On ϕ-ergodic property of Banach modules

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Abstract

Let $\mathcal{A}$ be a Banach algebra and let ϕ be a non-zero character on $\mathcal{A}$. We introduce the notion of ϕ-ergodic property for a Banach right $\mathcal{A}$-module X. This concept considerably generalizes the existence of ϕ-means of norm one on $\mathcal{A}^*$. We also show that the ϕ-ergodic property of X is related to some other properties such as a Hahn-Banach type extension property and the existence of ϕ-means of norm one on a certain subspace of $\mathcal{A}^*$. Finally, we give some characterizations for ϕ-amenability of a Banach algebra in terms of its closed ideals.

1 Introduction

Let $\mathcal{A}$ be Banach algebra and let $\phi : \mathcal{A} \to \mathbb{C}$ be a character, i.e., a non-zero homomorphism of $\mathcal{A}$. Recently, Kaniuth, Lau and Pym [4, 5] introduced and investigated a notion of amenability for Banach algebras called *ϕ-amenability*. Independently, Monfared introduced and studied in [10] the notion of character amenability for Banach algebras. Let $\Delta(\mathcal{A})$ be the set of all non-zero characters, bounded multiplicative linear functionals on Banach algebra $\mathcal{A}$ and let $\phi \in \Delta(\mathcal{A})$. Following [5], $\mathcal{A}$ is called *ϕ-amenable* if there exists a *ϕ-mean* on $\mathcal{A}^*$, that is a functional $m \in \mathcal{A}^{**}$ such that

$$m(\phi) = 1, \quad m(f \cdot a) = \phi(a)m(f) \quad (f \in \mathcal{A}^*, a \in \mathcal{A}).$$

Moreover, $\mathcal{A}$ is called *character amenable* if it has a bounded right approximate identity and it is ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$. There are many characterizations

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for ϕ-amenability of Banach algebras. For example, Nasr-Isfahani and the author in [11] characterized ϕ-amenability in terms of ϕ-ergodic anti representations. Recently, Sahami and Pourabbas in [13] has introduced and studied ϕ-homological concepts of Banach algebras which are closely related to ϕ-amenability.

Note that the notion of ϕ-amenability is a generalization of left amenability for Lau algebras $\mathcal{A}$ studied in [7]. In fact, ϕ-amenability coincides with left amenability in the case where the character ϕ is taken to be the identity of the von Neumann algebra $\mathcal{A}^*$.

Examples of Lau algebras include the predual algebras of a Hopf von Neumann algebra, in particular the class of quantum group algebras $L^1(\mathbb{G})$ and the Fourier-Stieltjes algebra $B(G)$ of a topological group G; see [7, 6]. They also include the measure algebra $M(S)$ of a locally compact semigroup S. Moreover, the hypergroup algebra $L^1(H)$ and the measure algebra $M(H)$ of a locally compact hypergroup H with a left Haar measure are Lau algebras.

In this work we aim to introduce and study a notion of amenability for Banach modules. We continue and generalize our investigation [12] as ϕ-ergodic property with respect to a character ϕ on $\mathcal{A}$. We also generalize major results in [5]. The article is organized as follows: After introducing some notations, we define the concept of ϕ-ergodic property for a Banach right $\mathcal{A}$-module X which is a generalization of ϕ-amenability for Banach algebras with a ϕ-mean of norm one. We show (in Corollary 2.11) that the ϕ-ergodic property of X is closely related to the existence of a ϕ-mean of norm one, on topologically left introverted subspace $\mathcal{L}_X(\mathcal{A})$ of $\mathcal{A}^*$. We shall prove the ϕ-ergodic property of X is also related to some other properties such as a Hahn-Banach type extension property (see Theorem 2.18). Finally we characterize ϕ-amenability of a Banach algebra, with a ϕ-mean of norm one, in terms of its ideals.

2 ϕ-ergodic property

For a normed space X, the dual space of X is denoted by X^* and the action of $\xi \in X^*$ at $x \in X$ is denoted either by $\xi(x)$ or by $\langle \xi, x\rangle$. Let $\mathcal{A}$ be a Banach algebra and let X be a Banach left, right or two-sided $\mathcal{A}$-module. Then X^* is respectively a Banach right, left or two-sided $\mathcal{A}$-module with the corresponding module action(s) defined naturally by

$$\langle \xi \cdot a, x\rangle = \langle \xi, a \cdot x\rangle, \quad \langle a \cdot \xi, x\rangle = \langle \xi, x \cdot a\rangle \quad (\xi \in X^*, x \in X, a \in \mathcal{A}).$$

For each $\phi \in \Delta(\mathcal{A})$ define the semigroup

$$S_\phi = \{a \in \mathcal{A} : \phi(a) = 1\}.$$

Let $N(\mathcal{A}^*, \phi)$ denote the set of all $f \in \mathcal{A}^*$ with the following property: for each $\delta > 0$, there exists a sequence (a_n) in S_ϕ such that $\|a_n\| \leq 1 + \delta$ for all n and $\|f \cdot a_n\| \to 0$.

