} \alpha((p,q)^0) \vee \alpha(r,s)] \vee t^* \\
&= \textstyle\bigvee_{\alpha \in T} \alpha((p,s)^0) \vee t^* \\
&= \alpha_t(p,s).
\end{aligned}$$

The proof of $q \leq 0 < s$ is similar.

(R3′). If $0 \notin \tau(p,q)$, then $0 \notin \tau(r,s)$, for every $r,s \in \mathbb{Q}$ with $p < r < s < q$. Hence

$$\bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \alpha_t(r,s) = \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \bigvee_{\alpha\in T} \alpha(r,s) = \bigvee_{\alpha\in T} \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \alpha(r,s) = \bigvee_{\alpha\in T} \alpha(p,q) = \alpha_t(p,q).$$

If $0 \in \tau(p,q)$, then

$$\begin{aligned}
\bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \alpha_t(r,s) &= \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q\\ 0\in\tau(r,s)}} [\textstyle\bigvee_{\alpha\in T} \alpha((r,s)^0) \vee t^*] \vee \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q\\ 0\notin\tau(r,s)}} \textstyle\bigvee_{\alpha\in T} \alpha(r,s) \\
&= \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \textstyle\bigvee_{\alpha\in T} \alpha((r,s)^0) \vee t^* \\
&= \textstyle\bigvee_{\alpha\in S} \alpha((p,q)^0) \vee t^* \\
&= \alpha_t(p,q).
\end{aligned}$$

(R4′). Since $\mathrm{Coz}\,L$ is a σ-frame and t is the countable subset of $\mathcal{R}L$, we conclude that $t \in \mathrm{Coz}\,L$, which implies that $t \vee t^* = \top$, because L is a P-frame. Therefore, we have

$$\begin{aligned}
\textstyle\bigvee_{r,s\in\mathbb{Q}} \alpha_t(r,s) &= \bigvee_{\substack{r,s\in\mathbb{Q},\\ 0\in\tau(r,s)}} [\textstyle\bigvee_{\alpha\in T} \alpha((r,s)^0) \vee t^*] \vee \bigvee_{\substack{r,s\in\mathbb{Q},\\ 0\notin\tau(r,s)}} \textstyle\bigvee_{\alpha\in T} \alpha(r,s) \\
&= \textstyle\bigvee_{r,s\in\mathbb{Q},} \bigvee_{\alpha\in T} \alpha((r,s)^0) \vee t^* \\
&= \textstyle\bigvee_{\alpha\in T} \bigvee_{r,s\in\mathbb{Q},} \alpha((r,s)^0) \vee t^* \\
&= \textstyle\bigvee_{\alpha\in T} \mathrm{coz}(\alpha) \vee t^* \\
&= t \vee t^* \\
&= \top.
\end{aligned}$$

Therefore, α_t is a real-valued continuous function. To prove the second part, we have $\mathrm{coz}(\alpha_t) = \alpha_t((-,0) \vee (0,-)) = \bigvee_{\alpha\in T} \alpha((-,0) \vee (0,-)) = \bigvee_{\alpha\in T} \mathrm{coz}(\alpha) = t$. ∎

Lemma 3.3. *Let $S \cup T \subseteq \mathcal{R}L$ be an orthogonal set with $S \cap T = \emptyset$. If $s = \bigvee_{\alpha\in S} \mathrm{coz}(\alpha)$ and $t = \bigvee_{\beta\in T} \mathrm{coz}(\beta)$, then $t \leq s^*$.*

Proof.

