Differential K-homology and explicit isomorphisms between $\mathbb{R}/\mathbb{Z}$-K-homologies

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Abstract

In this paper, we construct an explicit isomorphism between Deeley $\mathbb{R}/\mathbb{Z}$-K-homology and flat K-homology. We also describe $\mathbb{R}/\mathbb{Z}$-K-homology out of $\mathbb{Z}/k\mathbb{Z}$-bordism theories.

1 Introduction

K-homology is dual to topological K-theory. A geometric model was introduced by Baum-Douglas (see [5]), and proved to be an extremely important tool in index theory and physics (see [15]): one of the main advantages of this geometric formulation is that K-homology cycles encode the most primitive requisite objects that must be carried by any D-brane, such as a $Spin^c$-manifold and a Hermitian vector bundle.

Beside K-theory, there is also the so-called differential K-theory. It combines cohomological information with differential form information in a complicated way. A model for this theory was studied extensively by Freed and Lott (see [9]). Motivated by generalizing pairings between K-theory and K-homology to the case of differential K-theory and K-homology with $\mathbb{R}/\mathbb{Z}$-coefficients, we introduce an extension of geometric K-homology by continuous current data, called differential K-homology, which encodes Deeley $\mathbb{R}/\mathbb{Z}$-K-homology as a flat theory (Theorem 3.8), and so through Theorem 2.6 we obtain explicit realizations of this pairing.

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In the end of this paper, we describe the torsion part of Deeley $\mathbb{R}/\mathbb{Z}$-K-homology through framed $\mathbb{Z}/k\mathbb{Z}$-bordism theories (Proposition 4.4), and so the approach of Atiyah-Patodi-Singer to $\mathbb{R}/\mathbb{Z}$-K-theory presented in [2, 3] leads to another model for $\mathbb{R}/\mathbb{Z}$-K-homology.

2 Differential K-homology and its pairings with Freed-Lott differential K-theory

In this section we define a differential K-homology.

In all the following, we denote by X a smooth compact manifold.

Definition 2.1. A differential K-cycle over X is a quadruple $(M, (E, \nabla^E), f, \phi)$ consisting of :

- A smooth closed $Spin^c$-manifold M.
- A smooth Hermitian vector bundle E over M with a unitary connection ∇^E.
- A smooth map $f : M \to X$.
- A class of currents $\phi \in \frac{\Omega_*(X)}{img(\partial)}$.

There are no connectedness requirements made upon M, and hence the bundle E can have different fibre dimensions on the different connected components of M. It follows that the disjoint union,

$$(M, (E, \nabla^E), f, \phi) \sqcup (M', (E', \nabla^{E'}), f', \phi') := \\ (M \sqcup M', (E \sqcup E', \nabla^E \sqcup \nabla^{E'}), f \sqcup f', \phi + \phi'),$$

is a well-defined operation on the set of differential K-cycles over X.

A differential K-cycle $(M, (E, \nabla^E), f, \phi)$ is called even (resp. odd), if all connected components of M are of even (resp. odd) dimension and $\phi \in \frac{\Omega_{odd}(X)}{img(\partial)}$ (resp. $\phi \in \frac{\Omega_{ev}(X)}{img(\partial)}$).

We define an equivalence relation on differential K-cycles as follows. First, let $x := (M, (E, \nabla^E), f, \phi)$ be a differential K-cycle over X and V be a $Spin^c$-vector bundle of even rank over M with an Euclidean connection ∇^V. Let 1_M denote the trivial rank-one real vector bundle over M. We denote by $\hat{M}$ the boundary of the unit disk bundle $\mathbb{D}(V \oplus 1_M)$ of $V \oplus 1_M$. The $Spin^c$-structures on TM and $V \oplus 1_M$ induce a $Spin^c$-structure on $T\mathbb{D}(V \oplus 1_M)$ by a direct sum decomposition $T(V \oplus 1_M) \cong \pi^*(V \oplus 1_M) \oplus \pi^* TM$ where π is the bundle projection of $V \oplus 1_M$, and then taking the boundary of this $Spin^c$-structure to obtain a $Spin^c$-structure on $T\hat{M}$.

Denote by $S := S_+ \oplus S_-$ the $\mathbb{Z}_2$-graded spinor bundle associated with the $Spin^c$-structure on the vertical tangent bundle of $\hat{M}$ carrying a unitary connection

$\nabla^{S_+} \oplus \nabla^{S_-}$ induced by ∇^V. Define $\hat{V}$ to be the dual of S_+ and $\nabla^{\hat{V}}$ to be the unitary connection on $\hat{V}$ induced by ∇^{S_+}. We denote by x^V the quadruple $(\hat{M}, (\hat{V} \otimes \pi^* E, \nabla^{\hat{V}} \otimes \pi^* \nabla^E), f \circ \pi, \phi)$, called the modification of x by V, which is obviously a differential K-cycle over X.

Now two differential K-cycles ξ and ξ' over X are said to be *equivalent* if there exist a $Spin^c$-vector bundle V of even rank over the manifold in ξ', a smooth compact $Spin^c$-manifold W, a smooth Hermitian vector bundle ε over W with a unitary connection ∇^ε, and a smooth map $g : W \to X$ such that

$$\xi \sqcup \xi'^V_- = (\partial W, (\varepsilon|_{\partial W}, \nabla^\varepsilon|_{\partial W}), g|_{\partial W}, [\int_W Td(W) ch(\nabla^\varepsilon) g^*]),$$

where $\xi_- = (M^-, (E, \nabla^E), f, -\phi)$ when $\xi = (M, (E, \nabla^E), f, \phi)$ and M^- denotes M with its $Spin^c$ structure reversed, $Td(W)$ is the $Spin^c$-Todd form of the Levi-Civita connection on M and $ch(\nabla^\varepsilon)$ is the geometric Chern form of ∇^ε. In this situation, $(W, (\varepsilon, \nabla^\varepsilon), g)$ is called a K-chain over X with differential boundary $\xi \sqcup \xi'^V_-$.

Definition 2.2. *The differential K-homology group* $\check{K}_*(X)$ is the group of equivalence classes of differential K-cycles over X, for the equivalence relation generated by the above relation and the following identification:
Direct sum:
$(M, (E, \nabla^E), f, \phi) \sqcup (M, (E', \nabla^{E'}), f, \phi') \sim (M, (E \oplus E', \nabla^E \oplus \nabla^{E'}), f, \phi + \phi')$.

The group $\check{K}_*(X)$ is Abelian and naturally $\mathbb{Z}_2$-graded:

$$\check{K}_*(X) = \check{K}_{ev}(X) \oplus \check{K}_{odd}(X).$$

The construction of differential K-homology is functorial: for every smooth map $\rho : X \to Y$ between two smooth compact manifolds, the homomorphism $\rho_* : \check{K}_*(X) \to \check{K}_*(Y)$ is defined by

$$\rho_*[M, (E, \nabla^E), f, \phi] := [M, (E, \nabla^E), \rho \circ f, \phi \circ \rho^*].$$

Remark 2.3. If $(M, (E, \nabla^E_0), f, \phi)$ and $(M, (E, \nabla^E_1), f, \phi)$ are two differential K-cycles, then

$$[M, (E, \nabla^E_0), f, \phi] = \\ [M, (E, \nabla^E_1), f, \phi - [\int_{M \times [0,1]} Td(M \times [0,1]) ch(B) (f \circ p)^*]] \ (\in \check{K}_*(X)),$$

where B is the connection on the pullback of E by the projection $p : M \times [0,1] \to M$, given by $B = (1-t)\nabla^E_0 + t\nabla^E_1 + dt\frac{d}{dt}$.

Recall that a K-chain (of Baum-Douglas) over X is of the form $(W, (\varepsilon, \nabla^\varepsilon), g)$, where W is a smooth compact $Spin^c$-manifold, ε is a Hermitian vector bundle over W with a unitary connection ∇^ε, and g a smooth map from W to X. The boundary of a K-chain $(W, (\varepsilon, \nabla^\varepsilon), g)$ is defined by $\partial(W, (\varepsilon, \nabla^\varepsilon), g) := (\partial W, (\varepsilon|_{\partial W}, \nabla^\varepsilon|_{\partial W}), g|_{\partial W})$. A K-cycle is a K-chain without boundary. We refer the reader to [5] for the definition of K-homology group $K_*^{geo}(X)$ out of K-cycles and K-chains. Let $Ch_* : K_*^{geo}(X) \to H_*^{dR}(X)$ be the Chern character, $[M,(E,\nabla^E),f] \overset{Ch_*}{\mapsto} [\int_M Td(M)ch(\nabla^E)f^*]$, and $\Omega_*^0(X) := \{\phi \in \Omega_*(X) \mid [\phi] \in img(Ch_*)\}$. The group $\check{K}_*(X)$ fits into the exact sequence

$$0 \to \frac{\Omega_{*+1}(X)}{\Omega_{*+1}^0(X)} \xrightarrow{a} \check{K}_*(X) \xrightarrow{\imath} K_*^{geo}(X) \to 0$$

where $\imath$ is the forgetful map ($\imath[M, E^{\nabla^E}, f, \phi] = [M, E^{\nabla^E}, f]$), and a is induced by the map which associates to each $\phi \in \Omega_{*+1}(X)$ the class $[\emptyset,\emptyset,\emptyset,-\phi] \in \check{K}_*(X)$.

