Flag-transitive point-primitive non-symmetric 2-$(v,k,2)$ designs with alternating socle*

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Abstract

We prove that if $\mathcal{D}$ is a non-trivial non-symmetric 2-$(v,k,2)$ design admitting a flag-transitive point-primitive automorphism group G with $Soc(G) = A_n$ for $n \geq 5$, then $\mathcal{D}$ is a 2-$(6,3,2)$ or 2-$(10,4,2)$ design.

1 Introduction

A 2-(v,k,λ) *design* is a finite incidence structure $\mathcal{D}$=($\mathcal{P}$, $\mathcal{B}$) consisting of v points and b blocks such that every block is incident with k points, every point is incident with r blocks, and any two distinct points are incident with exactly λ blocks. The design $\mathcal{D}$ is called *symmetric* if $v = b$ (or equivalently $r = k$) and *non-trivial* if $1 < k < v$. A *flag* of $\mathcal{D}$ is an incident point-block pair (α, B) where α is a point and B is a block. An automorphism of $\mathcal{D}$ is a permutation of the points which also permutes the blocks. The group of all automorphisms of $\mathcal{D}$ is denoted by $Aut(\mathcal{D})$. A subgroup $G \leq Aut(\mathcal{D})$ is called *point-primitive* if it acts primitively on $\mathcal{P}$ and *flag-transitive* if it acts transitively on the set of flags of $\mathcal{D}$.

It was shown in [12] that the socle of the automorphism group of a flag-transitive point-primitive symmetric 2-$(v,k,2)$ design cannot be alternating or sporadic. Recently, we proved in [7] that, for a non-symmetric 2-$(v,k,2)$ design $\mathcal{D}$, if $G \leq Aut(\mathcal{D})$ is flag-transitive and point-primitive then G must be an affine

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or almost simple group. Moreover, if the socle of G is sporadic, then $\mathcal{D}$ is the unique 2-$(176, 8, 2)$ design with $G = HS$, the Higman-Sims simple group. Here we solve completely the case of almost simple groups in which $Soc(G)$ is an alternating group. Our main result is the following.

Theorem 1.1. *If $\mathcal{D}$ is a non-trivial non-symmetric 2-$(v, k, 2)$ design admitting a flag-transitive point-primitive automorphism group G with alternating socle A_n for $n \geq 5$, then*

(i) *$\mathcal{D}$ is a unique 2-$(6, 3, 2)$ design and $G = A_5$, or*

(ii) *$\mathcal{D}$ is a unique 2-$(10, 4, 2)$ design and $G = S_5$, A_6 or S_6.*

The structure of our paper is as follows. In Section 2, we give some preliminary lemmas on flag-transitive designs and permutation groups. In Section 3, we prove Theorem 1.1 in 5 steps.

2 Preliminaries

Lemma 2.1. *The parameters v, b, r, k, λ of a non-trivial 2-(v, k, λ) design satisfy the following arithmetic conditions:*

(i) *$vr = bk$;*

(ii) *$\lambda(v - 1) = r(k - 1)$;*

(iii) *$b \geq v$ and $k \leq r$.*

In particular, if the design is non-symmetric then $b > v$ and $k < r$. Note also that $k > 2$ as soon as $\lambda > 1$, otherwise two points could not be incident with more than one block.

Lemma 2.2. *Let $\mathcal{D}$ be a non-trivial 2-(v, k, λ) design. Let α be a point of $\mathcal{D}$ and G be a flag-transitive automorphism group of $\mathcal{D}$.*

(i) *$r^2 > \lambda v$ and $|G_\alpha|^3 > \lambda|G|$. In particular, $r^2 > v$.*

(ii) *$r \mid \lambda(v - 1, |G_\alpha|)$, where G_α is the stabilizer of α.*

(iii) *If d is any non-trivial subdegree of G, then $r \mid \lambda d$ (and so $\frac{r}{(r,\lambda)} \mid d$).*

Proof. (i) The equality $r = \frac{\lambda(v-1)}{k-1}$ implies $\lambda v = r(k-1) + \lambda \leq r(r-1) + \lambda = r^2 - r + \lambda$, and the non-triviality of $\mathcal{D}$ implies $r > \lambda$, and so $r^2 > \lambda v$. Combining this with $v = |G : G_\alpha|$ and $r \leq |G_\alpha|$ by the flag-transitivity of G, we have $|G_\alpha|^3 > \lambda|G|$. (ii) Since G is flag-transitive and $\lambda(v - 1) = r(k - 1)$, we have $r \mid \lambda(v - 1)$ and $r \mid |G_\alpha|$. It follows that r divides $(\lambda(v - 1), |G_\alpha|)$, and hence $r \mid \lambda(v - 1, |G_\alpha|)$. Part (iii) was proved in [2, p.91] and [3]. ∎

Lemma 2.3. ([8, p.366]) *If G is A_n or S_n, acting on a set Ω of size n, and H is any maximal subgroup of G with $H \neq A_n$, then H satisfies one of the following:*