$$s \wedge t = \bigvee_{\alpha\in S} \mathrm{coz}(\alpha) \wedge \bigvee_{\beta\in T} \mathrm{coz}(\beta) = \bigvee_{\alpha,\beta} \big(\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)\big) = \bigvee \mathrm{coz}(\alpha\beta) = \bot,$$

which implies $t \leq s^*$. ∎

Lemma 3.4. *If L is a P-frame, then the following statements hold.*

(1) *If $\alpha \in \mathcal{R}L$, then $(\mathrm{coz}(\alpha))^* = \alpha(r,s) \wedge (\mathrm{coz}(\alpha))^*$ and $\alpha(r,s) = \alpha((r,s)^0) \vee (\mathrm{coz}(\alpha))^*$, for every $r,s \in \mathbb{Q}$ with $r < 0 < s$.*

(2) *If $T \subseteq \mathcal{R}L$ is an orthogonal set with $t = \bigvee_{\alpha\in T} \mathrm{coz}(\alpha)$, then*

$$\alpha((r,s)^0) \vee t^* = \alpha(r,s) \wedge \bigwedge_{\alpha\neq\beta\in T} (\mathrm{coz}(\beta))^*,$$

for every $\alpha \in T$ and $r,s \in \mathbb{Q}$ with $r < 0 < s$.

Proof. (1). Consider $\alpha \in \mathcal{R}L$ and $r, s \in \mathbb{Q}$ with $r < 0 < s$. Since $\alpha(r,s) \vee \mathrm{coz}(\alpha) = \top$, we conclude that

$$(\mathrm{coz}(\alpha))^* = (\mathrm{coz}(\alpha))^* \wedge (\alpha(r,s) \vee \mathrm{coz}(\alpha)) = \alpha(r,s) \wedge (\mathrm{coz}(\alpha))^*.$$

To prove the second part, by hypothesis, $\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))^* = \top$, then

$$\alpha(r,s) = (\alpha(r,s) \wedge \mathrm{coz}(\alpha)) \vee \big(\alpha(r,s) \wedge (\mathrm{coz}(\alpha))^*\big) = \alpha((r,s)^0) \vee (\mathrm{coz}(\alpha))^*.$$

(2). If $\alpha \neq \beta \in T$, then $\alpha((r,s)^0) \leq \mathrm{coz}(\alpha) \leq (\mathrm{coz}(\beta))^*$, because $\mathrm{coz}(\beta) \wedge \mathrm{coz}(\alpha) = \bot$. Hence $\alpha((r,s)^0) \leq \bigwedge_{\alpha \neq \beta \in S} (\mathrm{coz}(\beta))^*$, which follows from statement (1) that

$$\begin{aligned} \alpha((r,s)^0) \vee t^* &= \alpha((r,s)^0) \vee \textstyle\bigwedge_{\beta \in T} (\mathrm{coz}(\beta))^* \\ &= (\alpha((r,s)^0) \vee (\mathrm{coz}(\alpha))^*) \wedge (\alpha((r,s)^0) \vee \textstyle\bigwedge_{\alpha \neq \beta \in T} (\mathrm{coz}(\beta))^* \\ &= \alpha(r,s) \wedge \textstyle\bigwedge_{\alpha \neq \beta \in T} (\mathrm{coz}(\beta))^*. \end{aligned}$$

This completes the proof. ■

Recall that the proof of the following proposition is concluded by [10, Theorem 2.2], and [11, Proposition 1.2].

Proposition 3.5. *Let R be a reduced ring, then the following statements are equivalent.*

(1) *The ring R is self-injective ($\aleph_0$-self-injective).*

(2) *The ring R is a regular ring and whenever $S \cup T$ is an orthogonal (countable) set with $S \cap T = \emptyset$, then there exists an element in R which separates S from T.*

For convenience, given any two generators (u,v) and (w,z), we shall write $\langle uvwzpq \rangle$ to signify that $\langle u,v \rangle . \langle w,z \rangle \subseteq \langle p,q \rangle$.

In the proof that follows we shall use the fact that if L is a regular frame and $h, g : L \to M$ are frame homomorphisms such that $h(x) \leq g(x)$ for every $x \in L$, then $h = g$.