Example 2.4.

- The above exact sequence, together with the fact that the only classes in $K_*^{geo}(pt)$ are $[pt, \mathbb{C}^k, id_{pt}]$ with $k \in \mathbb{N}$ implies that

$$\check{K}_{ev}(pt) \cong \mathbb{Z} \text{ and } \check{K}_{odd}(pt) \cong \mathbb{R}/\mathbb{Z}.$$

- Since $K_{ev}^{geo}(S^1) \cong \mathbb{Z} \cong K_{odd}^{geo}(S^1)$, we have two short exact sequences

$$0 \to \mathbb{R}/\mathbb{Z} \to \check{K}_{ev}(S^1) \to \mathbb{Z} \to 0$$

$$0 \to Hom_c(C^\infty(S^1), \mathbb{R})/\mathbb{Z} \to \check{K}_{odd}(S^1) \to \mathbb{Z} \to 0.$$

  It follows from the second exact sequence that the homomorphism which associates to each closed curve (in $C^\infty(S^1)$) the holonomy around it determines an element in $\check{K}_{odd}(S^1)$.

Definition 2.5. The curvature of a differential K-cycle $(M,(E,\nabla^E),f,\phi)$ is the real-valued current $R(M,(E,\nabla^E),f,\phi)$ given by

$$R(M,(E,\nabla^E),f,\phi) := \int_M Td(M)ch(\nabla^E)f^* - \partial\phi.$$

The assignment

$$(M,(E,\nabla^E),f,\phi) \mapsto R(M,(E,\nabla^E),f,\phi)$$

induces a homomorphism $\check{K}_*(X) \xrightarrow{R} \Omega_*(X)$.

Recall that the Freed-Lott differential K-group $\hat{K}(X)$ ([9]) is the abelian group coming from the following generators and relations: a generators is a pair $((F,\nabla^F),w)$, where F is a Hermitian vector bundle over X with a unitary connection ∇^F and $w \in \frac{\Omega^{odd}(X)}{img(d)}$ is a class of odd differential form. The relation is $((F_2,\nabla^{F_2}),w_2) = ((F_1 \oplus F_3, \nabla^{F_1} \oplus \nabla^{F_3}), w_1 + w_3)$ whenever there is a split short exact sequence of
Hermitian vector bundles over X,

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "i"] & F_2 \arrow[r] & F_3 \arrow[l, bend right, dotted, "s"'] \arrow[r] & 0 \end{tikzcd},$$

with $w_2 = w_1 + w_3 + CS((i \oplus s)^* \nabla^{F_2}, \nabla^{F_1} \oplus \nabla^{F_3})$ where $CS(\nabla, \nabla') \in \frac{\Omega^{odd}(X)}{img(d)}$ is the relative Chern-Simons form of two connections on a smooth complex vector bundle. It is related to the K-theory group $K(X)$ by the following short exact sequence

$$0 \to \frac{\Omega^{odd}(X)}{\Omega_0^{odd}(X)} \xrightarrow{b} \hat{K}(X) \xrightarrow{\jmath} K(X) \to 0$$

where $\Omega_0^{odd}(X)$ is the space of odd forms on X with integer K-periods, $\jmath$ is the forgetful map ($\jmath([(F, \nabla^F), w] - [(F', \nabla^{F'}), w']) = [F] - [F']$), and b is the map induced by $w \in \Omega^{odd}(X) \mapsto [(1_n, \nabla^{can}), 0] - [(1_n, \nabla^{can}), w]$. The curvature homomorphism $r : \hat{K}(X) \to \Omega^{ev}(X)$ is given by $[(F, \nabla^F), w] \mapsto ch(\nabla^F) - dw$. The kernel of r is isomorphic to the K-theory with $\mathbb{R}/\mathbb{Z}$-coefficients $K^{-1}(X, \mathbb{R}/\mathbb{Z})$, given through differential K-characters (see [7]).

Theorem 2.6. *There is a unique pairing $\mu : \hat{K}(X) \otimes \check{K}_{odd}(X) \to \mathbb{R}/\mathbb{Z}$ up to torsion in $K^{-1}(X, \mathbb{R}/\mathbb{Z})$, which satisfies*

(i)

$$\begin{tikzcd}
\Omega^{odd}(X) \otimes \check{K}_{odd}(X) \arrow[r, "b\otimes id"] \arrow[d, "id\otimes R"'] \arrow[dr, phantom, "\circlearrowleft"] & \hat{K}(X) \otimes \check{K}_{odd}(X) \arrow[d, "\mu"] \\
\Omega^{odd}(X) \otimes \Omega_{odd}(X) \arrow[r, "\alpha"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
\hat{K}(X) \otimes \Omega_{ev}(X) \arrow[r, "id\otimes a"] \arrow[d, "r\otimes id"'] \arrow[dr, phantom, "\circlearrowleft"] & \hat{K}(X) \otimes \check{K}_{odd}(X) \arrow[d, "\mu"] \\
\Omega^{ev}(X) \otimes \Omega_{ev}(X) \arrow[r, "\alpha"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

where $\alpha(w, \phi) = \phi(w) \mod \mathbb{Z}$ for all $(w, \phi) \in \Omega^*(X) \times \Omega_*(X)$.

Proof. For every $[(F, \nabla^F), w] \in \hat{K}(X)$ and $[M, (E, \nabla^E), f, \phi] \in \check{K}_{odd}(X)$, we set

$$\begin{aligned}\mu(((F, \nabla^F), w))((M, (E, \nabla^E), f, \phi)) := \bar{\eta}_{E \otimes f^*F} - \int_M Td(M) ch(\nabla^E) f^*(w) \\ - \phi(r[(F, \nabla^F), w]) \mod \mathbb{Z},\end{aligned}$$

where $\bar{\eta}_{E \otimes f^*F}$ is the eta (spectral) invariant of the Dirac operator $\mathcal{D}^{E \otimes f^*F}$ on M twisted by $E \otimes f^*F$ ([7]). It is apparent that μ is bi-additive. We show that μ is compatible with the equivalence relation on differential K-cycles. Compatibility with direct sum relation is straightforward. Let $((F, \nabla^F), w)$ be a differential K-cocycle over X, and let $(W, (\varepsilon, \nabla^\varepsilon), g)$ be a K-chain over X. The Atiyah-Patodi-Singer index theorem [2, 3, 4] implies that

$$\bar{\eta}_{(\varepsilon \otimes g^*F)|_{\partial W}} - \int_W Td(W) ch(\nabla^{\varepsilon \otimes g^*F}) = -Ind(\mathcal{D}_+^{\varepsilon \otimes g^*F}) \in \mathbb{Z},$$

and then

$$\mu((F,\nabla^F),w)(\partial W,(\varepsilon|_{\partial W},\nabla^\varepsilon|_{\partial W}),g|_{\partial W},[\int_W Td(W)ch(\nabla^\varepsilon)g^*])=0.$$

Now let $V\to M$ be an even $Spin^c$-vector bundle of dimension $2p$. We consider the smooth closed manifold $\hat{M}$ defined above, which is an $\mathbb{S}^{2p}$-fibration over M,

$$\pi:\hat{M}\to M.$$

If $S_{\mathbb{S}^{2p}}=S^+_{\mathbb{S}^{2p}}\oplus S^-_{\mathbb{S}^{2p}}$ and $S_M=S^+_M\oplus S^-_M$ are the spinor bundles associated with the $Spin^c$-structures on the tangent vector bundles $T\mathbb{S}^{2p}$ and TM respectively, then the spinor bundle $S_{\hat{M}}$ associated with the tangent vector bundle $T\hat{M}$ is isomorphic to the graded tensor product vector bundle $\tilde{S}_{\mathbb{S}^{2p}}\hat{\otimes}\tilde{S}_M$, where $\tilde{S}_{\mathbb{S}^{2p}}$ and $\tilde{S}_M$ are
corresponding lifts to $\hat{M}$. Let b be the Bott bundle over $\mathbb{S}^{2p}$ (see [1] for the construction of this element). We denote by $\mathcal{D}^b$ the self-adjoint Dirac operator on $\mathbb{S}^{2p}$ twisted by b. The index of $\mathcal{D}^b_+$ is equal to 1. According to [4], we get out of $\mathcal{D}^b$ a differential operator $\hat{\mathcal{D}}^b$ on $\hat{M}$ acting on smooth sections of the vector bundle $S_{\hat{M}}\otimes\hat{V}\otimes\pi^*E$. In the same way and following the same reference ([4]), we get out of the Dirac operator on M twisted by E, $\mathcal{D}^E$, a differential operator $\hat{\mathcal{D}}^E$ over $\hat{M}$ acting on smooth sections of $S_{\hat{M}}\otimes\hat{V}\otimes\pi^*E$.
The sharp product of $\hat{\mathcal{D}}^b$ and $\hat{\mathcal{D}}^E$ yields an elliptic differential operator $\hat{\mathcal{D}}^b\sharp\hat{\mathcal{D}}^E$ acting on sections of $S_{\hat{M}}\otimes\hat{V}\otimes\pi^*E$. This operator can be identified with the Dirac operator on $\hat{M}$ twisted by $\hat{V}\otimes\pi^*E$:

$$\mathcal{D}^{\hat{V}\otimes\pi^*E}=\hat{\mathcal{D}}^b\sharp\hat{\mathcal{D}}^E.$$