(i) $H = (S_\ell \times S_m) \cap G$, *with* $n = \ell + m$ *and* $\ell \neq m$ (*intransitive case*);

(ii) $H = (S_\ell \wr S_m) \cap G$, *with* $n = \ell m$, $\ell > 1$, $m > 1$ *and* $\ell \neq m$ (*imprimitive case*);

(iii) $H = AGL_m(p) \cap G$, *with* $n = p^m$ *and* p *a prime* (*affine case*);

(iv) $H = (T^m.(Out\ T \times S_m)) \cap G$, *with* T *a nonabelian simple group,* $m \geq 2$ *and* $n = |T|^{m-1}$ (*diagonal case*);

(v) $H = (S_\ell \wr S_m) \cap G$, *with* $n = \ell^m$, $\ell \geq 5$ *and* $m > 1$ (*wreath case*);

(vi) $T \trianglelefteq H \leq Aut(T)$, *with* T *a nonabelian simple group,* $T \neq A_n$ *and* H *acting primitively on* Ω (*almost simple case*).

Remark 1. This lemma does not deal with the groups M_{10}, $PGL_2(9)$ and $P\Gamma L_2(9)$ that have A_6 as socle. These exceptional cases will be handled in the first part of Section 3.

Lemma 2.4. ([9, Theorem (b)(I)]) *Let G be a primitive permutation group of odd degree n, acting on a set Ω with simple socle $X = Soc(G)$, and let $H = G_\alpha$, $\alpha \in \Omega$. If $X \cong A_c$, then one of the following holds:*

(i) *H is intransitive, and* $H = (S_a \times S_{c-a}) \cap G$ *where* $1 \leq a < \frac{1}{2}c$;

(ii) *H is transitive and imprimitive, and* $H = (S_a \wr S_{c/a}) \cap G$ *where* $a > 1$ *and* $a \mid c$;

(iii) *H is primitive,* $n = 15$ *and* $G \cong A_7$.

Lemma 2.5. ([5, Theorem 5.2A]) *Let $G = Alt(\Omega)$ where $n = |\Omega| \geq 5$, and let s be an integer with $1 \leq s \leq \frac{n}{2}$. Suppose that $K \leq G$ has index $|G : K| < \binom{n}{s}$. Then one of the following holds:*

(i) *For some* $\Delta \subset \Omega$ *with* $|\Delta| < s$ *we have* $G_{(\Delta)} \leq K \leq G_{\{\Delta\}}$;

(ii) $n = 2m$ *is even, K is imprimitive with two blocks of size m, and* $|G : K| = \frac{1}{2}\binom{n}{m}$; *or*

(iii) *one of six exceptional cases holds:*

  (a) *K is imprimitive on Ω and* $(n, s, |G : K|) = (6, 3, 15)$;

  (b) *K is primitive on Ω and* $(n, s, |G : K|, K) = (5, 2, 6, 5 : 2)$, $(6, 2, 6, PSL_2(5))$, $(7, 2, 15, PSL_3(2))$, $(8, 2, 15, AGL_3(2))$ *or* $(9, 4, 120, P\Gamma L_2(8))$.

Remark 2. (1) From part (i) of Lemma 2.5 we know that K contains the alternating group $G_{(\Delta)} = Alt(\Omega \backslash \Delta)$ of degree $n - s + 1$.

(2) A result similar to Lemma 2.5 holds for the finite symmetric groups $Sym(\Omega)$ [5, Theorem 5.2B].

Lemma 2.6. *Let s and t be two positive integers.*

(i) *If $t > s \geq 7$, then $\binom{s+t}{s} > 2s^2t^2$.*

(ii) *If $s \geq 6$ and $t \geq 2$, then $2^{(s-1)(t-1)} > 2s^4\binom{t}{2}^2$ implies $2^{s(t-1)} > 2(s+1)^4\binom{t}{2}^2$.*

(iii) *If $t \geq 6$ and $s \geq 2$, then $2^{(s-1)(t-1)} > 2s^4\binom{t}{2}^2$ implies $2^{(s-1)t} > 2s^4\binom{t+1}{2}^2$.*

(iv) *If $t \geq 4$, and $s \geq 3$, then $\binom{s+t}{s} > 2s^2t^2$ implies $\binom{s+t+1}{s} > 2s^2(t+1)^2$.*

Proof. (i) If $t > s = 7$, then $\binom{t+7}{7} > 2 \cdot 7^2 \cdot t^2$. If $t > s \geq 8$, then $[\frac{s+t}{2}] \geq s \geq 8$, and so $\binom{s+t}{s} \geq \binom{t+8}{8} > 2t^4 > 2s^2t^2$.

(ii) We have

$$2^{s(t-1)} = 2^{(s-1)(t-1)}2^{t-1} > 2s^4\binom{t}{2}^2 2^{t-1} = 2(s+1)^4\binom{t}{2}^2(1-\frac{1}{s+1})^4 2^{t-1}.$$

Combing this with $(1-\frac{1}{s+1})^4 2^{t-1} \geq 2 \times (\frac{6}{7})^4 > 1$ gives (ii).