Theorem 3.6. *Let L be a frame. Then L is a P-frame if and only if $\mathcal{R}L$ is an $\aleph_0$-self-injective ring.*

Proof. We begin with the necessity. Let $S \cup T \subseteq \mathcal{R}L$ be an orthogonal countable set with $S \cap T = \emptyset$ and $s = \bigvee_{\alpha \in S} \mathrm{coz}(\alpha)$. Now, we show that $\alpha_s \in Ann(T)$. Consider $\beta \in T$ and $t = \bigvee_{\beta \in T} \mathrm{coz}(\beta)$. Then, by Lemma 3.2, we have

$$\begin{aligned} \mathrm{coz}(\alpha_s \beta) &= \mathrm{coz}(\alpha_s) \wedge \mathrm{coz}(\beta) \\ &= s \wedge \mathrm{coz}(\beta) \\ &\leq s \wedge t \\ &= \bot \end{aligned}$$

which implies that $\alpha_s \beta = \mathbf{0}$. Therefore, $\alpha_s \in Ann(T)$.

Now, consider $\delta \in S$. We show that $\delta\alpha_s = \delta^2$. In order to approach this goal, let us assume that $p, q \in \mathbb{Q}$. If $0 \notin \tau(p,q)$, then

$$\begin{aligned}
\delta\alpha_s(p,q) &= \bigvee\{\delta(u,v) \wedge \alpha_s(w,z) : \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \textstyle\bigvee_{\alpha\in S} \alpha(w,z) : \langle uvwzpq\rangle\} \\
&= \bigvee\{\textstyle\bigvee_{\alpha\in S} \delta(u,v) \wedge \alpha(w,z) : \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \delta(w,z) : \langle uvwzpq\rangle\} \\
&= \delta^2(p,q),
\end{aligned}$$

because, if $\alpha \in S$ and $\delta \neq \alpha$, then since $0 \notin \tau(u,v) \cup \tau(w,z)$,

$$\delta(u,v) \wedge \alpha(w,z) \leq \operatorname{coz}(\delta) \wedge \operatorname{coz}(\alpha) = \operatorname{coz}(\delta\alpha) = \operatorname{coz}(\mathbf{0}) = \perp.$$

Now, if $0 \in \tau(p,q)$, then, by Lemma 3.4, we have

$$\begin{aligned}
\delta\alpha_s(p,q) &= \bigvee\{\delta(u,v) \wedge \alpha_s(w,z) : \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \textstyle\bigvee_{\alpha\in S} \alpha(w,z) : 0 \notin \tau(w,z),\ \langle uvwzpq\rangle\} \vee \\
&\quad\ \bigvee\{\delta(u,v) \wedge [\textstyle\bigvee_{\alpha\in S} \alpha((w,z)^0) \vee s^*] : 0 \in \tau(w,z),\ \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \delta(w,z) : 0 \notin \tau(w,z),\ \langle uvwzpq\rangle\} \vee \\
&\quad\ \bigvee\{\delta(u,v) \wedge \delta(w,z) \wedge \textstyle\bigwedge_{\delta\neq\alpha\in S} (\operatorname{coz}(\alpha))^* : 0 \in \tau(w,z),\ \langle uvwzpq\rangle\} \\
&\leq \bigvee\{\delta(u,v) \wedge \delta(w,z) : \langle uvwzpq\rangle\} \\
&= \delta^2(p,q).
\end{aligned}$$

Since δ and α_s are frame maps and $\mathcal{L}\mathbb{R}$ is a regular frame, we conclude that $\delta\alpha_s = \delta^2$, which means that α_s separates S from T. Now, by Proposition 3.5, we are through.