We can work locally and assume that the fibration $\pi:\hat{M}\to M$ is trivial: π is the projection $\mathbb{S}^{2p}\times M\to M$. The Hilbert space on which $\mathcal{D}^{\hat{V}\otimes\pi^*E}$ acts is the graded tensor product

$$L^2(\mathbb{S}^{2p}\times M,S_{\hat{M}}\otimes\hat{V}\otimes\pi^*E)=L^2(\mathbb{S}^{2p},S_{\mathbb{S}^{2p}}\otimes b)\ \hat{\otimes}\ L^2(M,S_M\otimes E).$$

We have

$$\begin{aligned}\int_{\hat{M}}Td(\hat{M})ch(\nabla^{\hat{V}})(f\circ\pi)^* &= \int_M\left(\int_{\hat{M}}Td(\hat{M})ch(\nabla^{\hat{V}})\right)ch(\nabla^E)f^*\\ &= \int_M\left(\int_{S^2}Td(S^2)ch(b)\right)ch(\nabla^E)f^*\\ &= index(\mathcal{D}^b_+)\times\int_M Td(M)ch(\nabla^E)f^*\\ &= \int_M Td(M)ch(\nabla^E)f^*.\end{aligned}$$

On the other hand, if we split the first factor, $L^2(\mathbb{S}^{2p},S_{\mathbb{S}^{2p}}\otimes b)$, as $\ker(\mathcal{D}^b_+)$ plus its orthogonal complement, then we obtain a corresponding direct sum decomposition of $L^2(\mathbb{S}^{2p}\times M,S_{\hat{M}}\otimes\hat{V}\otimes\pi^*E)$. We therefore obtain a decomposition of $\mathcal{D}^{\hat{V}\otimes\pi^*E}$ as a direct sum of two operators. Since the kernel of $\mathcal{D}^b_+$ is one-dimensional, the first operator acts on $\ker(\mathcal{D}^b_+)\hat{\otimes}L^2(M,S_M\otimes E)\cong L^2(M,S_M\otimes E)$ and is equal to $\mathcal{D}^E$. The second operator has a antisymmetric spectrum. To see this, if T is the partial isometry part of $\mathcal{D}^b_+$ in the polar decomposition, and if γ is the grading operator on $L^2(M,S_M\otimes E)$, then the odd-graded involution $iT\hat{\otimes}\gamma$ on the Hilbert space $\ker(\mathcal{D}^b_+)^{\perp}\hat{\otimes}L^2(M,S_M\otimes E)$ anticommutes with the restriction of $\mathcal{D}^{\hat{V}\otimes\pi^*E}$ to $\ker(\mathcal{D}^b_+)^{\perp}\hat{\otimes}L^2(M,S_M\otimes E)$. Furthermore, the kernel of $\mathcal{D}_+^{\hat{V}\otimes\pi^*E}$ coincides with the kernel of $\mathcal{D}^E_+$. Since the same relation holds for the adjoint, we deduce that

$$\bar{\eta}_{E\otimes f^*F}=\bar{\eta}_{\hat{V}\otimes\pi^*(E\otimes f^*F)}\,.$$

Then μ is defined up to the equivalence relation on differential K-cycles.

We show that μ is compatible with the equivalence relation used to define the Freed-Lott differential K-theory. Let $(M,(E,\nabla^E),f,\phi)$ be a differential K-cycle over X, and let $((F,\nabla^F),w)$ and $((F',\nabla^{F'}),w')$ be two K-cocycles over X which define the same class in $\hat{K}(X)$. Since the map $\mu(\cdot)(M,(E,\nabla^E),f,\phi)$ is additive, we can assume that there exists an isomorphism of Hermitian vector bundles $h:F\to F'$ such that $CS(\nabla^F,h^*\nabla^{F'})=w-w'$. It follows by Fubini and APS-index theorem ([2, 3, 4]) that

$$\begin{aligned}&\mu((F,\nabla^F),w)(M,(E,\nabla^E),f,\phi)-\mu((F',\nabla^{F'}),w')(M,(E,\nabla^E),f,\phi)\\&=\bar{\eta}_{E\otimes f^*F}-\bar{\eta}_{E\otimes f^*F'}-\int_M Td(M)ch(\nabla^E)\wedge CS(f^*\nabla^F,(h\circ f)^*\nabla^{F'}) \bmod \mathbb{Z}\\&=\bar{\eta}_{p^*(E\otimes f^*F)|_{\partial M\times[0,1]}}-\int_{M\times[0,1]}Td(M\times[0,1])ch(B) \bmod \mathbb{Z}=\bar{0},\end{aligned}$$

where B is the connection on the pullback of $E\otimes f^*F$ by the projection $p:M\times[0,1]\to M$ given by $B=t\left(\nabla^E\otimes f^*\nabla^F\right)+(1-t)\left(\nabla^E\otimes(h\circ f)^*\nabla^{F'}\right)+dt\frac{d}{dt}$.

It is clear that μ is natural map and satisfies (i) and (ii).

Now assume that we have two natural pairings μ^k, $k=0,1$, which satisfy (i) and (ii). We consider the bilinear map $B:\hat{K}(X)\otimes\check{K}_{odd}(X)\to\mathbb{R}/\mathbb{Z}$ given by

$$(x,\xi)\mapsto B(x,\xi):=\mu^0(x,\xi)-\mu^1(x,\xi).$$

For $w\in\Omega^{odd}(X)$, we have by $i)$,

$$B(b(w),\xi)=\alpha(w,\xi)-\alpha(w,\xi)=0.$$

Similarly, $B(x,a(\phi))=0$ for all $\phi\in\Omega_{ev}(X)$. Therefore, B factors over a pairing

$$\bar{B}:K(X)\to Hom(K^{geo}_{odd}(X),\mathbb{R}/\mathbb{Z})\overset{\theta}{\cong}K^{-1}(X,\mathbb{R}/\mathbb{Z}),$$

where $\theta^{-1} : K^{-1}(X, \mathbb{R}/\mathbb{Z}) \cong Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z})$ is the isomorphism constructed in [7]. Using the rational isomorphism $Ch_{\mathbb{R}/\mathbb{Q}}$ constructed in [13, p.9], $\bar{B}$ induces a natural transformation from $K(X)$ to the cohomology group $H^{odd}(X, \mathbb{R}/\mathbb{Q})$, denoted by

$$\widetilde{B} : K(X) \to H^{odd}(X, \mathbb{R}/\mathbb{Q}).$$

Let $Gr := \varinjlim G_n(\mathbb{C}^\infty)$ where $G_n(\mathbb{C}^\infty)$ are the complex Grassmannians of n-dimensional vector subspaces. Since $K(X) \cong [X, \mathbb{Z} \times Gr]$, then from Yoneda's lemma $\widetilde{B}$ is necessarily induced by a class $\mathcal{N} \in H^{odd}(\mathbb{Z} \times Gr, \mathbb{R}/\mathbb{Q}) = 0$, and hence B vanishes up to torsion in $K^{-1}(X, \mathbb{R}/\mathbb{Z})$. ∎

Remark 2.7. If μ' is a natural pairing such that (i) and (ii) from Theorem 2.6 hold and μ is the pairing defined in the same theorem, upon eta invariant, then the Atiyah-Singer index theorem and the surjectivity of the usual Atiyah-Singer homomorphism $K(S^1 \times X) \to Hom(K^{geo}_{odd}(X), \mathbb{Z})$, imply that for each $[(F, \nabla^F), w] \in \hat{K}(X)$, the homomorphism $(\mu' - \mu)([(F, \nabla^F), w])(\cdot)$ is identified with an odd form on X with periods in the image of an injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$: for certain $q \in \mathbb{N}^*$ and $v \in \Omega^{odd}_0(X)$,

$$(\mu' - \mu)([(F, \nabla^F), w], [M, (E, \nabla^E), f, \phi]) = \frac{1}{q} \int_M Td(M) ch(\nabla^E) f^*(v) \mod \mathbb{Z}$$

for all $[M, (E, \nabla^E), f, \phi] \in \check{K}_{odd}(X)$.