(iii) We have

$$2^{(s-1)t} = 2^{(s-1)(t-1)}2^{s-1} > 2s^4\binom{t}{2}^2 2^{s-1} = 2s^4\binom{t+1}{2}^2(1-\frac{2}{t+1})^2 2^{s-1}.$$

Combing this with $(1-\frac{2}{t+1})^2 2^{s-1} \geq 2 \times (\frac{5}{7})^2 > 1$ gives (iii).

(iv) We have

$$\binom{s+t+1}{s} = \binom{s+t}{s}\frac{s+t+1}{t+1} > 2s^2t^2\frac{s+t+1}{t+1} = 2s^2(t+1)^2\frac{(s+t+1)t^2}{(t+1)^3}.$$

The fact that $(s+t+1)t^2 > (t+1)^3$ gives (iv). ∎

3 Proof of Theorem 1.1

In this section, unless otherwise specified, $\mathcal{D}$ denotes always a non-trivial non-symmetric 2-$(v,k,2)$ design, and $G \leq Aut(\mathcal{D})$ is flag-transitive point-primitive with $Soc(G) = A_n$. Let α be a point of $\mathcal{D}$ and $H = G_\alpha$. Since G is point-primitive, H is a maximal subgroup of G by [14, Theorem 8.2]. Furthermore, by the flag-transitivity of G, we have $v = |G : H|$, $b \mid |G|$, $r \mid |H|$ and $r^2 > 2v$ by Lemma 2.2 (i).

If r is odd, Zhou and Wang [13] proved the following:

Proposition 3.1. *Let $\mathcal{D}$ be a non-trivial non-symmetric 2-$(v,k,2)$ design admitting a flag-transitive point-primitive automorphism group G with $Soc(G) = A_n$, $n \geq 5$. If the replication number r is odd, then $\mathcal{D}$ is the unique 2-$(6,3,2)$ design and $G = A_5$.*

From now on, we will assume that r is even.

Suppose first that $n = 6$ and $G \cong M_{10}$, $PGL_2(9)$ or $P\Gamma L_2(9)$. Each of these groups has exactly three maximal subgroups with index greater than 2, and their indices are 45, 36 and 10. Using the computer algebra system GAP [6] for $v = 45$, 36 or 10, we have computed the parameters (v, b, r, k) that satisfy the following conditions:

$$r \mid (2(v-1), |H|); \tag{3.1}$$

$$r^2 > 2v; \tag{3.2}$$

$$2 \mid r; \tag{3.3}$$

$$r(k-1) = 2(v-1); \tag{3.4}$$

$$r > k > 2; \tag{3.5}$$

$$b = \frac{vr}{k}. \tag{3.6}$$

It turns out that the only possible parameters (v, b, r, k) are:

$$(10, 15, 6, 4) \text{ and } (36, 45, 10, 8).$$

Now we consider the possible existence of flag-transitive point-primitive non-symmetric designs with these parameters.

Suppose first that there exists a 2-$(10, 4, 2)$ design $\mathcal{D}$ with a flag-transitive point-primitive automorphism group G. Let $\mathcal{P} = \{1, 2, ..., 10\}$ and $G = M_{10}$, $PGL_2(9)$ or $P\Gamma L_2(9)$ be the primitive permutation group of degree 10 acting on $\mathcal{P}$. Since G is flag-transitive, G acts block-transitively on $\mathcal{B}$, so $|G|/b = |G_B|$, where B is a block. For each case, using the command `Subgroups(G:OrderEqual:=n)` where `n` $= |G|/b$ by Magma [1], it turns out that G has no subgroup of order `n`, which contradicts the fact that G_B is a subgroup of order $|G|/b$.

Assume next that there exists a 2-$(36, 8, 2)$ design $\mathcal{D}$ with a flag-transitive point-primitive automorphism group $G = M_{10}$, $PGL_2(9)$ or $P\Gamma L_2(9)$.

When $(v, G) = (36, P\Gamma L_2(9))$, by the Magma-command `Subgroups(G:OrderEqual:=n)` where `n` $= |G|/b$, we get the block stabilizer G_B. Since G is flag-transitive, G_B is transitive on B, and so B is an orbit of G_B acting on $\mathcal{P}$. Using the Magma-command `Orbits(GB)` where `GB` $= G_B$, it turns out that G_B has no orbit of length k, a contradiction.

Now assume that $(v, G) = (36, M_{10})$ or $(36, PGL_2(9))$. Every pair of distinct points must be contained in 2 blocks. However, for each case, the command `PairwiseBalancedLambda(D)` contradicts this condition.