To prove the sufficiency, consider $\alpha \in \mathcal{R}L$, and let I be the ideal of $\mathcal{R}L$ generated by α^2. Since $f : I \to \mathcal{R}L$ given by $\beta\alpha^2 \mapsto \beta\alpha$ is a $\mathcal{R}L$-homomorphism, we conclude from statement (2) that there exists a $\mathcal{R}L$-homomorphism $\bar{f} : \mathcal{R}L \to \mathcal{R}L$ such that $\bar{f}|_I = f$. Hence

$$\alpha = f(\alpha^2) = \bar{f}(\mathbf{1}\alpha^2) = \bar{f}(\mathbf{1})\alpha^2.$$

Then $\mathcal{R}L$ is a regular ring. Therefore, by Proposition 3.1, L is a P-frame. ∎

We denote the Hewitt realcompactification and universal Lindelöfication of L by $vL \to L$ and $\lambda L \to L$ respectively, (see [3] for details). It is shown in [5] that a frame L is a P-frame if and only if vL is a P-frame if and only if λL is a P-frame. We therefore have the following:

Corollary 3.7. *The following are equivalent for a frame L.*

1. *$\mathcal{R}L$ is an $\aleph_0$-self-injective ring.*
2. *$\mathcal{R}(vL)$ is an $\aleph_0$-self-injective ring.*
3. *$\mathcal{R}(\lambda L)$ is an $\aleph_0$-self-injective ring.*

As remarked in [5, p. 126], every pseudocompact P-frame is finite. We therefore have the following corollary.

Corollary 3.8. *If L is pseudocompact and $\mathcal{R}L$ is $\aleph_0$-self-injective, then L is finite.*

4 extremally disconnected frames

In the first part of this section, our aim is to give alternative proofs of several algebraic characterizations of extremally disconnected frames that were established in [6]. To do this we need two propositions. In the first proposition for a complemented element a in L is defined a idempotent element e_a of $\mathcal{R}L$, but we omit its proof for it can be easily deduced from the proof of [1, Theorem 8. 3. 3]. In the second proposition, we calculate the multiplication αe_a for a element $\alpha \in \mathcal{R}L$, but we also omit its proof for it can be easily checked.

Proposition 4.1. *Let a be a complemented element of L. Then $e_a : \mathcal{L}R \to L$ by*

$$e_a(U) = \begin{cases} \top & \text{if } 0, 1 \in \tau(U) \\ a' & \text{if } 0 \in \tau(U) \text{ and } 1 \notin \tau(U) \\ a & \text{if } 0 \notin \tau(U) \text{ and } 1 \in \tau(U) \\ \bot & \text{if } 0 \notin \tau(U) \text{ and } 1 \notin \tau(U), \end{cases}$$

is a continuous real-valued function, $e_a^2 = e_a$, and $\operatorname{coz}(e_a) = a$.

Proposition 4.2. *If a is complemented in L and $\alpha \in \mathcal{R}L$, then*

$$\alpha e_a(p, q) = \begin{cases} \alpha(p, q) \vee a' & \text{if } 0 \in \tau(p, q) \\ \alpha(p, q) \wedge a & \text{if } 0 \notin \tau(p, q). \end{cases}$$

Before the following proposition is proposed, we first recall some definitions. If A and B are ideals in a ring R we say A is essential in B if $A \subseteq B$ and every nonzero ideal inside B intersects A nontrivially, and when we say A is essential, we mean it is essential in R. An ideal A in a ring R is called closed ideal (complement) if it is not essential in a larger ideal and a ring R is said to be CS-ring if every closed ideal is a direct summand, see [14]. A ring R is called a Baer ring if for any subset S of R, we have $Ann_R(S) = eR$, where $e^2 = e$.

As in the introduction it is stated that Dube proved this proposition in [6, Proposition 2.4], but here, we indicate the different proof about that based on the foregoing proposition.