We shall use the following characterization of ϕ-amenable Banach algebras involving the set $N(\mathcal{A}^*, \phi)$ to extend the notion of ϕ-amenability over Banach modules.

Theorem 2.1. [5, Theorem 2.8] *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Then the following two conditions are equivalent.*

(i) *There exists a ϕ-mean with $\|m\| = 1$.*

(ii) *$N(\mathcal{A}^*, \phi)$ is a subspace of $\mathcal{A}^*$ and $f \cdot a - f \in N(\mathcal{A}^*, \phi)$ for all $f \in \mathcal{A}^*$ and all $a \in S_\phi$.*

Similarly, for a Banach right $\mathcal{A}$-module X, we denote by $N(X, \phi)$ the set of all $x \in X$ with the following property: for each $\delta > 0$, there exists a sequence (a_n) in S_ϕ such that $\|a_n\| \leq 1 + \delta$ for all n and $\|x \cdot a_n\| \to 0$.

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then the following statements hold.*

(i) *$N(X, \phi)$ is closed in X and closed under scaler multiplication.*

(ii) *If $\mathcal{A}$ is commutative, then $N(X, \phi)$ is a closed subspace of X.*

(iii) $N(X, \phi) \subseteq \{x - x \cdot a : x \in X, a \in S_\phi\}^{-\|\cdot\|}$

Proof. (i). Let $x \in \overline{N(X, \phi)}^{\|\cdot\|}$. Then there is a sequence $(x_n) \subseteq N(X, \phi)$ such that $x_n \to x$. Since $x_n \in N(X, \phi)$, for each n there exists $a_n \in S_\phi$ such that $\|a_n\| \leq 1 + \frac{1}{n}$ and $\|x_n \cdot a_n\| \leq \frac{1}{n}$. Thus,

$$\|x \cdot a_n\| \leq \|x \cdot a_n - x_n \cdot a_n\| + \|x_n \cdot a_n\| \leq \|x - x_n\| \|a_n\| + \frac{1}{n},$$

for all $n \in \mathbb{N}$ which implies that $x \in N(X, \phi)$.

(ii). It suffice to show that $N(X, \phi)$ is closed under addition. Suppose that $x_1, x_2 \in N(X, \phi)$ and $\delta > 0$. Then there are $a_j \in S_\phi$, $j = 1, 2$, such that $\|a_j\| \leq 1 + \delta$ and $\|x_j \cdot a_j\| \leq \delta$. Since $\mathcal{A}$ is commutative it follows that

$$\|(x_1 + x_2) \cdot (a_1 a_2)\| \leq \|x_1 \cdot a_1\| \|a_2\| + \|x_2 \cdot a_2\| \|a_1\| \leq 2\delta(1 + \delta).$$

Hence, $x_1 + x_2 \in N(X, \phi)$.

(iii). Let

$$Y := \{x - x \cdot a : x \in X, a \in S_\phi\}$$

and fix $x \in N(X)$. Then for each $\delta > 0$, there exists a sequence (a_n) in S_ϕ such that $\|a_n\| \leq 1 + \delta$ for all n and $\|x \cdot a_n\| \to 0$. Hence,

$$\|x - (x - x \cdot a_n)\| \to 0.$$

whence $N(X) \subseteq Y^{-\|\cdot\|}$. ∎

Let X be a Banach right $\mathcal{A}$-module. For subsets $Y \subseteq X$ and $F \subseteq \mathcal{A}$ let

$$YF = \{y \cdot b : y \in Y, b \in F\}.$$

Then we say that Y is F-invariant if $YF \subseteq Y$.

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then $N(X, \phi)$ is closed under addition if it is an S_ϕ-invariant subset of X.*

Proof. Let $x_1, x_2 \in N(X,\phi)$ and $\delta > 0$. Then there are $a_j \in S_\phi$, $j = 1, 2$, such that $\|a_j\| \le 1+\delta$, $\|x_1 \cdot a_1\| \le \delta$ and $\|(x_2 \cdot a_1)\cdot a_2\| \le \delta$. Thus,

$$\|(x_1 + x_2)\cdot(a_1a_2)\| \le \|x_1 \cdot a_1\|\|a_2\| + \|x_2 \cdot a_1a_2\| \le \delta(2+\delta).$$

Therefore, $x_1 + x_2 \in N(X)$. ■

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then $N(X,\phi)$ is an S_ϕ-invariant subset of X if it is a subspace of X and $x - x\cdot a \in N(X,\phi)$ for all $x \in X$ and $a \in S_\phi$.*

Proof. Let $x \in N(X,\phi)$ and $a \in S_\phi$. Then, $x\cdot a = x - (x - x\cdot a) \in N(X,\phi)$ by assumption. ■

Definition 2.5. Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. We say that X has *the ϕ-ergodic property* if there exists a net (u_α) in S_ϕ such that $\|u_\alpha\| \to 1$ and $\|(x - x\cdot a)\cdot u_\alpha\| \to 0$ for all $x \in X$ and $a \in S_\phi$.