If $(v, G) = (36, M_{10})$, the orbits of G_B are:

$$\begin{aligned}
\Delta_0 &= \{3, 17, 18, 21\}, \\
\Delta_1 &= \{1, 4, 12, 14, 16, 22, 26, 34\}, \\
\Delta_2 &= \{2, 6, 7, 9, 15, 23, 29, 36\}, \\
\Delta_3 &= \{5, 8, 10, 11, 13, 19, 20, 24, 25, 27, 28, 30, 31, 32, 33, 35\}.
\end{aligned}$$

As $k = 8$, we take $B = \Delta_1$ or $B = \Delta_2$. Using the GAP-command `D1 := BlockDesign(36, [[1,4,12,14,16,22,26,34]],G)`, we get $|\Delta_1^G| = 45 = b$. We take

$\mathcal{P} = \{1, 2, \ldots, 36\}$, $B = \Delta_1$ and $\mathcal{B} = B^G$. Now, we just need to check that each pair of distinct points is contained in 2 blocks. However, `PairwiseBlancedLambda(D1)` shows that this is not true, and so $B \neq \Delta_1$. Similarly, $B \neq \Delta_2$. So the case $(v, G) = (36, M_{10})$ cannot occur.

Now we consider $G = A_n$ or S_n with $n \geq 5$. The point stabilizer $H = G_\alpha$ acts both on $\mathcal{P}$ and on the set $\Omega_n = \{1, 2, \ldots, n\}$. Then by Lemma 2.3 one of the following holds:

(i) H is primitive in its action on Ω_n;

(ii) H is transitive and imprimitive in its action on Ω_n;

(iii) H is intransitive in its action on Ω_n.

We analyse each of these actions separately, under the following assumption:

Hypothesis 1. *$\mathcal{D}$ is a non-trivial non-symmetric 2-$(v, k, 2)$ design admitting a flag-transitive point-primitive automorphism group G with $Soc(G) = A_n$ ($n \geq 5$) and r is even.*

3.1 H acts primitively on Ω_n

Proposition 3.2. *If Hypothesis 1 holds and the point stabilizer H acts primitively on Ω_n, then there are 10 possible parameters (n, v, b, r, k), which are listed in Table 3.*

Proof. We claim that $2 \| r$. Otherwise $4 \mid r$, and the equality $r(k-1) = 2(v-1)$ implies that v is odd. Thus by Lemma 2.4, $v = 15$, $G = A_7$ and $|H| = |G|/v = 168$. Since $r \mid (2(v-1), |H|)$, $r^2 > 2v$ and $k \geq 3$, it follows that $r = 7$ or 14, which contradicts $4 \mid r$.

Thus $2 \| r$. Let $r = 2r'$. Since $r > 2$, there exists an odd prime p that divides r', then $p \mid (v-1)$, and so $(p, v) = 1$. Thus H contains a Sylow p-subgroup P of G. Let $g \in G$ be a p-cycle, then there is a conjugate of g belonging to H. This implies that H acting on Ω_n contains an even permutation with exactly one cycle of length p and $n-p$ fixed points. By a result of Jordan [14, Theorem 13.9], $n - p \leq 2$. Therefore, $n-2 \leq p \leq n$, $p^2 \nmid |G|$, and so $p^2 \nmid r'$. It follows that r' is either a prime, namely $n-2$, $n-1$ or n, or the product of two twin primes, namely $(n-2)n$. Moreover, the primitivity of H acting on Ω_n and $H \ngeq A_n$ imply that $v \geq \frac{[\frac{n+1}{2}]!}{2}$ by [14, Theorem 14.2]. Combining with $r^2 > 2v$, we get

$$r^2 > [\frac{n+1}{2}]!.$$

Therefore, $(n, r) =$ (5, 6), (5, 10), (5, 30), (6, 10), (7, 10), (7, 14), (7, 70), (8, 14), (9, 14) or (13, 286). By Lemmas 2.1 and 2.2, using $v \geq \frac{[\frac{n+1}{2}]!}{2}$ and $[b, v] \mid |G|$, we obtain exactly 10 possible parameters (n, v, b, r, k):

$(5, 10, 15, 6, 4)$, $(6, 16, 40, 10, 4)$, $(6, 36, 45, 10, 8)$, $(7, 15, 70, 14, 3)$, $(7, 16, 40, 10, 4)$, $(7, 21, 42, 10, 5)$, $(7, 36, 45, 10, 8)$, $(7, 36, 84, 14, 6)$, $(8, 15, 70, 14, 3)$, $(8, 36, 84, 14, 6)$.

They are listed in Table 3. ∎

3.2 H acts transitively and imprimitively on Ω_n

Proposition 3.3. *If Hypothesis 1 holds and the point stabilizer H acts transitively but imprimitively on Ω_n, then there are 2 possible parameters $(n, v, b, r, k) = (6, 10, 15, 6, 4)$ or $(10, 126, 1050, 50, 6)$, which are listed in Table 3.*

Proof. Suppose on the contrary that $\Sigma = \{\Delta_0, \Delta_1, \ldots, \Delta_{t-1}\}$ is a non-trivial partition of Ω_n preserved by H, where $|\Delta_i| = s$, $0 \leq i \leq t-1$, $s, t \geq 2$ and $st = n$. Then

$$v = \binom{ts-1}{s-1}\binom{(t-1)s-1}{s-1}\cdots\binom{3s-1}{s-1}\binom{2s-1}{s-1}.$$

Moreover, the set O_j of j-cyclic partitions with respect to X (a partition of Ω_n into t classes each of size s) is a union of orbits of H on $\mathcal{P}$ for $j = 2, \ldots, t$ (see [4, 15] for definitions and details).