A natural pairing $m : \hat{K}(X) \otimes \check{K}_*(X) \to \check{K}_*(X)$ can be defined as follows: for every K-cocycle $((F, \nabla^F), w)$ over X and differential K-cycle $(M, (E, \nabla^E), f, \phi)$ over X,

$$m([(F, \nabla^F), w], [M, (E, \nabla^E), f, \phi]) := [M, (E \otimes f^*F, \nabla^{E \otimes f^*F}), f, \int_M Td(M) ch(\nabla^E) \wedge$$
$$\wedge f^*(w \wedge \cdot) + \phi(r[(F, \nabla^F), w] \wedge \cdot) + \partial(\phi(w \wedge \cdot))].$$

Let us consider the collapse map $\epsilon : X \to pt$. It is obvious that the pairing $\epsilon_* \circ m_{odd} : \hat{K}(X) \otimes \check{K}_{odd}(X) \to \check{K}_{odd}(pt) \cong \mathbb{R}/\mathbb{Z}$ satisfies (i) and (ii) from Theorem 2.6. Following the same Theorem, for all $[M, (E, \nabla^E), f, 0] \in \check{K}_*(X)$ we have

$$\bar{\eta}_E \mod \mathbb{Z} = \mu([(1, \nabla^{can}), 0], [M, (E, \nabla^E), f, 0]) =$$
$$\epsilon_* \circ m_{odd}([(1, \nabla^{can}), 0], [M, (E, \nabla^E), f, 0]) + \frac{1}{q} Ch_{odd}([M, (E, \nabla^E), f])(v) \mod \mathbb{Z}$$

for certain $q \in \mathbb{N}^*$ and $v \in \Omega^{odd}_0(X)$. By the Hopkins theorem [12, Theorem 8.1]), the form v is with periods in $q\mathbb{Z}$, and then the following diagram commutes:

$$\begin{tikzcd} \check{K}_{odd}(X) \arrow[r, "\eta'"] \arrow[d, "\epsilon_*"'] & \mathbb{R}/\mathbb{Z} \cong \frac{\Omega_{ev}(pt)}{\Omega^0_{ev}(pt)} \arrow[dl, "\cong"', "a"] \\ \check{K}_{odd}(pt) & \end{tikzcd}$$

where $\eta'[M, (E, \nabla^E), f, \phi] = \bar{\eta}_E - \phi(1) \mod \mathbb{Z}$.

Definition 2.8. The flat K-homology group $\check{K}_*^f(X)$ is defined as the kernel of $R : \check{K}_*(X) \to \Omega_*(X)$.

The construction of flat K-homology is functorial. Let $\rho_0, \rho_1 : X \mapsto Y$ be two smooth homotopic maps between two smooth compact manifolds. If $\rho : X \times [0,1] \mapsto Y$ is a smooth homotopy between ρ_0 and ρ_1, then for all differential K-cycle $(M,(E,\nabla^E),f,\phi)$ over X with trivial curvature we can easily check that $\check{\rho_0}(M,(E,\ \nabla^E),f,\phi)$ and $\check{\rho_1}(M,(E,\nabla^E),f,\phi)$ are equivalent under $(M \times [0,1]$, $({p_M}^*E, {p_M}^*\nabla^E)$,$\rho \circ (f \times Id_{[0,1]}))$ where $p_M : M \times [0,1] \to M$ is the natural projection, and then $X \mapsto \check{K}_*^f(X)$ is a homotopy invariant.

Note that we have the exact sequences

$$0 \to \check{K}_*^f(X) \hookrightarrow \check{K}_*(X) \xrightarrow{R} \Omega_*^0(X) \to 0$$

$$K_{*+1}^{geo}(X) \xrightarrow{Ch_{*+1}} H_{*+1}^{DR}(X) \to \check{K}_*^f(X) \to \mathcal{T}(K_*^{geo}(X)) \to 0,$$

where $\Omega_*^0(X)$ denote the group of closed continuous currents whose de Rham homology class lie in the image of the Chern character $Ch_* : K_*^{geo}(X) \to H_*^{dR}(X)$, $\mathcal{T}(K_*^{geo}(X))$ is the torsion subgroup of $K_*^{geo}(X)$, which can be identified with the torsion subgroup of K-theory $K^{*-1}(X)$ ([7]).

Example 2.9.
- The group $\check{K}_{ev}^f(pt)$ is trivial and $\check{K}_{odd}(pt) \cong \mathbb{R}/\mathbb{Z}$.
- Since $K(S^1) \cong \mathbb{Z} \cong K^1(S^1)$, we have

$$\check{K}_{ev}^f(S^1) \cong \check{K}_{odd}^f(S^1) \cong \mathbb{R}/\mathbb{Z}.$$

We will define a homomorphism $\check{Ch}_* : \check{K}_*^f(X) \to H_{*+1}(X,\mathbb{R}/\mathbb{Q})$ where $H_{*+1}(X,\mathbb{R}/\mathbb{Q})$ is a certain homology group of X with $\mathbb{R}/\mathbb{Q}$-coefficients.
We define $\check{Ch}_*$. First, we construct $H_*(X,\mathbb{R}/\mathbb{Q})$. Denote by $\bar{\Omega}_*(X)$ the cartesian product $\Omega_*(X,\mathbb{R}) \times \Omega_{*-1}(Y,\mathbb{Q})$. The boundary map $\bar{\partial}_* : \bar{\Omega}_*(X) \to \bar{\Omega}_{*-1}(X)$ is defined by

$$\bar{\partial}_*(\phi,\psi) = (\partial\phi - j \circ \psi, -\partial\psi),$$

where $j : \mathbb{Q} \hookrightarrow \mathbb{R}$ is the inclusion. We set

$$H_*(X,\mathbb{R}/\mathbb{Q}) := \frac{Ker(\bar{\partial}_*)}{img(\bar{\partial}_{*+1})}.$$

It fits into the following long exact sequence

$$\cdots \longrightarrow H_{*+1}^{DR}(X,\mathbb{R}) \longrightarrow H_{*+1}(X,\mathbb{R}/\mathbb{Q}) \longrightarrow H_*^{DR}(X,\mathbb{Q}) \longrightarrow \cdots$$

where the homomorphisms $H_*^{DR}(X,\mathbb{R}) \to H_*(X,\mathbb{R}/\mathbb{Q})$ and $H_*(X,\mathbb{R}/\mathbb{Q}) \to H_{*-1}^{DR}(X,\mathbb{Q})$ are induced respectively by

$$\phi \mapsto (\phi, 0) \textit{ and } (\phi,\psi) \mapsto \psi.$$

Now let $(M,(E,\nabla^E),f,\phi)$ be a differential K-cycle over X with trivial curvature. Then the class of $(M,(E,\nabla^E),f)$ in $K_*^{geo}(X)$ has vanishing Chern character. Thus there is a positive integer k such that $(M,(kE,k\nabla^E),f)$ is the boundary of a K-chain $(W,(\varepsilon,\nabla^\varepsilon),g)$. It follows from the definitions that $\frac{1}{k}[\int_W Td(W)ch(\nabla^\varepsilon)g^*]-\phi\in H_{*+1}^{DR}(X,\mathbb{R})$. Let $\check{Ch}_*(M,E^{\nabla^E},f,\phi)$ be the image of $\frac{1}{k}[\int_W Td(W)ch(\nabla^\varepsilon)g^*]-\phi$ under the homomorphism $H_{*+1}^{DR}(X,\mathbb{R})\longrightarrow H_{*+1}(X,\mathbb{R}/\mathbb{Q})$. We show that $\check{Ch}_*(M,(E,\nabla^E),f,\phi)$ is independent of the choice of $(W,(\varepsilon,\nabla^\varepsilon),g)$. Suppose that k' is another positive integer such that $(M,(k'E,k'\nabla^E),f)$ is the boundary of a K-chain $(W',(\varepsilon',\nabla^{\varepsilon'}),g')$. Then

$$\begin{aligned}
&(kk')\left(\tfrac{1}{k}[\textstyle\int_W Td(W)ch(\nabla^\varepsilon)g^*]-\tfrac{1}{k'}[\int_{W'} Td(W')ch(\nabla^{\varepsilon'})g'^*]\right)\\
&\quad=[\int_{k'W} Td(k'W)ch(k'\nabla^\varepsilon)k'g^*]-[\int_{kW'} Td(kW')ch(\nabla^{k\varepsilon'})kg'^*]\\
&\quad=Ch_*[P,(V,\nabla^V),j],
\end{aligned}$$