The next theorem is one of the main results of the paper which is a characterization of ϕ-ergodic property for Banach modules, using the set $N(X,\phi)$. Note that the following theorem generalizes [5, Theorem 2.8].

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then the following statements are equivalent.*

(i) *There exists a net (u_β) in S_ϕ such that $\|u_\beta\| \to 1$ and $(x - x\cdot a)\cdot u_\beta \to 0$ in the weak topology of X for all $x \in X$ and $a \in S_\phi$.*

(ii) *$N(X,\phi)$ is a subspace of X and $x - x\cdot a \in N(X,\phi)$ for all $x \in X$ and $a \in S_\phi$.*

(iii) *X has the ϕ-ergodic property.*

Proof. (i) $\Rightarrow$ (ii). Fix $x \in N(X,\phi)$ and $a \in S_\phi$. Given $\delta > 0$. Then there exists $a_1 \in S_\phi$ such that $\|a_1\| \le 1+\delta$ and $\|x\cdot a_1\| \le \delta$. By assumption there exists a net (u_β) in S_ϕ such that $\|u_\beta\| \to 1$ and $(0,0,0)$ is in weak closure of the set

$$C := \{((x - x\cdot a)\cdot u_\beta, (x - x\cdot a_1)\cdot u_\beta, (x\cdot a - x\cdot aa_1)\cdot u_\beta)\}.$$

Therefore, $(0,0,0)$ is in the norm closure of the convex hull of C. Since the set $\{a_\beta\}$ being contained in the closed hyperplane S_ϕ, we easily can find $a_2 \in S_\phi$ such that $\|a_2\| \le 1+\delta$ and

$$\|(x - x\cdot a)\cdot a_2\| \le \delta, \quad \|(x - x\cdot a_1)\cdot a_2\| \le \delta, \quad \|(x\cdot a - x\cdot aa_1)\cdot a_2\| \le \delta.$$

Thus,

$$\begin{aligned}\|(x\cdot a)\cdot a_1a_2\| &\le \|(x\cdot a - x\cdot aa_1)\cdot a_2\| + \|x\cdot aa_2 - x\cdot a_2\| \\ &+ \|x\cdot a_2 - x\cdot a_1a_2\| + \|x\cdot a_1a_2\| \\ &\le \delta(4+\delta).\end{aligned}$$

Since $\phi(a_1a_2) = 1$, $\|a_1a_2\| \le (1+\delta)^2$ and $\delta > 0$ is arbitrary, it follows that $x\cdot a \in N(X,\phi)$. Thus, $N(X,\phi)$ is a closed subspace of X by Lemma 2.3. Similarly, we can prove that $x - x\cdot a \in N(X,\phi)$ for all $x \in X$ and $a \in S_\phi$ which completes the proof.

(ii) $\Rightarrow$ (iii). We claim that for every finite subsets F and Y of S_ϕ and X, respectively and $\varepsilon > 0$, there exists $u_{F,Y,\varepsilon} \in S_\phi$ such that $\|u_{F,Y,\varepsilon}\| \leq 1+\varepsilon$ and

$$\|(x - x \cdot a) \cdot u_{F,Y,\varepsilon}\| \leq \varepsilon$$

for all $x \in Y$ and $a \in F$. Let $Y = \{x_1, ..., x_k\}$ and $F = \{a_1 ... a_m\}$, say. Fix $a \in F$ and choose $\delta > 0$ such that $(1+\delta)^{m+k+1} \leq 1+\varepsilon$. By assumption, there exists $v_1 \in S_\phi$ such that $\|v_1\| \leq 1+\delta$ and

$$\|(x_1 \cdot a - x_1) \cdot v_1\| \leq \delta.$$

Since $(x_2 - x_2 \cdot a) \cdot v_1 \in N(X, \phi)$ by Lemma 2.4, again by (ii) there exists $v_2 \in S_\phi$ such that $\|v_2\| \leq 1+\delta$ and

$$\|(x_2 \cdot a - x_2) \cdot v_1 v_2\| \leq \delta.$$

For $j = 1, 2$ we have $v_j \in S_\phi$, $\|v_j\| \leq 1+\delta$ and

$$\|(x_j - x_j \cdot a) \cdot v_1 v_2\| \leq \delta(1+\delta).$$

By induction, there exist $v_j \in S_\phi$, $1 \leq j \leq k$, such that $\|v_j\| \leq 1+\delta$ and

$$\|(x_j - x_j \cdot a) \cdot v_1 ... v_j\| \leq \delta(1+\delta)^{j-1} \leq \varepsilon.$$