Proposition 4.3. *The following statements are equivalent.*

(1) *L is an extremally disconnected frame.*

(2) *$\mathcal{R}L$ is a Baer ring.*

(3) *Every nonzero ideal in $\mathcal{R}L$ is essential in a principal ideal generated by an idempotent.*

(4) *$\mathcal{R}L$ is a CS-ring.*

Proof. (1)$\Rightarrow$(2). Let $S \subseteq \mathcal{R}L$ be any subset, we are to show that $AnnS = e\mathcal{R}L$, where $e^2 = e$. We put $s = \bigvee_{\alpha \in S} \operatorname{coz}(\alpha)$. Since L is extremally disconnected, we infer that $s^* \vee s^{**} = \top$, which implies that s^* is a complemented element in L.

Consider $\beta \in Ann(S)$. Then $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \mathrm{coz}(\alpha\beta) = \mathrm{coz}(\mathbf{0}) = \perp$, which implies that $\mathrm{coz}(\alpha) \leq (\mathrm{coz}(\beta))^*$, for every $\alpha \in S$. Hence $s \leq (\mathrm{coz}(\beta))^*$ and so

$$\mathrm{coz}(\beta) \leq (\mathrm{coz}(\beta))^{**} \leq s^*.$$

Consider $v \in \mathcal{L}\mathbb{R}$ and $\alpha \in S$. If $0 \notin \tau(v)$, then, by Proposition 4.2, $\beta e_{s^*}(v) = s^* \wedge \beta(v) = \beta(v)$, because $\beta(v) \leq \mathrm{coz}(\beta) \leq s^*$. If $0 \in \tau(v)$, then $\beta(v^0) \leq \mathrm{coz}(\beta) \leq s^*$, which implies that $\beta e_{s^*}(v) = s^{**} \vee \beta(v) \geq \beta(v)$, by Proposition 4.2. Since β and e_{s^*} are frame maps and $\mathcal{L}\mathbb{R}$ is the regular frame, we conclude that $\beta e_{s^*} = \beta$ which means that $\beta \in e_{s^*}\mathcal{R}L$. Hence $Ann(S) \subseteq e_{s^*}\mathcal{R}L$. Now, suppose that $\alpha \in S$, then

$$\mathrm{coz}(\alpha) \leq s \Rightarrow \mathrm{coz}(\alpha e_{s^*}) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(e_{s^*}) \leq (\mathrm{coz}(\alpha))^{**} \wedge s^* \leq s^{**} \wedge s^* = \perp,$$

it follows that $\alpha e_{s^*} = \mathbf{0}$. Hence $e_{s^*} \in Ann(S)$. Therefore, $Ann(S) = e_{s^*}\mathcal{R}L$ and so $\mathcal{R}L$ is a Baer ring.

(2)⇒(3). Let I be a nonzero ideal in $\mathcal{R}L$, then there is an idempotent element e in $\mathcal{R}L$ such that $Ann(I) = e\mathcal{R}L = Ann((1-e)\mathcal{R}L)$, which implies that $\alpha = \alpha(1-e) \in (1-e)\mathcal{R}L \cap I$, for every $\alpha \in I$. Hence I is essential in $(1-e)\mathcal{R}L$.

(3)⇒(4). Let I be a closed ideal in $\mathcal{R}L$, then there is an idempotent element e in $\mathcal{R}L$ such that I is essential in $e\mathcal{R}L$.

(4)⇒(2). Consider $S \subseteq \mathcal{R}L$ and $I = Ann(S)$. We claim that the ideal $Ann(S)$ is a closed ideal in $\mathcal{R}L$. Let $Ann(S)$ be essential in a larger ideal J, then $SJ \neq (\mathbf{0})$ implies that $SJ \cap Ann(S) \neq (\mathbf{0})$, but $(SJ \cap Ann(S))^2 = (\mathbf{0})$, which is impossible, since $\mathcal{R}L$ is a reduced ring. This shows that $Ann(S)$ is a closed ideal and by statement (4), I is generated by an idempotent..