Case (1): Suppose first that $s = 2$. Then $t \geq 3$, $v = (2t-1)(2t-3)\cdots 5 \cdot 3$, and

$$d_j = |O_j| = \frac{1}{2}\binom{t}{j}\binom{s}{1}^j = 2^{j-1}\binom{t}{j}.$$

If $t \geq 7$, then $v = (2t-1)(2t-3)\cdots 5 \cdot 3 > 5t^2(t-1)^2$. On the other hand, since r divides $2d_2 = 2t(t-1)$, $2t(t-1) \geq r$, and so $v < 2t^2(t-1)^2$, a contradiction. Thus $t < 7$. For $t = 3, 4, 5$ or 6, the values of $d = 2\gcd(d_2, d_3)$ are listed in Table 1 below.

Table 1: Possible d when $s = 2$

t	n	v	d_2	d_3	d
3	6	15	6	4	4
4	8	105	12	16	8
5	10	945	20	40	40
6	12	10395	30	80	20

In each line $r \leq d$, which contradicts the fact that $r^2 > 2v$.

Case (2): Thus $s \geq 3$. So O_j is an orbit of H on $\mathcal{P}$, and $d_j = |O_j| = \binom{t}{j}\binom{s}{1}^j = s^j\binom{t}{j}$. In particular, $d_2 = s^2\binom{t}{2}$ and $r \mid 2d_2$. Moreover, from $\binom{is-1}{s-1} = \frac{is-1}{s-1} \cdot \frac{is-2}{s-2} \cdots \frac{is-(s-1)}{1} > i^{s-1}$, for $i = 2, 3, \ldots, t$, we have that $v > 2^{(s-1)(t-1)}$. Then

$$2 \cdot 2^{(s-1)(t-1)} < 2v < r^2 \leq 4s^4\binom{t}{2}^2,$$

and so

$$2^{(s-1)(t-1)} < 2s^4\binom{t}{2}^2. \tag{3.7}$$

Now we determine all pairs (s, t) satisfying (3.7). Clearly, the pair $(s, t) = (6, 6)$ does not satisfy (3.7), but it satisfies the conditions (ii) and (iii) of Lemma 2.6. Thus, either $s < 6$ or $t < 6$. It is not hard to get the 36 pairs (s,t) satisfying (3.7), namely

$(3,2)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$, $(3,7)$, $(3,8)$, $(3,9)$, $(3,10)$, $(4,2)$, $(4,3)$, $(4,4)$, $(4,5)$, $(4,6)$, $(5,2)$, $(5,3)$, $(5,4)$, $(5,5)$, $(6,2)$, $(6,3)$, $(6,4)$, $(7,2)$, $(7,3)$, $(8,2)$, $(8,3)$, $(9,2)$, $(9,3)$, $(10,2)$, $(11,2)$, $(12,2)$, $(13,2)$, $(14,2)$, $(15,2)$, $(16,2)$, $(17,2)$, $(18,2)$.

For each pair (s,t), we compute the parameters (v,b,r,k) satisfying Lemmas 2.1, 2.2, $2 \mid r$ and $r \mid 2d_2$. There are only 2 possible such parameters, namely

$$\begin{aligned}(s,t) &= (3,2) \ \text{ with } \ (n,v,b,r,k) = (6,10,15,6,4),\\ (s,t) &= (5,2) \ \text{ with } \ (n,v,b,r,k) = (10,126,1050,50,6),\end{aligned}$$

which are listed in Table 3. ∎

3.3 H acts intransitively on Ω_n

Proposition 3.4. *If Hypothesis 1 holds and the point stabilizer H acts intransitively on Ω_n, then there are 15 possible parameters (n,v,b,r,k), which are listed in Table 3.*

Proof. Since H acts intransitively on Ω_n, we have $H = (Sym(S) \times Sym(\Omega_n \backslash S)) \cap G$ and, without loss of generality, we may assume that $|S| = s < \frac{n}{2}$ by Lemma 2.3 (i). By the flag-transitivity of G, H is transitive on the blocks through α, and so H fixes exactly one point in $\mathcal{P}$. Since H stabilizes only one s-subset of Ω_n, we can identify the point α with S. As the orbit of S under G consists of all the s-subsets of Ω_n, we can identify $\mathcal{P}$ with the set of s-subsets of Ω_n. So $v = \binom{n}{s}$, G has rank $s+1$ and the subdegrees are:

$$d_0 = 1,\ d_{i+1} = \binom{s}{i}\binom{n-s}{s-i},\ i = 0,1,2,\ldots,s-1.$$

It follows from $r \mid 2d_s$ and $d_s = s(n-s)$ that $r \mid 2s(n-s)$. Combining this with $r^2 > 2v$, we have $2s^2(n-s)^2 > \binom{n}{s}$. Since $s < \frac{n}{2}$ is equivalent to $s < t = n-s$, we have

$$2s^2t^2 > \binom{s+t}{s}.$$

Combining this with Lemma 2.6 (i), we have $s \leq 6$.