Thus, if we put $v_{Y,\varepsilon} = v_1 ... v_k$, then we have $v_{Y,\varepsilon} \in S_\phi$, $\|v_{Y,\varepsilon}\| \leq 1+\varepsilon$ and

$$\|(x - x \cdot a) \cdot v_{Y,\varepsilon}\| \leq \varepsilon \qquad (*)$$

for all $x \in Y$. Now, by $(*)$, there exists $u_1 \in S_\phi$ such that $\|u_1\| \leq 1+\delta$ and

$$\|(x - x \cdot a_1) \cdot u_1\| \leq \delta$$

for all $x \in Y$. By assumption $(x - x \cdot a_2) \cdot u_1 \in N(X, \phi)$ for all $x \in Y$. Again by (ii) and using methods similar to those employed in the proof of $(*)$ we can find $u_2 \in S_\phi$ such that $\|u_2\| \leq 1+\delta$ and

$$\|(x - x \cdot a_2) u_1 u_2\| \leq \delta$$

for all $x \in Y$. Proceeding inductively, we see that there exist u_i, $1 \leq i \leq m$, such that $\|u_i\| \leq 1+\delta$ and

$$\|(x - x \cdot a_i) \cdot u_1 ... u_i\| \leq \delta(1+\delta)^{i-1} \leq \varepsilon$$

for all $y \in Y$. Thus, if we set $u_{F,Y,\varepsilon} = u_1 ... u_m$, then we have $u_{F,Y,\varepsilon} \in S_\phi$, $\|u_{F,Y,\varepsilon}\| \leq 1+\varepsilon$ and

$$\|(x - x \cdot a) \cdot u_{F,Y,\varepsilon}\| \leq \varepsilon$$

for all $x \in Y$ and $a \in F$.

Now, let Γ be the set of all $\gamma := (F, Y, \varepsilon)$ for which $\varepsilon > 0$, $F \subseteq S_\phi$ and $Y \subseteq X$ are finite sets. Then Γ is a directed set in the obvious manner and (u_γ) is the required net.

The implication (iii) $\Rightarrow$ (i) is trivial. ∎

Corollary 2.7. *Let $\mathcal{A}$ be a commutative Banach algebra, $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then X has the ϕ-ergodic property if and only if $x - x \cdot a \in N(X, \phi)$ for all $x \in X$ and $a \in S_\phi$.*

Corollary 2.8. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one if and only if* $\ker\phi = N(\mathcal{A}, \phi)$.

Proof. Suppose that $\mathcal{A}$ has a ϕ-mean of norm one. Then there is a net $(u_\alpha) \subseteq S_\phi$ such that $\|u_\alpha\| \to 1$ and

$$\|au_\alpha - \phi(a)u_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. In particular, $\|au_\alpha\| \to 0$ for all $a \in \ker\phi$. It follows that $\ker\phi \subseteq N(\mathcal{A}, \phi)$. The reverse inclusion follows from this fact that

$$N(\mathcal{A}, \phi) \subseteq \{b - ba : \ b \in \mathcal{A}, a \in S_\phi\}^{-\|\cdot\|} \subseteq \ker\phi.$$

Conversely, first note that $\ker\phi$ is a closed ideal in $\mathcal{A}$ and $b - ba \in \ker\phi$ for all $b \in \mathcal{A}$ and $a \in S_\phi$ which implies that $\mathcal{A}$ has the ϕ-ergodic property as a Banach right $\mathcal{A}$-module by Theorem 2.6. Thus, there is a net $(u_\alpha) \subseteq S_\phi$ such that $\|u_\alpha\| \to 1$ and

$$\|(b - ba)u_\alpha\| \to 0$$

for all $b \in \mathcal{A}$ and $a \in S_\phi$. By assumption $\ker\phi = \{b - ba : \ b \in \mathcal{A}, a \in S_\phi\}^{-\|\cdot\|}$ and therefore $\|bu_\alpha\| \to 0$ for all $b \in \ker\phi$. Thus $\mathcal{A}$ has a ϕ-mean of norm one by [5, Theorem 2.4(iv)]. ■