(2)⇒(1). Consider $a \in L$, then there are $\{\alpha_t\}_{t\in T} \subseteq \mathcal{R}L$ such that $a = \bigvee_{t\in T} \mathrm{coz}(\alpha_t)$. Since $\mathcal{R}L$ is a Baer ring, we conclude that there is an idempotent element $e \in \mathcal{R}L$ such that $Ann(\{\alpha_t\}_{t\in T}) = e\mathcal{R}L$, which implies that for every $t \in T$

$$\mathrm{coz}(e) \wedge \mathrm{coz}(\alpha_t) = \mathrm{coz}(e\alpha_t) = \mathrm{coz}(\mathbf{0}) = \perp \Rightarrow \forall t \in T\big(\mathrm{coz}(e) \leq (\mathrm{coz}(\alpha_t))^*\big),$$

and so $\mathrm{coz}(e) \leq \bigwedge_{t\in T}(\mathrm{coz}(\alpha_t))^* = a^*$. Since $\mathrm{coz}(e) \vee \mathrm{coz}(1-e) = \top$ and $\mathrm{coz}(e) \wedge \mathrm{coz}(1-e) = \perp$, we conclude that $a^{**} \leq (\mathrm{coz}(e))^* = \mathrm{coz}(1-e)$. Suppose that $\{\beta_k\}_{k\in K} \subseteq \mathcal{R}L$ such that $a^* = \bigvee_{k\in K} \mathrm{coz}(\beta_k)$. For every $(t,k) \in T \times K$, we have

$$\mathrm{coz}(\alpha_t\beta_k) = \mathrm{coz}(\alpha_t) \wedge \mathrm{coz}(\beta_k) \leq (\mathrm{coz}(\alpha_t))^{**} \wedge a^* \leq a^{**} \wedge a^* = \perp,$$

and so $\alpha_t\beta_k = \mathbf{0}$. Then $\beta_k \in Ann(\{\alpha_t\}_{t\in T}) = e\mathcal{R}L$, which implies that there is a $\gamma_k \in \mathcal{R}L$ such that $\beta_k = e\gamma_k$, for every $k \in K$. Therefore, $\mathrm{coz}(\beta_k) = \mathrm{coz}(e\gamma_k) \leq \mathrm{coz}(e)$ and so $a^* = \bigvee_{k\in K} \mathrm{coz}(\beta_k) \leq \mathrm{coz}(e)$. Hence $a^* = \mathrm{coz}(e)$ and $a^* \vee a^{**} = \mathrm{coz}(e) \vee \mathrm{coz}(1-e) = \top$. This completes the proof. ∎

We need the following two lemmas which give algebraic characterizations of extremally disconnected P-frames, but we omit the proof of the second lemma for it is similar to the proof of Lemma 3.2.

Lemma 4.4. *Let $\alpha, \delta \in \mathcal{R}L$ and $u, v, w, z \in \mathbb{Q}$. If L is a P-frame, then*

$$(\delta(u,v))^{**} = \delta(u,v) \text{ and } (\delta(u,v) \wedge \alpha(w,z))^{**} = \delta(u,v) \wedge \alpha(w,z).$$

Proof. By[2, Lemma 6], $\delta(u,v) = \mathrm{coz}((\delta - u)^+ \wedge (v - \delta)^+)$ and $\alpha(w,z) = \mathrm{coz}((\alpha - w)^+ \wedge (z - \alpha)^+)$. Since L is a P-frame, every cozero element in L is complemented. Hence $(\delta(u,v))^{**} = \delta(u,v)$ and if $\beta_1 = (\delta - u)^+ \wedge (v - \delta)^+$ and $\beta_2 = (\alpha - w)^+ \wedge (z - \alpha)^+$, then

$$\begin{aligned}
(\delta(u,v) \wedge \alpha(w,z))^{**} &= (\mathrm{coz}(\beta_1) \wedge \mathrm{coz}(\beta_2))^{**} \\
&= (\mathrm{coz}(\beta_1\beta_2))^{**} \\
&= \mathrm{coz}(\beta_1\beta_2) \\
&= \mathrm{coz}(\beta_1) \wedge \mathrm{coz}(\beta_2) \\
&= \delta(u,v) \wedge \alpha(w,z).
\end{aligned}$$