Case (1): If $s = 1$, then $v = n \geq 5$ and the subdegrees are $1, n-1$. If $k = v-1$, then $r(v-2) = 2(v-1)$, and so $v-2 \mid v-1$ since $(r,2) = 2$, a contradiction. Therefore, $2 < k \leq v-2$. Since G is $(v-2)$-transitive on $\mathcal{P}$, G acts k-transitively on $\mathcal{P}$, and so $b = |\mathcal{B}| = |B^G| = \binom{n}{k}$ for every block $B \in \mathcal{B}$. From the equality $bk = vr$, we obtain $\binom{n}{k}k = nr$. On one hand, by $r(k-1) = 2(n-1)$ and $k > 2$, we have $r \leq n-1$, and so $\binom{n}{k}k \leq n(n-1)$; on the other hand, by $2 < k \leq n-2$, we have $n-i \geq k-i+2 > k-i+1$ for $i = 2,3,\ldots,k-1$. Thus,

$$\binom{n}{k}k = n(n-1)\cdot\frac{n-2}{k-1}\cdot\frac{n-3}{k-2}\cdots\frac{n-k+1}{2} > n(n-1),$$

a contradiction.

Case (2): If $s = 2$, then $v = \frac{n(n-1)}{2}$ and the subdegrees are 1, $\binom{n-2}{2}$, $2(n-2)$. By Lemma 2.2 (iii), r divides $2(\binom{n-2}{2}, 2(n-2)) = (n-2)(n-3, 4)$.

(a) If $n \equiv 0$ or 2(mod 4), then r divides $n-2$, and so $n(n-1) = 2v < r^2 \leq (n-2)^2$, which is impossible.

(b) If $n \equiv 1$(mod 4), then r divides $2(n-2)$.

Let $r = \frac{2(n-2)}{u}$ for some integer u. Since $r^2 > 2v$, we have $4 > \frac{4(n-2)^2}{n(n-1)} > u^2$, which forces $u = 1$. Therefore, $r = 2(n-2)$. By Lemma 2.1, $k = \frac{n+3}{2}$ and $b = \frac{2n(n-1)(n-2)}{n+3}$. Since b is an integer, $n+3$ divides 120 with $n \equiv 1$(mod 4), and so n=5, 9, 17, 21, 37, 57 or 117. For each such n, we compute the parameters (v, b, r, k). If $n \in \{17, 21, 37, 57, 117\}$, then $|G : G_B| = b < \binom{n}{3}$. By Lemma 2.5 and [5, Theorem 5.2B], G has no subgroup of index b, a contradiction. So we obtain only 2 possible parameters (n, v, b, r, k), namely

$$(5, 10, 15, 6, 4),\ (9, 36, 84, 14, 6).$$

(c) If $n \equiv 3$(mod 4), then r divides $4(n-2)$.

Let $r = \frac{4(n-2)}{u}$ for some integer u. Since $r^2 > 2v$, we have $16 > \frac{16(n-2)^2}{n(n-1)} > u^2$, and so $u = 1, 2$ or 3.

If $u = 1$, then $r = 4(n-2)$, $k = \frac{n+5}{4}$ and $b = \frac{8n(n-1)(n-2)}{n+5}$. As b is an integer, $n+5$ divides 1680 with $n \equiv 3$(mod 4), and so n=7, 11, 15, 19, 23, 35, 43, 51, 55, 75, 79, 107, 115, 135, 163, 235, 275, 331, 415, 555, 835 or 1675. By Lemma 2.5 and [5, Theorem 5.2B], $n \in \{7, 11, 15, 19, 23, 35, 43\}$ and we obtain 7 possible parameters (n, v, b, r, k), namely

$$(7, 21, 140, 20, 3),\ (11, 55, 495, 36, 4),\ (15, 105, 1092, 52, 5),\ (19, 171, 1938, 68, 6),$$
$$(23, 253, 3036, 84, 7),\ (35, 595, 7854, 132, 10),\ (43, 903, 12341, 164, 12).$$

If $u = 2$, then $r = 2(n-2)$, $k = \frac{n+3}{2}$ and $b = \frac{2n(n-1)(n-2)}{n+3}$, and so $n+3$ divides 120 with $n \equiv 3$(mod 4). Therefore $n = 7$ or 27. By Lemma 2.5 and [5, Theorem 5.2B], $n \neq 27$ and we get $(n, v, b, r, k) = (7, 21, 42, 10, 5)$.