Given a Banach algebra $\mathcal{A}$ with $\phi \in \Delta(\mathcal{A})$, for each $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$ define elements $f \cdot a$ of $\mathcal{A}^*$ by $\langle f \cdot a, b\rangle = \langle f, ab\rangle$ for all $b \in \mathcal{A}$. We recall that a closed subspace X of $\mathcal{A}^*$ is *left invariant* if $X \cdot \mathcal{A} \subseteq X$. Note that a closed left invariant subspace of $\mathcal{A}^*$ is thus a Banach right $\mathcal{A}$-module. Suppose that X is a left invariant subspace of $\mathcal{A}^*$. Then $m \in \mathcal{A}^{**}$ is called a *ϕ-mean* on X if

$$m(\phi) = 1, \quad m(f \cdot a) = \phi(a)m(f) \quad (f \in X, a \in \mathcal{A}).$$

Let $X \subseteq \mathcal{A}^*$ be a left invariant subspace. For each $m \in X^*$ and $f \in X$ define $m \cdot f \in \mathcal{A}^*$ by $\langle m \cdot f, a\rangle = \langle m, f \cdot a\rangle$ for all $a \in \mathcal{A}$. The subspace X is called *topologically left introverted* if $X^* \cdot X \subseteq X$. Define the Arens product $\odot$ on the topological left introverted subspace X by

$$\langle m \odot n, f\rangle = \langle m, n \cdot f\rangle$$

for all $m, n \in X^*$ and $f \in X$. This product makes X^* into a Banach algebra.

Corollary 2.9. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a closed topological left introverted subspace of $\mathcal{A}^*$. Then X has the ϕ-ergodic property if and only if there is a ϕ-mean of norm one on X.*

Proof. Let X has the ϕ-ergodic property. Then $N(X, \phi)$ is a closed subspace of $\mathcal{A}^*$ and $f - f \cdot a \in N(X, \phi)$ for all $f \in X$ and $a \in S_\phi$. Since $\phi \notin N(X, \phi)$ and $\|\phi\| = 1$, by the Hahn-Banach theorem there exists $m \in \mathcal{A}^*$ such that $m = 0$ on $N(X, \phi)$

and $\|m\| = m(\phi) = 1$. Thus $m(f \cdot a) = m(f)$ for all $f \in X$ and $a \in S_\phi$. Therefore, $m(f \cdot a) = \phi(a)m(f)$ for all $f \in X$ and $a \in \mathcal{A}$.

Conversely, let $m \in \mathcal{A}^*$ be a ϕ-mean of norm one on X. Then there exists a net $(u_\beta) \subseteq S_\phi$ such that $\|u_\beta\| \to 1$ and $u_\beta \to m$ in the weak* topology of $\mathcal{A}^*$. For each $n \in X^*$ and $f \in X$ we have

$$\begin{aligned} \lim_\beta \langle n, (f - f \cdot a) \cdot u_\beta \rangle &= \langle m \odot n, f - f \cdot a \rangle \\ &= \langle m, (n \cdot f) - (n \cdot f) \cdot a \rangle \\ &= 0, \end{aligned}$$

for all $a \in S_\phi$ whence X has the ϕ-ergodic property by Theorem 2.6. ■

Remark 2.10. Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$. Suppose that $\mathcal{A}^*$ is equipped with its natural Banach right $\mathcal{A}$-module action. Then $\mathcal{A}^*$ has the ϕ-ergodic property if and only if $\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one by Corollary 2.9.

Suppose that X is a Banach right $\mathcal{A}$-module. For each $\xi \in X^*$ and $x \in X$ consider the functional $\xi \circ x \in \mathcal{A}^*$ defined by

$$(\xi \circ x)(a) = \xi(x \cdot a)$$

for all $a \in \mathcal{A}$. Now, define $\mathcal{L}_X(\mathcal{A})$ to be the closed linear span of the following set

$$\{\xi \circ x : \ x \in X, \xi \in X^*\}.$$

Then, it is clear that $(\xi \circ x) \cdot a = \xi \circ (x \cdot a)$ for all $a \in \mathcal{A}$. Thus, $\mathcal{L}_X(\mathcal{A})$ is a left invariant subspace of $\mathcal{A}^*$. Also $\mathcal{L}_X(\mathcal{A})$ is a topologically left introverted subspace of $\mathcal{A}^*$. Indeed, for each $m \in \mathcal{A}^{**}$, $\xi \in X^*$ and $x \in X$ we have $m \cdot (\xi \circ x) = (m \bullet \xi) \circ x$, where $m \bullet \xi \in X^*$ is defined by $m \bullet \xi(x) = m(\xi \circ x)$ for all $x \in X$.

