An explicit formula for the cup-length of the rotation group

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Abstract

This paper gives an explicit formula for the $\mathbb{Z}_2$-cup-length of the rotation group $\mathrm{SO}(n)$.

1 Introduction and the main result

As is well known, the $\mathbb{Z}_2$-*cup-length* $\mathrm{cup}(X;\mathbb{Z}_2)$ of a compact path-connected topological space X is the maximum of all integers c such that there exist reduced cohomology classes $a_1,\dots,a_c \in \widetilde{H}^*(X;\mathbb{Z}_2)$ such that their cup product $a_1 \cup \cdots \cup a_c$ does not vanish. Instead of the usual notation $a \cup b$, we shall write ab, $H^*(X;\mathbb{Z}_2)$ will be abbreviated to $H^*(X)$, and $\mathrm{cup}(X;\mathbb{Z}_2)$ will be shortened to $\mathrm{cup}(X)$ in the sequel (we shall only consider cohomology with coefficients in $\mathbb{Z}_2$). The Elsholz inequality $\mathrm{cat}(X) \geq \mathrm{cup}(X)$ relates $\mathrm{cup}(X)$ to another important homotopy invariant, the *Lyusternik-Shnirel'man category* $\mathrm{cat}(X)$; the latter is defined to be the least positive integer k such that X can be covered by $k+1$ open subsets each of which is contractible in X.

For the *rotation* (or *special orthogonal*) *group* $\mathrm{SO}(n)$, the $\mathbb{Z}_2$-cohomology algebra is known due to A. Borel [1]. We recall its description by A. Hatcher [2]:

$$H^*(\mathrm{SO}(n)) \cong \otimes_{i\,\mathrm{odd},\, i<n} \mathbb{Z}_2[\beta_i]/(\beta_i^{p_i}), \tag{1}$$

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where the degree of β_i is equal to i and p_i is the smallest power of 2 such that the degree of $\beta_i^{p_i}$ is at least n. This cohomology algebra looks quite simple but, to the best of the author's knowledge, only recursive formulas for $\mathrm{cup}(\mathrm{SO}(n))$ were known up to now: the formula $\mathrm{cup}(\mathrm{SO}(2n)) = 2\mathrm{cup}(\mathrm{SO}(n)) + n$, $\mathrm{cup}(\mathrm{SO}(2n)) = \mathrm{cup}(\mathrm{SO}(2n-1)) + 1$, known to the author thanks to Mamoru Mimura, from a 2008-preprint by Kei Sugata, and the formula (perhaps folkloric) $\mathrm{cup}(\mathrm{SO}(n+1)) = \mathrm{cup}(\mathrm{SO}(n)) + 2^{\nu_2(n)}$, where $2^{\nu_2(n)}$ is the highest power of 2 dividing n.

But the problem of finding an explicit formula for $\mathrm{cup}(\mathrm{SO}(n))$ was open thus far. The main aim of this note is to solve it by proving that the cup-length of $\mathrm{SO}(n)$ can be expressed in the following surprisingly concise way.

Theorem 1.1. *For any positive integer n,*

$$\mathrm{cup}(\mathrm{SO}(n)) = n - 1 + (n-1)',$$

where $(n-1)' = \sum_{i=1}^{k} i n_i 2^{i-1}$ *if* $n-1$ *has the dyadic expansion* $\sum_{i=0}^{k} n_i 2^i$.

In view of the Elsholz inequality, Theorem 1.1 immediately implies a global lower bound for the Lyusternik-Shnirel'man category of rotation groups.

Corollary 1.1. *We have*

$$\mathrm{cat}(\mathrm{SO}(n)) \geq n - 1 + (n-1)'.$$

Due to I. James and W. Singhof [5], N. Iwase, M. Mimura, and T. Nishimoto [3], and N. Iwase, K. Kikuchi, and T. Miyauchi [4], it is known that this lower bound is sharp for $n = 1, 2, \ldots, 10$. Of course, our formula for $\mathrm{cup}(\mathrm{SO}(n))$ (Theorem 1.1) is of interest in its own right. But it also enables us to transform the conjecture worded in [4], "this would suggest that $\mathrm{cat}(\mathrm{SO}(n)) = \mathrm{cup}(\mathrm{SO}(n))$ for all n," into the following explicit problem.

Question 1.1. *Is it true that* $\mathrm{cat}(\mathrm{SO}(n)) = n - 1 + (n-1)'$ *for* $n \geq 1$*?*

For odd n ($n \geq 3$), let q be the unique integer such that $2^{q-1} < n < 2^q$. Write $n = 1 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_t}$ ($1 \leq \nu_1 < \nu_2 < \cdots < \nu_t$) the dyadic expansion of n. Then we have $\nu_t < q$ and Theorem 1.1 yields that $\mathrm{cup}(\mathrm{SO}(n)) < \frac{(n-1)(q+2)}{2}$. In a similar way, one verifies that $\mathrm{cup}(\mathrm{SO}(n)) \leq \frac{(n-2)(q+2)}{2}$ for even n. [It is easy to compare these bounds with $\frac{n(n-1)}{2} = \binom{n}{2} = \dim(\mathrm{SO}(n))$.] We thus may state the following weaker (but presumably still very hard) question (whose answer by "No" would of course mean that also Question 1.1 must be answered by "No").