■

Lemma 4.5. *Let L be an extremally disconnected P-frame and $T \subseteq \mathcal{R}L$ with $t = \bigvee_{\alpha \in T} \mathrm{coz}(\alpha)$. If $\mu_t : \mathcal{L}\mathbb{R} \to L$ given by*

$$\mu_t(v) = \begin{cases} \left(\bigvee_{\alpha \in T} \alpha(v^0)\right)^{**} \vee t^* & \text{if } 0 \in \tau(v) \\ \left(\bigvee_{\alpha \in T} \alpha(v)\right)^{**} & \text{if } 0 \notin \tau(v) \end{cases}$$

for every $v \in \mathcal{L}\mathbb{R}$, then $\mu_t \in \mathcal{R}L$ and $\mathrm{coz}(\mu_t) = t^{**}$.

Proof. Similar to the proof of Lemma 3.2, because $a^{**} \wedge b^{**} = (a \wedge b)^{**}$ and $a^{**} \vee b^{**} = (a \vee b)^{**}$, for every $a, b \in L$. ■

In what follows, our aim is that extremally disconnected P-frames characterize in terms of ring-theoretic properties of the ring $\mathcal{R}L$, such as Baer, self-injective, continuous, complete, and regular ring. We first recall some definitions and propositions. A lattice A is called upper continuous if A is complete and $a \wedge (\vee b_i) = \vee(a \wedge b_i)$ for all $a \in A$ and all linearly ordered subset $\{b_i\} \subseteq A$. A regular ring R is called continuous if the lattice of all principal ideals is upper continuous.

We recall from [9, Corollary 13.4] that a regular ring R is continuous if and only if every ideal of R is essential in a principal right ideal of R. Also, we recall from [9, Corollary 13.5] that every regular self-injective ring is continuous. Also, every reduced self-injective ring is regular ring which is Baer ring, see [12, Proposition 1.7].

Proposition 4.6. [4] *The following statements are equivalent.*

(1) *R is a Baer ring.*

(2) *R is a p.p. ring which is also the Boolean algebra $B(R)$ of idempotents in R is complete.*

(3) *R is a p.p. ring and every set of orthogonal idempotents in R has a supremum.*

Theorem 4.7. *The following statements are equivalent.*

(1) *$\mathcal{R}L$ is a Baer regular ring.*

(2) *$\mathcal{R}L$ is a continuous regular ring.*

(3) *$\mathcal{R}L$ is a complete regular ring.*

(4) *L is an extremally disconnected P-frame.*

(5) *$\mathcal{R}L$ is a self-injective ring.*

Proof. (1)$\Rightarrow$(2). It is clear by [9, Corollary 13.4] and Proposition 4.6.

(2)$\Rightarrow$(3). It is evident.

(3)$\Rightarrow$(4). Since every regular ring is a p.p. ring, we conclude from Proposition 4.6 that $\mathcal{R}L$ is a Baer regular ring. Then, by Proposition 4.3, L is an extremally disconnected frame.

(4)$\Rightarrow$(5). Let $S \cup T \subseteq \mathcal{R}L$ be an orthogonal set with $S \cap T = \emptyset$ and $s = \bigvee_{\alpha \in S} \mathrm{coz}(\alpha)$. Similar to the proof of Proposition 3.6, $\mu_s \in Ann(T)$.