Corollary 2.11. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then the following statements are equivalent.*

(i) *X has the ϕ-ergodic property.*

(ii) *$\mathcal{L}_X(\mathcal{A})$ has the ϕ-ergodic property.*

(iii) *There is a ϕ-mean of norm one on $\mathcal{L}_X(\mathcal{A})$.*

Proof. (i) $\Rightarrow$ (ii). Fix $x \in X$, and $\xi \in X^*$ and let $f := \xi \circ x$. Then by assumption there exists a net $(u_\alpha) \subseteq S_\phi$ such that $\|u_\alpha\| \to 1$ and $\|(x - x \cdot a) \cdot u_\alpha\| \to 0$ for all $a \in S_\phi$. Thus

$$\|(f - f \cdot a) \cdot u_\alpha\| \leq \|\xi\| \|(x - x \cdot a) \cdot u_\alpha\| \to 0.$$

This shows that, $\mathcal{L}_X(\mathcal{A})$ has the ϕ-ergodic property.

(ii) $\Rightarrow$ (iii). This follows from Corollary 2.9.

(iii) $\Rightarrow$ (i). Suppose that $m \in \mathcal{A}^*$ is a ϕ-mean of norm one on $\mathcal{L}_X(\mathcal{A})$. Then there exists a net $(u_\alpha) \subseteq S_\phi$ such that $\|u_\alpha\| \to 1$ and $u_\alpha \to m$ in the weak* topology of $\mathcal{A}^*$. It follows that $\langle f, au_\alpha - u_\alpha \rangle \to 0$ for all $a \in S_\phi$ and $f \in \mathcal{L}_X(\mathcal{A})$. In particular, for each $x \in X$ and $\xi \in X^*$ we have

$$\langle \xi, (x - x \cdot a) \cdot u_\alpha \rangle = \langle \xi \circ x, au_\alpha - u_\alpha \rangle \to 0$$

which implies that X has the ϕ-ergodic property by Theorem 2.6. ■

Example 2.12. (1). Let H be a locally compact hypergroup with the convolution product $*$, defined on $M(H)$, the space of bounded Radon measures on H. Concerning the general theory of hypergroups we refer the reader to [2]. Suppose that ω is a left Haar measure on H and let $x \mapsto \bar{x}$ be an involution of H. Thus the convolution product on hypergroup algebra $L^1(H)$ is naturally defined to make it a Banach algebra. Therefore, for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q}$, we can identify each $L^p(H)$ and $L^q(H)$ with the dual space of the other via

$$\langle f, g \rangle = \int_H f(x) g(x) d\omega(x), \quad (f \in L^p(H), g \in L^q(H)).$$

By [2, Theorem 6.2C] for each $1 < p < \infty$ the Banach spaces $L^p(H)$ is a Banach left $L^1(H)$-module with respect to the convolution product. Thus $L^q(H)$ the dual space of $L^p(H)$ is a Banach right $L^1(H)$-module. Define $\mathfrak{e} : L^1(H) \to \mathbb{C}$ by $\mathfrak{e}(f) = \int_H f(x) d\omega(x)$ for all $f \in L^1(H)$. It is routine to show that $\mathfrak{e}$ is the identity of the von Neumann algebra $L^\infty(H)$ such that $\mathfrak{e} \in \Delta(L^1(H))$ and induces $L^1(H)$ to a Lau algebra. One may consider amenability of H in terms of the existence of a $\mathfrak{e}$-mean on $L^\infty(H)$ [14].

We claim that $\mathcal{L}_{L^q(H)}(L^1(H), \mathfrak{e}) \subseteq C_0(H)$, where $C_0(H)$ is the Banach space of complex continuous functions on H vanishing at infinity. Indeed, for each $f \in L^p(H), g \in L^q(H)$ and $h \in L^1(H)$ we have

$$\begin{aligned}
\langle f \circ g, h \rangle = \langle g, h * f \rangle &= \int_H g(x)(h * f)(x) d\omega(x) \\
&= \int_H \int_H g(x) h(y) f(\bar{y} * x) d\omega(x) d\omega(y) \\
&= \int_H \int_H g(x) h(y) \bar{f}(\bar{x} * y) d\omega(x) d\omega(y) \\
&= \langle g * \bar{f}, h \rangle
\end{aligned}$$

where $\bar{f}(x) = f(\bar{x})$ for all $x \in H$. But $g * \bar{f} \in C_0(H)$ by [2, Theorem 6.2E and 6.2F] and so $f \circ g \in C_0(H)$. Thus $\mathcal{L}_{L^q(H)}(L^1(H), \mathfrak{e}) \subseteq C_0(H)$. So if H where not compact, then $\mathfrak{e} \notin C_0(H)$. It follows that $L^q(H)$ always has the $\mathfrak{e}$-ergodic property. On the other hand, since $L^1(H)$ has a bounded approximate identity the $\mathfrak{e}$-ergodic property of $L^1(H)$ is the same as left amenability of the Lau algebra $L^1(H)$ and this is equivalent to amenability of the hypergroup H [14].