Question 1.2. *For a positive integer n, let q denote the unique integer such that* $2^{q-1} < n \leq 2^q$. *Is it true that* $\mathrm{cat}(\mathrm{SO}(n)) < \frac{(n-1)(q+2)}{2}$ *for all odd n,* $n \geq 11$, *and* $\mathrm{cat}(\mathrm{SO}(n)) \leq \frac{(n-2)(q+2)}{2}$ *for all even n,* $n \geq 12$*?*

2 Proof of the main result

Poincaré duality implies that the cup-length of $\mathrm{SO}(n)$ is realized by a cohomology class in the top degree; note that we may identify $H^{\frac{n(n-1)}{2}}(\mathrm{SO}(n)) = \mathbb{Z}_2$. Obviously, if *n is odd*, then $\mathrm{cup}(\mathrm{SO}(n))$ equals the sum of the exponents in

$$\beta_1^{p_1-1}\beta_3^{p_3-1}\cdots\beta_{n-4}^{p_{n-4}-1}\beta_{n-2} \in H^{\frac{n(n-1)}{2}}(\mathrm{SO}(n)). \tag{2}$$

Thus by (2), for odd n, we see that $\mathrm{cup}(\mathrm{SO}(n)) = (p_1-1)+(p_3-1)+\cdots+(p_{n-4}-1)+1$; consequently, $\mathrm{cup}(\mathrm{SO}(n+1))$ is obviously the sum of the exponents in the product $\beta_1^{p_1-1}\beta_3^{p_3-1}\cdots\beta_{n-4}^{p_{n-4}-1}\beta_{n-2}\beta_n$, since $\dim(\mathrm{SO}(n+1)) - \dim(\mathrm{SO}(n)) = n$ (this difference equals the degree in which we have the generator β_n). Thus indeed, for odd n, $\mathrm{cup}(\mathrm{SO}(n+1)) = [(p_1-1)+(p_3-1)+\cdots+(p_{n-4}-1)+1]+1 = \mathrm{cup}(\mathrm{SO}(n))+1$, as claimed. For even n, a proof is omitted. We have come to the following fact.

Fact 2.1. *Let $c(n) = \mathrm{cup}(\mathrm{SO}(n))$, $n \geq 1$, and $\nu_2(n)$ be the exponent of the highest power of 2 dividing n. Then (i) $c(1) = 0$; (ii) $c(n+1) = c(n) + 2^{\nu_2(n)}$.*

The key observation is the following.

Lemma 2.1. *We have $c(m+2^k) - c(m) = c(2^k-1) + 2^k + 1$, if $1 \leq m \leq 2^k$, $k \geq 1$.*

Proof. If $m = 1$, we have $c(1+2^k) - c(1) = c(1+2^k) = c(2^k) + 2^k = c(2^k-1) + 2^k + 1$ by Fact 2.1 (*i*) and (*ii*), and so we assume $1 < m \leq 2^k$. By Fact 2.1 (ii), we have $c(m+2^k) - c(m-1+2^k) = 2^{\nu_2(m-1)} = c(m) - c(m-1)$, and thus obtain $c(m+2^k) - c(m) = c(m-1+2^k) - c(m-1) = ... = c(1+2^k) - c(1)$ and is equal to $c(2^k-1) + 2^k + 1$. ∎

To show the main result, we need the following proposition.

Proposition 2.1. *For any $k \geq 1$, $c(2^k-1) = k2^{k-1} - 1$.*

Proof. By Lemma 2.1 with $m = 2^k - 1$, we obtain $1 \leq m \leq 2^k$ and $c(2^{k+1}-1) = c(2^k-1+2^k) = c(2^k-1) + c(2^k-1) + 2^k + 1$, which yields the following recurrence relation by taking $a_k = \frac{c(2^k-1)+1}{2^k}$:

$$a_{k+1} = a_k + \frac{1}{2}, \; k \geq 1,$$

which is an arithmetic sequence starting with $a_1 = \frac{c(2^1-1)+1}{2^1} = \frac{1}{2}$, and hence $a_k = \frac{k}{2}$ and $c(2^k-1) = k2^{k-1} - 1$, $k \geq 1$. ∎

Under the above observation, we obtain the main result as follows.

Theorem. *We have* $c(n) = n - 1 + (n-1)'$, $n \geq 1$, *where* $(n-1)' = \sum_{i=1}^{k} i n_i 2^{i-1}$ *if* $n - 1$ *has the dyadic expansion* $\sum_{i=0}^{k} n_i 2^i$.

Proof. If $n = 1$, it is clear by Fact 2.1 (i), and so we assume $n \geq 2$ and $n - 1 \geq 1$ has the dyadic expansion $\sum_{i=0}^{k} n_i 2^i$, with $n_k = 1$. We show the formula by induction on $k \geq 0$.

$k = 0$: Then $n = 2$ and $c(2) = 1 = 1 + 1'$ by Fact 2.1 (i) and (ii).

$k \geq 1$: Let $m = n - 2^k$, to obtain $0 \leq m < 2^k$ and $c(m) = m - 1 + (m-1)'$ by induction hypothesis. Then by Lemma 2.1, we have $c(n) = c(m + 2^k) = c(m) + c(2^k - 1) + 2^k + 1 = m - 1 + (m-1)' + k2^{k-1} + 2^k = (n-1) + (n-1)'$. This completes the proof of Theorem. ∎

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