Now, consider $\delta \in S$. We show that $\delta\mu_s = \delta^2$. In order to approach this goal, let us assume that $p, q \in \mathbb{Q}$. If $0 \notin \tau(p,q)$, then

$$\begin{aligned}
\delta\alpha_s(p,q) &= \bigvee\{\delta(u,v) \wedge \mu_s(w,z) : \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \big(\textstyle\bigvee_{\alpha\in S} \alpha(w,z)\big)^{**} : \langle uvwzpq\rangle\} \\
&= \bigvee\{\textstyle\bigvee_{\alpha\in S} \big(\delta(u,v) \wedge \alpha(w,z)\big)^{**} : \langle uvwzpq\rangle\} \\
&= \bigvee\{\big(\delta(u,v) \wedge \delta(w,z)\big)^{**} : \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \delta(w,z) : \langle uvwzpq\rangle\}, \qquad \text{by Lemma 4.4} \\
&= \delta^2(p,q),
\end{aligned}$$

because, if $\alpha \in S$ and $\delta \neq \alpha$, then $\delta(u,v) \wedge \alpha(w,z) \leq \mathrm{coz}(\delta) \wedge \mathrm{coz}(\alpha) = \mathrm{coz}(\delta\alpha) = \mathrm{coz}(\mathbf{0}) = \bot$, since $0 \notin \tau(u,v) \cup \tau(w,z)$. If $0 \in \tau(p,q)$, then, by Lemma 3.4 and 4.4, we have

$$\begin{aligned}
\delta\mu_s(p,q) &= \bigvee\{\delta(u,v) \wedge \mu_s(w,z) : \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \big(\textstyle\bigvee_{\alpha\in S} \alpha(w,z)\big)^{**} : 0 \notin \tau(w,z),\ \langle uvwzpq\rangle\} \vee \\
&\quad\ \bigvee\{\delta(u,v) \wedge [\big(\textstyle\bigvee_{\alpha\in S} \alpha((w,z)^0)\big)^{**} \vee s^*] : 0 \in \tau(w,z),\ \langle uvwzpq\rangle\} \\
&= \bigvee\{\big(\delta(u,v) \wedge \delta(w,z)\big)^{**} : 0 \notin \tau(w,z),\ \langle uvwzpq\rangle\} \vee \\
&\quad\ \bigvee\{\big(\delta(u,v) \wedge \delta(w,z) \wedge \textstyle\bigwedge_{\delta\neq\alpha\in S} (\mathrm{coz}(\alpha))^*\big)^{**} : \\
&\qquad\qquad 0 \in \tau(w,z),\ \langle uvwzpq\rangle\} \\
&= \bigvee\{\delta(u,v) \wedge \delta(w,z) : 0 \notin \tau(w,z),\ \langle uvwzpq\rangle\} \vee \\
&\quad\ \bigvee\{\delta(u,v) \wedge \delta(w,z) \wedge \big(\textstyle\bigwedge_{\delta\neq\alpha\in S} (\mathrm{coz}(\alpha))^*\big)^{**} : \\
&\qquad\qquad 0 \in \tau(w,z),\ \langle uvwzpq\rangle\} \\
&\leq \bigvee\{\delta(u,v) \wedge \delta(w,z) : \langle uvwzpq\rangle\} \\
&= \delta^2(p,q).
\end{aligned}$$

Since δ and α_s are frame maps and $\mathcal{L}\mathbb{R}$ is the regular frame, we conclude that $\delta\alpha_s = \delta^2$. which means that α_s separates S from T. Now, by Proposition 3.5, we are through.

(5)$\Rightarrow$(1). By [12, Proposition 1.7.], $\mathcal{R}L$ is a Baer regular ring. ∎

It is shown in [6] that a frame L is extremally connected if and only if vL is extremally connected if and only if λL extremally connected. We therefore, by paragraph before Corollary 3.7, have the following:

Corollary 4.8. *The following are equivalent for a frame L.*

1. *$\mathcal{R}L$ is an self-injective ring.*
2. *$\mathcal{R}(vL)$ is an self-injective ring.*
3. *$\mathcal{R}(\lambda L)$ is an self-injective ring.*

Acknowledgment: We thank the referee for comments that have improved the readability of the paper, and for providing a simpler proof of Lemma 3.3

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