(2). Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \Delta(\mathcal{A})$ with $\|\phi\| = 1$. Suppose that X is a Banach right $\mathcal{A}$-module with the following module action:

$$x \cdot a = \phi(a) x \quad (a \in \mathcal{A}, x \in X).$$

Clearly, $\mathcal{L}_X(\mathcal{A}) = \{\lambda\phi : \lambda \in \mathbb{C}\}$ which implies that X has the ϕ-ergodic property.

For a Banach right $\mathcal{A}$-module X, let $UC_r(X, \mathcal{A})$ be the *right uniformly continuous elements of* X, that is the closed linear span of the set $X\mathcal{A}$ in X. Then $UC_r(X, \mathcal{A})$ is a closed subspace of X which is also a Banach right $\mathcal{A}$-module.

Corollary 2.13. *Let $\mathcal{A}$ be a Banach algebra with a bounded left approximate identity, $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then X has the ϕ-ergodic property if and only if $UC_r(X, \mathcal{A})$ has the ϕ-ergodic property.*

Proof. Suppose that $UC_r(X, \mathcal{A})$ has the ϕ-ergodic property. Fix $x_i \in X$, $a_i \in S_\phi$ and $\delta > 0$ for $i = 1, 2$. Set $x_i' := x_i - x_i \cdot a_i$ and let $a_0 \in S_\phi$ be such that $\|a_0\| \leq 1 + \delta$. Then by hypothesis, there exists $e \in \mathcal{A}$ such that

$$\|a_0 - ea_0\| < \delta/(\|x_1\| + \|x_2\|) \quad \text{and} \quad \|a_i - ea_i\| < \delta/2\|x_i\|.$$

Thus, we may define y_i by setting $y_i = (x_i \cdot e) - (x_i \cdot e) \cdot a_i$ for $i = 1, 2$. Now, Lemma 2.4 implies that $(y_1 + y_2) \cdot a_0 \in N(UC_r(X, \mathcal{A}), \phi)$. In fact, there exists $u_0 \in S_\phi$ such that $\|u_0\| \leq 1 + \delta$ and

$$\|(y_1 + y_2) \cdot a_0 u_0\| < \delta.$$

Thus,

$$\begin{aligned}
\|(x_1' + x_2') \cdot a_0 u_0\| &\leq \|[x_1 \cdot (a_1 - ea_1) + x_2 \cdot (a_2 - ea_2)] \cdot a_0 u_0\| \\
&+ \|(y_1 + y_2) \cdot a_0 u_0\| \\
&+ \|(x_1 + x_2) \cdot (ea_0 - a_0) u_0\| \\
&< \delta(1 + \delta)^2 + \delta + \delta(1 + \delta).
\end{aligned}$$

Since $a_0 u_0 \in S_\phi$, $\|a_0 u_0\| \leq (1 + \delta)^2$ and δ is arbitrary, it follows that $x_1' + x_2' \in N(X, \phi)$. Therefore, X has the ϕ-ergodic property by Theorem 2.6(ii). ■

Remark 2.14. (1). Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$. Regard $X = \mathcal{A}$ as the Banach right $\mathcal{A}$-module with the module action being given by the product of $\mathcal{A}$. Then $\mathcal{L}_X(\mathcal{A}) = UC_r(\mathcal{A}^*, \mathcal{A})$. Thus, X has the ϕ-ergodic property if and only if there is a ϕ-mean of norm one on $UC_r(\mathcal{A}^*, \mathcal{A})$ by Corollary 2.11. On the other hand, in the case where $\mathcal{A}$ has a bounded left approximate identity the existence of a ϕ-mean of norm one on $UC_r(\mathcal{A}^*, \mathcal{A})$ is equivalent to the ϕ-amenability of $\mathcal{A}$ with a ϕ-mean of norm one by Corollary 2.13.

(2). Corollary 2.13 is false when $\mathcal{A}$ is not assumed to have a bounded left approximate identity. Indeed, let X be a Banach space with dimension more than one and let $\phi \in X^*$ with $\|\phi\| = 1$. Define a product on X by

$$ab = \phi(b)a \quad (a, b \in X).$$

With this product X is a Banach algebra which we denote it by $\mathcal{A}$. It is clear that $\Delta(\mathcal{A}) = \{\phi\}$ and $\mathcal{A}$ can not have a left approximate identity. Also

$$UC_r(\mathcal{A}^*, \mathcal{A}) = \{\lambda\phi : \lambda \in \mathbb{C}\}.$$

Trivially, there is a ϕ-mean of norm one on $UC_r(\mathcal{A}^*, \mathcal{A})$ but there is not a ϕ-mean on $\mathcal{A}^*$.