where $(P,(V,\nabla^V),j)$ is the K-cycle obtained by gluing together the two K-chains $(W,(\varepsilon,\nabla^\varepsilon),g)$ and $(W',(\varepsilon',\nabla^{\varepsilon'}),g')$ along their common boundary via the composed isomorphism $k'\partial(W,(\varepsilon,\nabla^\varepsilon),g)\xrightarrow{\cong}kk'(M,(E,\nabla^E),f)\xrightarrow{\cong}k\partial(W',(\varepsilon',\nabla^{\varepsilon'}),g')$. Then $\frac{1}{k}[\int_W Td(W)ch(\nabla^\varepsilon)g^*]-\frac{1}{k'}[\int_{W'} Td(W')ch(\nabla^{\varepsilon'})g'^*]$ is the same, up to multiplication by rational numbers, as the image of $Ch_*[P,(V,\nabla^V),j]$ ($\in H_{*+1}^{DR}(X,\mathbb{Q})$), and so vanishes when mapped into $H_{*+1}(X,\mathbb{R}/\mathbb{Q})$ ($(Ch_*[P,(V,\nabla^V),j],0)=\bar{\partial}(0,-Ch_*[P,(V,\nabla^V),j])$). Thus, $\check{Ch}_*(M,(E,\nabla^E),f,\phi)$ does not depend on k and $(W,(\varepsilon,\nabla^\varepsilon),g)$. The assignment

$$(M,(E,\nabla^E),f,\phi)\mapsto\check{Ch}_*(M,(E,\nabla^E),f,\phi)$$

induces a well-defined odd homomorphism

$$\check{Ch}_*:\check{K}_*^f(X)\to H_{*+1}(X,\mathbb{R}/\mathbb{Q}),$$

called the flat Chern character. It fits into the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{*+1}^{DR}(X,\mathbb{R}) \arrow[r, "a"] \arrow[d, "-Id"] \arrow[dr, phantom, "\circlearrowleft"] & \check{K}_*^f(X) \arrow[r, "\imath"] \arrow[d, "\check{Ch}_*"] \arrow[dr, phantom, "\circlearrowleft"] & K_*^{geo}(X) \arrow[r] \arrow[d, "Ch_*"] & \cdots \\
\cdots \arrow[r] & H_{*+1}^{DR}(X,\mathbb{R}) \arrow[r] & H_{*+1}(X,\mathbb{R}/\mathbb{Q}) \arrow[r] & H_*^{DR}(X,\mathbb{Q}) \arrow[r] & \cdots
\end{tikzcd}$$

Upon tensoring everything with $\mathbb{Q}$, it follows from the five-lemma that $\check{Ch}_*$ is a rational isomorphism.

3 An isomorphism between flat K-homology and Deeley $\mathbb{R}/\mathbb{Z}$-K-homology

We recall the construction of the Deeley $\mathbb{R}/\mathbb{Z}$-K-homology (see [8]) with some additional remarks.

In all the following, we denote by N a II_1-factor and τ a faithful normal trace on N.

Definition 3.1. An $\mathbb{R}/\mathbb{Z}$-K-cycle over X is a triple $(W, ((H, \varepsilon, \alpha), (\nabla^H, \nabla^\varepsilon)), g)$, where

- W is a smooth compact $Spin^c$-manifold;
- H is a fiber bundle over W with fibers are finitely generated projective Hermitian Hilbert N-modules with a unitary connection ∇^H;
- ε is a Hermitian vector bundle over ∂W with a unitary connection ∇^ε;
- α is an isomorphism from $H|_{\partial W}$ to $\varepsilon \otimes_{\mathbb{C}} \mathsf{N}$;
- $g : W \to X$ is a smooth map.

An $\mathbb{R}/\mathbb{Z}$-K-cycle $(W, ((H, \varepsilon, \alpha), (\nabla^H, \nabla^\varepsilon)), g)$ is called even (resp. odd), if all connected components of W are of even (resp. odd) dimension.

The addition operation on the set of $\mathbb{R}/\mathbb{Z}$-K-cycles is defined using disjoint union operations. The semigroup of $\mathbb{R}/\mathbb{Z}$-K-cycles over X will be denoted by $\Gamma_*(X)$.

A bordism of $\mathbb{R}/\mathbb{Z}$-K-cycles over X consists of the following data :

- Z is a smooth compact $Spin^c$-manifold;
- $W \subseteq \partial Z$ is a regular domain;
- V is a fiber bundle over Z with fibers are finitely generated projective Hermitian Hilbert N-modules with a unitary connection ∇^V, and ϑ is a Hermitian vector bundle over $\partial Z - int(W)$ with a unitary connection ∇^ϑ, such that $V|_{\partial Z - int(W)} \overset{\beta}{\cong} \vartheta \otimes_{\mathbb{C}} \mathsf{N}$;
- $h : Z \to X$ is a smooth map.

Here, a regular domain W of ∂Z means a closed submanifold of ∂Z such that $int(W) \neq \emptyset$ and if $x \in \partial W$, then there exists a coordinate chart $\psi : U \to \mathbb{R}^n$ centred at x with $\psi(W \cap U) = \{(y_i) \in \psi(U) \mid y_n \geq 0\}$.
The boundary of a bordism $(Z, W, ((V, \vartheta, \beta), (\nabla^V, \nabla^\vartheta)), h)$ is the $\mathbb{R}/\mathbb{Z}$-K-cycle

$$\partial(Z, W, ((V, \vartheta, \beta), (\nabla^V, \nabla^\vartheta)), h) := (W, ((V|_W, \vartheta|_{\partial W}, \beta), (\nabla^V|_W, \nabla^\vartheta|_{\partial W})), h|_W).$$

Remark 3.2. If $(Z, W, ((V, \vartheta, \beta), (\nabla^V, \nabla^\vartheta)), h)$ is a bordism, then

$$\partial(\partial Z - int(W), (\vartheta, \nabla^\vartheta), h|_{\partial Z - int(W)}) = (\partial W, (\vartheta|_{\partial W}, \nabla^\vartheta|_{\partial W}), h|_{\partial W}).$$

The modification of an $\mathbb{R}/\mathbb{Z}$-K-cycle y by a $Spin^c$-vector bundle V of even rank with an Euclidean connection ∇^V, is denoted by y^V, and is defined in the same way as that on differential K-cycles.

Definition 3.3. Two $\mathbb{R}/\mathbb{Z}$-K-cycles $(W_0,((H_0,\varepsilon_0,\alpha_0),(\nabla^{H_0},\nabla^{\varepsilon_0})),g_0)$ and $(W_1,((H_1,\varepsilon_1,\alpha_1),(\nabla^{H_1},\nabla^{\varepsilon_1})),g_1)$ are equivalent if there exist a $Spin^c$-vector bundle $V \to W_1$ of even rank and a bordism ζ over X such that

$$(W_0,((H_0,\varepsilon_0,\alpha_0),(\nabla^{H_0},\nabla^{\varepsilon_0})),g_0) \sqcup (W_1^-,((H_1,\varepsilon_1,\alpha_1),(\nabla^{H_1},\nabla^{\varepsilon_1})),g_1)^V = \partial\zeta.$$

Remark 3.4. (i) If $(W,((H,\varepsilon,\alpha),(\nabla^H,\nabla^\varepsilon)),g)$ and $(W,((H',\varepsilon',\alpha'),(\nabla^{H'},\nabla^{\varepsilon'}))$ $,g)$ are two $\mathbb{R}/\mathbb{Z}$-cycles over X with the same $Spin^c$-manifold W and map g, then $\Big((W,((H,\varepsilon,\alpha),(\nabla^H,\nabla^\varepsilon)),g) \sqcup (W,((H',\varepsilon',\alpha'),(\nabla^{H'},\nabla^{\varepsilon'})),g)\Big)^{1^2_{W\sqcup W}}$ and $(W,((H\oplus H',\varepsilon\oplus\varepsilon',\alpha\oplus\alpha'),(\nabla^H\oplus\nabla^{H'},\nabla^\varepsilon\oplus\nabla^{\varepsilon'})),g)$ are equivalent ([8, Proposition 4.11]).

(ii) If $(M,(E,\nabla^E),f)$ is a cycle of Baum-Douglas over X, then the $\mathbb{R}/\mathbb{Z}$-K-cycle $(M,((E\otimes \mathsf{N},\emptyset,\emptyset),(\nabla^E,\emptyset)),f)$ is equivalent to the trivial $\mathbb{R}/\mathbb{Z}$-K-cycle, $(\emptyset,(\emptyset,\emptyset),\emptyset)$, where a bordism is given by $(M\times[0,1],M,((p_M^*E\otimes\mathsf{N},E,id_M),(p_M^*\nabla^E,\nabla^E)),f\circ p_M)$ with $p_M : M\times[0,1]\to M$ is the natural projection.