If $u = 3$, then $r = \frac{4(n-2)}{3}$, $k = \frac{3n+7}{4}$ and $b = \frac{8n(n-1)(n-2)}{3(3n+7)}$, and so $3n+7$ divides 7280. Since r is an integer with $n \equiv 3$(mod 4), it follows that $n \equiv 11$(mod 12). Therefore n =11, 35, 119 or 1211. For each n, $|G : G_B| = b < \binom{n}{3}$. By Lemma 2.5 and [5, Theorem 5.2B], it is easy to know that G has no subgroup of index b.

Case (3): Suppose that $3 \leq s \leq 6$. For each value of s, there is a value of t such that $\binom{s+t}{s} > 2s^2t^2$ and so, by Lemma 2.6 (iv), t is bounded (hence so $n = s + t$). For example, let $s = 3$, since $\binom{3+102}{3} > 2 \cdot 3^2 \cdot 102^2$, we must have $4 \leq t \leq 101$, and so $7 \leq n \leq 104$. The bounds for n are listed in Table 2 below.

Note that $v = \binom{n}{s}$, and $d_1 = \binom{n-s}{s}$, $d_2 = s\binom{n-s}{s-1}$, $d_3 = \binom{s}{2}\binom{n-s}{s-2}$ are three non-trivial subdegrees of G acting on $\mathcal{P}$. Therefore, the 5-tuple (n, v, b, r, k) satisfies the arithmetical conditions: (3.1)-(3.6) and $r \mid 2d_i$, $i \in \{1, 2, 3\}$.

If $s = 3$, `GAP` outputs only five 5-tuples, namely

$$(13, 286, 429, 30, 20),\ (14, 364, 2002, 66, 12),\ (22, 1540, 6270, 114, 28),$$
$$(32, 4960, 14880, 174, 58),\ (50, 19600, 39480, 282, 140).$$

Table 2: Bounds of n when $3 \leq s \leq 6$

s	t	n
3	$4 \leq t \leq 101$	$7 \leq n \leq 104$
4	$5 \leq t \leq 22$	$9 \leq n \leq 26$
5	$6 \leq t \leq 12$	$11 \leq n \leq 17$
6	$7, 8, 9$	$13, 14, 15$

If $s = 4$, 5 or 6, using GAP, there is no parameter (n, v, b, r, k) satisfying these conditions.

Thus, we obtain exactly 15 possible parameters (n, v, b, r, k), listed in Table 3.

■

3.4 Ruling out potential parameters

Now, we will rule out the 23 potential cases listed in Table 3.

(i) Ruling out CASES 6, 7, 11 and 12.

The GAP-command `PrimitiveGroup(v,nr)` returns the primitive group with degree v in position nr in the list of the library of primitive permutation groups. For each CASE, the command shows that there is no primitive group corresponding to v.

(ii) Ruling out CASES 1 and 8.

Since G is flag-transitive, $|H| = |G|/v$. For each case, H is primitive on Ω_n. However, the command `PrimitiveGroup(v,nr)`, where `v` $= n$, shows that there is no such group of order $|G|/v$.

(iii) Ruling out CASES 15, 16, 18, 19, 21, 23 and 25.

Since G is flag-transitive, G acts transitively on $\mathcal{B}$, so $|G|/b = |G_B|$, where B is a block. For each case, using the Magma-command `Subgroups(G:OrderEqual:=n)` where `n` $= |G|/b$, it turns out that G has no subgroup of order `n`. When `v` ≥ 2500, the GAP-command `PrimitiveGroup(v,nr)` does not know the group of degree `v`. For CASE 25, $G = A_{50}$ or S_{50}, we use the Magma-command `G := Alt(50)` or `G := Sym(50)` to get the group G, and `Subgroups(G:OrderEqual:=n)` where `n` $= |G|/b$ to conclude that G does not have such a subgroup of order $|G|/b$.

(iv) Ruling out CASES 13, 14, 17, 20 and 22.

Since G_B is transitive on B, B is an orbit of G_B acting on the point set $\mathcal{P}$. Using the Magma-command `Orbits(GB)`, where `GB` $= G_B$, it turns out that G_B has no orbit of length b, a contradiction.

(v) Ruling out CASES 3 and 5.

Using the command `Orbits(GB)`, we get the orbits of G_B. As $|B| = k$, we take the orbit of length k as B. Since G acts transitively on $\mathcal{B}$, $|B^G| = b$. However, using the GAP-command `OrbitLength(G,B,OnSets)`, we get that $|B^G| < b$.

(vi) Ruling out CASES 9 and 10.

For each case, the GAP-command `PairwiseBalancedLambda(D)` concludes that D is not pairwise balanced, a contradiction.