Definition 3.5. The Deeley $\mathbb{R}/\mathbb{Z}$-K-homology group $K_*(X,\mathbb{R}/\mathbb{Z})$ is the quotient of $\Gamma_*(X)$ by the equivalence relation on $\mathbb{R}/\mathbb{Z}$-K-cycles.

The group $K_*(X,\mathbb{R}/\mathbb{Z})$ is Abelian and naturally $\mathbb{Z}_2$-graded.

Remark 3.6. If moreover X is a $Spin$-manifold, then $K_*(X,\mathbb{R}/\mathbb{Z})$ is identified with the Kasparov K-homology group $KK^{*-1}(C(X),\mathcal{C})$ where $\mathcal{C}$ is the mapping cone of the inclusion $\mathbb{C}\hookrightarrow\mathsf{N}$ ([8, Theorem 5.2]).

Example 3.7. Note that the trace τ on N extends to $M_n(\mathsf{N})\cong\mathsf{N}\otimes M_n(\mathbb{C})$, also denoted by τ, with the property that two projections $p,q\in M_n(\mathsf{N})$ are Murray-von Neumann equivalent if and only if $\tau(p)=\tau(q)$. Then it induces an isomorphism from the K-theory group $K_0(\mathsf{N})$ to $\mathbb{R}$. Moreover, $K_1(\mathsf{N})$ is trivial.
Let $K^{an,*}(A)$ denote the analytic K-homology group of a C^*-algebra A (for more details we refer the reader to [11]). Following the universal coefficients theorem for K-homology,

$$0\to Ext(K_*(\mathsf{N}),\mathbb{R})\to K^{an,*+1}(\mathsf{N})\to Hom(K_{*+1}(\mathsf{N}),\mathbb{R})\to 0,$$

together with $\mathbb{R}$ is divisible, we get

$$K^{an,0}(\mathsf{N})=\mathbb{R} \text{ and } K^{an,1}(\mathsf{N})=0.$$

Because the $*$-algebra $\mathcal{C}$ is null-homotopic, we have $K^{an,0}(\mathcal{C})=0$. On the other hand, the six-term exact sequence for K-homology associated to the short exacte sequence $0\to C_0(]0,1[)\otimes_{\mathbb{C}}\mathsf{N}\hookrightarrow\mathcal{C}\overset{ev_1}{\to}\mathbb{C}\to 0$, implies that $K^{an,1}(\mathcal{C})\cong\mathbb{R}/\mathbb{Z}$. From the above remark, we obtain that

$$K_{ev}(pt,\mathbb{R}/\mathbb{Z})\cong\mathbb{R}/\mathbb{Z} \text{ et } K_{odd}(pt,\mathbb{R}/\mathbb{Z})=0.$$

Note that from [8] and [16], cocycles in $KK^*(C(X), \mathsf{N})$ can be described by geometric cycles of the form $(M, (H, \nabla^H), f)$, where M is a smooth closed $Spin^c$-manifold, H is a fiber bundle over M with fibers are finitely generated projective Hermitian Hilbert N-modules, with a unitary connection ∇^H, and $f : M \to X$ is a smooth map. The group $KK^*(C(X), \mathsf{N})$ is nothing more than an analytic model for the real K-homology of X. An isomorphism between $K_*^{geo}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $KK^*(C(X), \mathsf{N})$ is given at level of cycles by

$$\nu((M, (E, \nabla^E), f), t) = [M, (E \otimes p_t \mathsf{N}^n, \nabla^E), f],$$

where $p_t \in M_n(\mathsf{N})$ is a projection with $\tau(p_t) = t$.

The Chern character $Ch_{\tau,*} : KK^*(C(X), \mathsf{N}) \to H_*^{dR}(X, \mathbb{R})$ is giving by

$$Ch_{\tau,*}[M, (H, \nabla^H), f] := [\int_M Td(M) ch_\tau(\nabla^H) f^*(\cdot)],$$

where $ch_\tau(\nabla^H) := \tau_* \left(Tr(e^{\frac{-\nabla^{H^2}}{2i\pi}}) \right) \in \Omega^{2*}(X, \mathbb{R})$ and $\tau_* : \Omega^*(X, \mathsf{N}) \to \Omega^*(X, \mathbb{R})$ is the homomorphism associated by fonctoriality to the trace $\tau : \mathsf{N} \to \mathbb{R}$. It fits into the commutative diagram

$$\begin{tikzcd}
K_*^{geo}(X) \otimes \mathbb{R} \arrow[d, "Ch_*^{\mathbb{R}}"'] \arrow[dr, "\nu"] & \\
H_*^{dR}(X, \mathbb{R}) & KK^*(C(X), \mathsf{N}) \arrow[l, "Ch_{\tau,*}"']
\end{tikzcd}$$

where $Ch_*^{\mathbb{R}} : K_*^{geo}(X) \otimes \mathbb{R} \overset{Ch_* \times \cdot}{\to} H_*^{dR}(X, \mathbb{R})$, and then $Ch_{\tau,*}$ turns out to be an isomorphism.

Using the above commutative diagram, Remark 3.2 and the Atiyah-Singer index theorem on even spheres, we obtain that $\gamma : K_*(X, \mathbb{R}/\mathbb{Z}) \to \check{K}_*^f(X)$ given by

$$\gamma[W, ((H, \varepsilon, \alpha), (\nabla^H, \nabla^\varepsilon)), g] := [\partial W, (\varepsilon, \nabla^\varepsilon), g|_{\partial W}, [\int_W Td(W) ch_\tau(\nabla^H) g^*]]$$

is a well-defined homomorphism.

Theorem 3.8. *The homomorphism γ is an isomorphism.*

Proof. We construct the inverse of γ, denoted by $v : \check{K}^f_*(X) \to K_{*+1}(X, \mathbb{R}/\mathbb{Z})$, as follows. Let $(M, (E, \nabla^E), f, \phi)$ be a differential K-cycle over X with trivial curvature. Since the diagram relating $Ch_{\tau,*}$ with $Ch^{\mathbb{R}}_*$ is commutatif and $Ch_{\tau,*}$ is an isomorphism, there exist a smooth compact $Spin^c$-manifold W, a fiber bundle H over W with fibers are finitely generated projective Hermitian Hilbert N-modules with a unitary connection ∇^H, and a smooth map $g : W \to X$ such that

$$(M, (E \otimes \mathsf{N}, \nabla^E), f) \overset{h}{\cong} (\partial W, (H|_{\partial W}, \nabla^H|_{\partial W}), g|_{\partial W}).$$

This implies that

$$\begin{aligned}\partial(\phi - \int_W Td(W) ch_\tau(\nabla^H) g^*) = \int_M Td(M) ch(\nabla^E) f^* \\ - \int_{\partial W} Td(\partial W) ch_\tau(\nabla^H|_{\partial W}) g|_{\partial W}{}^* = 0.\end{aligned}$$

Let then $[N, (F, \nabla^F), j] \in KK^*(C(X), \mathsf{N})$ with

$$Ch_{\tau,*}([N, (F, \nabla^F), j]) = \phi - [\int_W Td(W) ch_\tau(\nabla^H) g^*].$$

We set

$$v(M, (E, \nabla^E), f, \phi) := [W \sqcup N, ((H \sqcup F, \alpha^* E, \beta), (\nabla^H \sqcup \nabla^F, \alpha^* \nabla^E)), g \sqcup j],$$

where $\alpha : \partial W \to M$ and $\beta : H|_{\partial W} \to \alpha^* E \otimes \mathsf{N}$ are isomorphisms induced by h.
We show that v is well defined on $\check{K}^f_*(X)$. From (i) in Remark 3.4, v is compatible with the relation of direct sum in Definition 2.2, and from definitions, the image of every modification of $(M, (E, \nabla^E), f, \phi)$ under v is equal to the modification of $v(M, (E, \nabla^E), f, \phi)$.
Let $(W, (\varepsilon, \nabla^\varepsilon), g)$ be a K-chain over X. We have

$$\begin{aligned}v(\partial W, (\varepsilon|_{\partial W}, \nabla^\varepsilon|_{\partial W}), g_{|\partial W}, \int_W Td(W) ch(\nabla^\varepsilon) g^*) = [W, ((\varepsilon \otimes \mathsf{N}, \varepsilon|_{\partial W}, (id^*_{\partial W} \otimes 1)), \\ (\nabla^\varepsilon, \nabla^\varepsilon|_{\partial W})), g].\end{aligned}$$

If $p : W \times [0,1] \to W$ is the projection and $i : (W \sqcup W^-) \times]0,1] \sqcup (\partial W \times [0,1]) \sqcup \partial W \hookrightarrow W \times [0,1]$ the inclusion, then $(W \times [0,1], W, ((p^*\varepsilon, (p \circ i)^*\varepsilon), (p^*\nabla^\varepsilon, (p \circ i)^*\nabla^\varepsilon)), g \circ p)$ is a bordism between $(W, ((\varepsilon \otimes \mathsf{N}, \varepsilon|_{\partial W}, (id^*_{\partial W} \otimes id_{\mathsf{N}})), (\nabla^\varepsilon, \nabla^\varepsilon|_{\partial W})), g)$ and the trivial cycle, and then the class $v(\partial W, (\varepsilon|_{\partial W}, \nabla^\varepsilon|_{\partial W}), g_{|\partial W}, [\int_W Td(W) ch(\nabla^\varepsilon) g^*])$ is trivial.

Now we show that $v(M, (E, \nabla^E), f, \phi)$ does not depend on choice of $(W, (H, \nabla^H), g)$. Let $(W', (H', \nabla^{H'}), g')$ be an N-K-chain over X such that

$$\begin{aligned}(M, (E \otimes \mathsf{N}, \nabla^E), f) \overset{h}{\cong} (\partial W, (H|_{\partial W}, \nabla^H|_{\partial W}), g|_{\partial W}) \overset{h'}{\cong} \\ (\partial W', (H'|_{\partial W'}, \nabla^{H'}|_{\partial W'}), g'|_{\partial W'}),\end{aligned}$$

and let $[N', (F', \nabla^{F'}), j'] \in KK^*(C(X), \mathsf{N})$ with

$$Ch_{\tau,*}([N', (F', \nabla^{F'}), j']) = \phi - [\int_{W'} Td(W')ch_\tau(\nabla^{H'})g'^*].$$

We claim that $x := (W \sqcup N, ((H \sqcup F, \alpha^*E, \beta), (\nabla^H \sqcup \nabla^F, \alpha^*\nabla^E)), g \sqcup j)$ and $y := (W' \sqcup N', ((H' \sqcup F', \alpha'^*E, \beta'), (\nabla^{H'} \sqcup \nabla^{F'}, \alpha'^*\nabla^E)), g' \sqcup j')$ are equivalent. We consider the $\mathbb{R}/\mathbb{Z}$-K-cycle

$$(\widetilde{W}, ((\widetilde{H}, \widetilde{E}, \widetilde{\beta}), (\nabla^{\widetilde{H}}, \nabla^{\widetilde{E}})), \widetilde{g}) := x \sqcup y^-,$$

and let $(Z, (\zeta, \nabla^\zeta), h)$ be the N-K-cycle where,

$$Z := \widetilde{W} \bigcup_{\partial W \cong M \times \{0\}; \partial W' \cong M \times \{1\}} M \times [0,1],\ \zeta := \widetilde{H} \bigcup_{\partial W \cong M \times \{0\}; \partial W' \cong M \times \{1\}} p_M^* E \otimes_{\mathbb{C}} \mathsf{N},$$

$$\nabla^\zeta := \nabla^{\widetilde{H}} \cup p_M^* \nabla^E, \text{ and } h := \widetilde{g} \cup (f \circ p_M).$$

Here, $p_M : M \times [0,1] \to M$ denotes the canonical projection.
A bordism between $(Z, ((\zeta, \emptyset, \emptyset), (\nabla^\zeta, \emptyset)), h)$ and $(\widetilde{W}, ((\widetilde{H}, \widetilde{E}, \widetilde{\beta}), (\nabla^{\widetilde{H}}, \nabla^{\widetilde{E}})), \widetilde{g})$ is given by the following quadruple

$$(Z \times [0,1], Z \sqcup \widetilde{W}, ((p_Z^* \zeta, p_M^* E), (p_Z^* \nabla^\zeta, p_M^* \nabla^E)), h \circ p_Z).$$

Furthermore,

$$\begin{aligned} Ch_{\tau,*}([Z, (\zeta, \nabla^\zeta), h]) &= [\int_W Td(W)ch_\tau(\nabla^H)g^*] + Ch_{\tau,*}([N, (F, \nabla^F), j]) \\ &\quad - [\int_{W'} Td(W')ch_\tau(\nabla^{H'})g'^*] - Ch_{\tau,*}([N', (F', \nabla^{F'}), j']) \\ &= \phi - \phi = 0. \end{aligned}$$

Hence, $v(M, (E, \nabla^E), f, \phi)$ depends only on $(M, (E, \nabla^E), f, \phi)$.

We check that $v \circ \gamma = id_{K_*(X, \mathbb{R}/\mathbb{Z})}$ and $\gamma \circ v = id_{\check{K}_*^f(X)}$. The first equality is straightforward, and the second is obtained as follows. For all $[M, (E, \nabla^E), f, \phi] \in \check{K}_*^f(X)$,

$$\begin{aligned} &\gamma(v[M, (E, \nabla^E), f, \phi]) \\ &\quad = \gamma([W \sqcup N, ((H \sqcup F, \alpha^*E, \beta), (\nabla^H \sqcup \nabla^F, \alpha^*\nabla^E)), g \sqcup j]) \\ &\quad = [\partial W, (\alpha^*E, \alpha^*\nabla^E), g|_{\partial W}, [\int_{W \sqcup N} Td(W \sqcup N)ch_\tau(\nabla^H \sqcup \nabla^F)(g \sqcup j)^*]] \\ &\quad = [\partial W, (\alpha^*E, \alpha^*\nabla^E), g|_{\partial W}, [\int_W Td(W)ch_\tau(\nabla^H)g^*] + Ch_{\tau,*}[N, (F, \nabla^F), j]]. \end{aligned}$$

Since $Ch_{\tau,*}([N, (F, \nabla^F), j]) = \phi - [\int_W Td(W)ch_\tau(\nabla^H)g^*]$, we have

$$\gamma(v[M, (E, \nabla^E), f, \phi]) = [M, (E, \nabla^E), f, \phi]. \qquad \blacksquare$$

4 The torsion part of Deeley $\mathbb{R}/\mathbb{Z}$-K-homology

The aim of this section is to describe the torsion subgroup of $K_*(X, \mathbb{R}/\mathbb{Z})$ via $\mathbb{Q}/\mathbb{Z}$-bordism theory.

We start by recalling the notions of $\mathbb{Z}_k$-manifold and $\mathbb{Z}_k$-vector bundle.

Definition 4.1.
- A $\mathbb{Z}_k$-manifold is a triple (M, N, k) where (M, N) is a pair of smooth compact manifolds such that $\partial M = kN$. We often drop the integers from this notation and denote a $\mathbb{Z}_k$-manifold by (M, N).
- A $\mathbb{Z}_k$-vector bundle over (M, N) is a pair of vector bundles, (E, F), over M and N respectively such that $E|_{\partial M}$ decomposes into k copies of F.

Additionally, we have natural definitions of (Hermitian) connections on (Hermitian) $\mathbb{Z}_k$-vector bundles, $Spin^c$-$\mathbb{Z}_k$-manifolds, and framed $\mathbb{Z}_k$-manifolds. We refer the reader to [10] for supplementary details.

Now, if Y is any paracompact Hausdorff space then we shall denote by $\Omega_n^{F,k}(Y)$ the n-th framed $\mathbb{Z}_k$-bordism group of Y. Thus $\Omega_n^{F,k}(Y)$ is the set of all bordism classes of maps from framed n-dimensional $\mathbb{Z}_k$-manifolds into Y. Here, a smooth map f from a $\mathbb{Z}_k$-manifold (M, N) to Y is a pair of smooth maps $f_M : M \to Y$ and $f_N : N \to Y$ where $f_M : M \to Y$ is an extension of f_N.
The set $\Omega_n^{F,k}(Y)$ is an Abelian group under the disjoint union operation.
For $l \in \mathbb{N}^*$, the assignment

$$((f_M, f_N) : (M, N^n) \to Y) \mapsto ((l.f_M, f_N) : (l.M, N^n) \to Y)$$

induces a well-defined homomorphism $L_l : \Omega_n^{F,k}(Y) \to \Omega_n^{F,lk}(Y)$. Denote by $\widetilde{\Omega}_n^F(Y)$ the limit of the direct system $(\Omega_n^{F,k!}(Y), L_{k+1})$.
Let $(S^{3,k}, S^2)$ be the $Spin^c$-$\mathbb{Z}_k$-manifold obtained by removing k open balls $int(D^3)$ from the 3-sphere, equipped with its standard $Spin^c$-structure, $\mathcal{S} \to (M, N)$, as the boundary of the couple of balls (D^4, D^3): $\partial_k(D^4, D^3) := (\partial D^4 - k.int(D^3), \partial D^3) = (S^{3,k}, S^2)$. It is also a framed manifold. Denote by $c : (S^{3,k}, S^2) \to \mathcal{B}U$, where $\mathcal{B}U$ is the classifying space of the unitary group $U(\infty)$, a basepoint-preserving map which, under the isomorphism $[(S^{3,k}, S^2), \mathcal{B}U] \cong \widetilde{K}(S^{3,k}, S^2)$, corresponds to the difference $[\mathcal{S}_+^*] - 1_{(S^{3,k}, S^2)}$. Consider the direct system of Abelian groups

$$\Omega_n^{F,k}(\mathcal{B}U \times X) \to \Omega_{n+2}^{F,k}(\mathcal{B}U \times X) \to \Omega_{n+4}^{F,k}(\mathcal{B}U \times X) \to \cdots$$

given as follows: for $f : (M, N^n) \to \mathcal{B}U \times X$ be a smooth map from a framed n-dimensional $\mathbb{Z}_k$-manifold (M, N) to Y, the composition

$$(M, N) \times (S^{3,k}, S^2) \overset{c \times f}{\to} \mathcal{B}U \times \mathcal{B}U \times X \overset{m \times id_X}{\to} \mathcal{B}U \times X$$

is a cycle for $\Omega_{n+2}^{F,k}(\mathcal{B}U \times X)$ where m is the map defined through tensor product of Hermitian vector spaces. This defines a map from $\widetilde{\Omega}_n^F(\mathcal{B}U \times X)$ to