Table 3: Potential parameters

CASE	(v, b, r, k)	$Soc(G)$ or G	Proposition	Step/Reference
1	$(10, 15, 6, 4)$	A_5	3.2	(ii)
2		A_6	3.3	$\mathcal{D}$
3		$G = A_5$	3.4	(v)
4		$G = S_5$	3.4	$\mathcal{D}$
5	$(15, 70, 14, 3)$	$G = A_7$ or A_8	3.2	(v)
6		$G = S_7$ or S_8	3.2	(i)
7	$(16, 40, 10, 4)$	A_6, A_7	3.2	(i)
8	$(21, 42, 10, 5)$	A_7	3.2	(ii)
9		A_7	3.4	(vi)
10	$(21, 140, 20, 3)$	A_7	3.4	(vi)
11	$(36, 45, 10, 8)$	A_6, A_7	3.2	(i)
12	$(36, 84, 14, 6)$	A_7, A_8	3.2	(i)
13		A_9	3.4	(iv)
14	$(55, 495, 36, 4)$	A_{11}	3.4	(iv)
15	$(105, 1092, 52, 5)$	A_{15}	3.4	(iii)
16	$(126, 1050, 50, 6)$	A_{10}	3.3	(iii)
17	$(171, 1938, 68, 6)$	A_{19}	3.4	(iv)
18	$(253, 3036, 84, 7)$	A_{23}	3.4	(iii)
19	$(286, 429, 30, 20)$	A_{13}	3.4	(iii)
20	$(364, 2002, 66, 12)$	A_{14}	3.4	(iv)
21	$(595, 7854, 132, 10)$	A_{35}	3.4	(iii)
22	$(903, 12341, 164, 12)$	A_{43}	3.4	(iv)
23	$(1540, 6270, 114, 28)$	A_{22}	3.4	(iii)
24	$(4960, 14880, 174, 58)$	A_{32}	3.4	(vii)
25	$(19600, 39480, 282, 140)$	A_{50}	3.4	(iii)

For CASE 9, take $(v, G) = (21, A_7)$ for example. The orbits of G_B are:

$$\Delta_0 = \{13\}, \qquad \Delta_1 = \{2, 7, 12, 14, 15\},$$
$$\Delta_2 = \{4, 9, 16, 19, 20\}, \quad \Delta_3 = \{1, 3, 5, 6, 8, 10, 11, 17, 18, 21\}.$$

As $k = 5$, we take $B = \Delta_1$ or $B = \Delta_2$. Using the GAP-command `D := BlockDesign(21, [[2, 7, 12, 14, 15]], G)`, we get $|\Delta_1^G| = 42$ and $\Delta_2 \in \Delta_1^G$. Without loss of generality, we take $\mathcal{P} = \{1, 2, \ldots, 21\}$, $B = \Delta_1$ and $\mathcal{B} = \Delta_1^G$. Now, we just need check that `D` is pairwise balanced. However, `PairwiseBalancedLambda(D)` shows that this is not true. So the case $(v, G) = (21, A_7)$ cannot occur.

(vii) Ruling out CASE 24.

Consider first $(v, G) = (4960, A_{32})$. Let $\Omega_n = \{1, 2, \ldots, 32\}$, then G acts primitively on Ω_n. Let $\mathcal{P} = \Omega_n^{\{3\}}$ denote the set of all 3-subsets of Ω_n. Then G acts on $\mathcal{P}$ in a natural way and $|\mathcal{P}| = \binom{32}{3} = 4960$. Using the Magma-command `G := Alt(32)` and `Subgroups(G:OrderEqual:=n)` where `n` $= |G|/b$, we get that G

contains only one conjugacy class of subgroups of order $|G|/b$, with K as representative, so the block stabilizer G_B is conjugate to K, and then there is a block B_0 such that $K = G_{B_0}$. Since G is flag-transitive, B_0 is an orbit of K acting on $\mathcal{P}$. Take $S = \{1, 2, 3\} \in \mathcal{P}$. Using the command $\texttt{OrbitLength(G,S,OnSets)}$, G acts transitively on $\mathcal{P}$, and using the command $\texttt{OrbitLength(K,S}'\texttt{,OnSets)}$ for all $\texttt{S}' \in \mathcal{P}$, K acting on $\mathcal{P}$ has exactly one orbit Γ of length 58. As $k = 58$, we take $B_0 = \Gamma$. Furthermore, the Magma-command $\texttt{O} := \Gamma\texttt{\^{}G}$ shows out that $|\texttt{O}| = 14880 = b$, and so we take $\mathcal{B} = \texttt{O}$. Now, we just need to check that each pair of distinct points is contained in 2 blocks. Let $S_1 = \{1, 2, 3\}$, $S_2 = \{5, 6, 9\} \in \mathcal{P}$. Magma shows that there is no block in $\mathcal{B}$ containing both S_1 and S_2 , a contradiction. So the case $(v, G) = (4960, A_{32})$ cannot occur.

The analysis of $(v, G) = (4960, S_{32})$ is similar. ■

3.5 The unique non-symmetric 2-$(10, 4, 2)$ design

For CASE 2 and CASE 4, the parameters $(v, b, r, k) = (10, 15, 6, 4)$. It is well-known that, up to isomorphism, there are exactly three 2-$(10, 4, 2)$ designs, see [10] or [11]. Moreover, it is not hard to know that, among these 3 designs, only one has a flag-transitive point-primitive automorphism group $G = S_5$, A_6 or S_6, which is denoted by $\mathcal{D}$.

This completes the proof of Theorem 1.1. ■

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