$\widetilde{\Omega}^F_{n+2}(\mathcal{B}U \times X)$. Denote by $\widetilde{\Omega}^F_*(\mathcal{B}U \times X)$, with $* \in \{ev, odd\}$, the direct limit of the above directed system.
Let $\beta : \Omega^F(\mathcal{B}U \times X) \to \widetilde{\Omega}^F_*(\mathcal{B}U \times X)$ be the homomorphism from the framed bordism group of $\mathcal{B}U \times X$ to $\widetilde{\Omega}^F_*(\mathcal{B}U \times X)$ which associates to each $[f : M \to \mathcal{B}U \times X]$ the class $[(f, \emptyset) : (M, \emptyset) \to \mathcal{B}U \times X]$.

Definition 4.2. Denote by $\widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z})$ the cokernel of β.

Remark 4.3. By the Pontrjagin-Thom isomorphism [14], we can identify $\widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z})$ with the stable homotopy group of the (base-pointed) topological space $\mathcal{B}U \times X$.

Let $f : (M, N^n) \to \mathcal{B}U \times X$ be a cycle in $\Omega^{F,k}_n(\mathcal{B}U \times X)$. It determines a Hermitian $\mathbb{Z}_k$-vector bundle (E, F) over (M, N) and a smooth map $(f'_M, f'_N) : (M, N) \to X$. We choose a unitary connection (∇^E, ∇^F) on (E, F). Recall that the framing $T(M, N^n) \oplus 1^k \cong 1^{n+k}$ of the framed $\mathbb{Z}_k$-manifold (M, N) defines a $Spin^c$-structure on (M, N). We obtain that the quadruple $(N, (F, \nabla^F), f'_N, [\frac{1}{k}\int_M Td(M)ch(\nabla^E){f'_M}^*])$ is a differential K-cycle over X. Moreover,

$$k\left(\int_N Td(N)ch(\nabla^F){f'_N}^* - \frac{1}{k}\partial(\int_M Td(M)ch(\nabla^E){f'_M}^*)\right) =$$
$$\int_{kN} Td(kN)ch(\nabla^{kF})(kf'_N)^* - \int_{\partial M} Td(\partial M)ch(\nabla^{E|_{\partial M}}) \wedge \wedge(f'_M|_{\partial M})^* = 0.$$

Then the class $[N, (F, \nabla^F), f_N, \frac{1}{k}([\int_M Td(M)ch(\nabla^E){f'_M}^*])]$ lies in $\check{K}^f_{n[2]}(X)$, and from Remark 2.3, it is independent of the choice of geometry.

Proposition 4.4. *The correspondence*

$$[f : (M, N) \to \mathcal{B}U \times X] \mapsto [N, (F, \nabla^F), f_N, \frac{1}{k}[\int_M Td(M)ch(\nabla^E){f'_M}^*]]$$

determines an injective homomorphism

$$\tau : \widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z}) \to \check{K}^f_*(X).$$

Proof. It is clear that τ is an additive map and well-defined on $\widetilde{\Omega}^F_n(\mathcal{B}U \times X)$. Since every cycle in differential K-homology is identified with its modifications, τ is also well-defined on $\widetilde{\Omega}^F_*(\mathcal{B}U \times X)$ ($* \in \{ev, odd\}$). Moreover, τ sends $img(\beta)$ to the trivial subgroup of $\check{K}^f_*(X)$.
τ is injective. In fact let $f : (M, N^n) \to \mathcal{B}U \times X$ be a cycle in $\Omega^{F,k}_n(\mathcal{B}U \times X)$ where $[N, (F, \nabla^F), f_N, \frac{1}{k}[\int_M Td(M)ch(\nabla^E){f'_M}^*]] = 0$. Without loss of generality, we reduce to the case when there is a smooth compact $Spin^c$-manifold W, a smooth Hermitian vector bundle ε over W with a unitary connection ∇^ε, and a smooth map $g : W \to X$ such that

$$(N, (F, \nabla^F), f_N, \frac{1}{k}[\int_M Td(M)ch(\nabla^E){f'_M}^*]) = (\partial W, (\varepsilon|_{\partial W}, \nabla^\varepsilon|_{\partial W}), g|_{\partial W},$$
$$[\int_W Td(\nabla^W)ch(\nabla^\varepsilon)g^*]).$$

As $Spin^c$-bordism relation is equivalent to framed bordism relation ([6, p.21]), we will consider W as a framed manifold. Let $(P,(V,\nabla^V),h)$ be the K-cycle obtained by gluing together the two K-chains of Baum-Douglas $(M,(E,\nabla^E),f'_M)$ and $(kW,(k\varepsilon,\nabla^{k\varepsilon}),kg)$ along their common boundary. Let $(\widetilde{P},(\widetilde{V},\nabla^{\widetilde{V}}),\widetilde{h})$ be a bordism between two copies of $(P,(V,\nabla^V),h)$. We have

$$\begin{aligned}\partial\widetilde{P} &= P \sqcup P^- \\ &= P \sqcup M^- \cup_{\partial M^- \cong \partial(k.W^-)} k.W^-.\end{aligned}$$

Denote by $h_{\widetilde{V}} : (\widetilde{P},W^-) \to \mathcal{B}U$ a map which determines the class in the $\mathbb{Z}_k$-K-theory of $(\widetilde{P},W^-)$ represented by $\widetilde{V}$.
Since $(\widetilde{P},W^-)$ is a framed $\mathbb{Z}_k$-manifold with $\partial_k(\widetilde{P},W^-) = (P \sqcup M^-, N^-)$, and the fiber bundle $\widetilde{V}$ and map $\widetilde{h}$ respect this $\mathbb{Z}_k$-structure so that $(h_{\widetilde{V}} : (\widetilde{P},W^-) \to \mathcal{B}U, \widetilde{h} : (\widetilde{P},W^-) \to X)$ is a bordism between $(h_V : (P,\emptyset) \to \mathcal{B}U, (h,\emptyset) : (P,\emptyset) \to X)$ and $f : (M,N) \to \mathcal{B}U \times X$, which implies that $[f : (M,N) \to \mathcal{B}U \times X] \in Img(\beta)$, and this finishes the proof. ■

Theorem 3.8 leads to an injective homomorphism

$$\bar{\tau} : \widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z}) \to K_*(X, \mathbb{R}/\mathbb{Z}).$$

Furthermore, from the construction of the flat Chern character $\check{C}h_* : \check{K}^f_*(X) \to H_{*+1}(X, \mathbb{R}/\mathbb{Q})$ we have the exact sequence

$$0 \longrightarrow \widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\bar{\tau}} K_*(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\check{C}h_* \circ \gamma} H_{*+1}(X, \mathbb{R}/\mathbb{Q}).$$

Corollary 4.5. *The torsion part of $K_*(X,\mathbb{R}/\mathbb{Z})$ is isomorphic to $\widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z})$.*

Remark 4.6. We can use the approach of Atiyah-Patodi-Singer [2, 3] to $\mathbb{R}/\mathbb{Z}$-K-theory, to obtain a third model for $\mathbb{R}/\mathbb{Z}$-K-homology by regarding $\widetilde{\Omega}^F_*(\mathcal{B}U \times X, \mathbb{Q}/\mathbb{Z})$ as a K-homology of X with $\mathbb{Q}/\mathbb{Z}$-coefficients and $H_*(X,\mathbb{R}/\mathbb{Q})$ as the cokernel of the natural injection $K^{geo}_*(X,\mathbb{Q}) \to K^{geo}_*(X) \otimes \mathbb{R}$.

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