Corollary 2.15. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$. Then $\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one if and only if any Banach right $\mathcal{A}$-module X has the ϕ-ergodic property.*

Proof. Suppose that $m \in \mathcal{A}^{**}$ is a ϕ-mean of norm one on $\mathcal{A}^*$ and X is a Banach right $\mathcal{A}$-module. Then the restriction of m to $\mathcal{L}_X(\mathcal{A})$ is a ϕ-mean of norm one. Hence, X has the ϕ-ergodic property. The converse follows from Example 2.10. ∎

For a Banach right $\mathcal{A}$-module X, a linear functional $\xi \in X^*$ is called *ϕ-invariant* if $a \cdot \xi = \phi(a)\xi$ for all $a \in \mathcal{A}$.

Definition 2.16. Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. We say that X has *the Hahn-Banach ϕ-extension property* if p is a seminorm on X such that $p(x \cdot a) \leq \|a\| p(x)$ for all $a \in \mathcal{A}$ and $x \in X$, and if ξ is a ϕ-invariant linear functional on an $\mathcal{A}$-invariant subspace Y of X such that $|\xi| \leq p$, then there exists a ϕ-invariant extension $\widetilde{\xi}$ of ξ to X such that $|\widetilde{\xi}| \leq p$.

Lemma 2.17. *Let $\mathcal{A}$ be a Banach algebra, $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module with the ϕ-ergodic property. Then X has the Hahn-Banach ϕ-extension property.*

Proof. Let X, Y, p and ξ be as in Definition 2.16. By the Hahn-Banach theorem, there exists $\eta \in X^*$ such that $|\eta| \leq p$ and $\eta(x) = \xi(x)$ for all $x \in Y$. Now, consider $\widetilde{\xi} \in X^*$ defined by

$$\widetilde{\xi} = m \bullet \eta,$$

where $m \in \mathcal{A}^{**}$ is a ϕ-mean of norm one on $\mathcal{L}_X(\mathcal{A})$. Obviously, $|\widetilde{\xi}| \leq p$ and for each $x \in X$ and $a \in \mathcal{A}$ we have

$$\begin{aligned} \widetilde{\xi}(x \cdot a) &= m(\eta \circ (x \cdot a)) \\ &= m((\eta \circ x) \cdot a) \\ &= \phi(a) m(\eta \circ x) \\ &= \phi(a)\widetilde{\xi}(x). \end{aligned}$$

Moreover, if $x \in Y$, then

$$\begin{aligned} (\eta \circ x)(a) &= \eta(x \cdot a) \\ &= \xi(x \cdot a) \\ &= \phi(a)\xi(x). \end{aligned}$$

That is, $\eta \circ x = \xi(x)\phi$ which implies that

$$\widetilde{\xi}(x) = m(\eta \circ x) = \xi(x) m(\phi) = \xi(x),$$

as required. ∎

Theorem 2.18. *Let $\mathcal{A}$ be a Banach algebra, $\phi \in \Delta(\mathcal{A})$ with $\|\phi\| = 1$ and let X be a Banach right $\mathcal{A}$-module. Then the following statements are equivalent.*

(i) *X has the ϕ-ergodic property.*

(ii) *$\mathcal{L}_X(\mathcal{A})$ has the Hahn-Banach ϕ-extension property.*

Proof. (i) $\Rightarrow$ (ii). This follows from Corollary 2.11 and Lemma 2.17.

(ii) $\Rightarrow$ (i). If $\phi \notin \mathcal{L}_X(\mathcal{A})$, then by the Hahn-Banach theorem we can find $m \in \mathcal{A}^{**}$ such that $\|m\| = m(\phi) = 1$ and $m = 0$ on $\mathcal{L}_X(\mathcal{A})$. Thus X has the ϕ-ergodic property. Now, suppose that $\phi \in \mathcal{L}_X(\mathcal{A})$ and let Y be the subspace of $\mathcal{L}_X(\mathcal{A})$ generated by ϕ. Let p be the seminorm on $\mathcal{L}_X(\mathcal{A})$ defined by $p(f) = \|f\|$ for all $f \in \mathcal{L}_X(\mathcal{A})$. Then

$$p(f \cdot a) = \|f \cdot a\| \leq \|a\|\|f\| = \|a\|p(f)$$

for all $f \in \mathcal{L}_X(\mathcal{A})$ and $a \in \mathcal{A}$. We may consider $\xi \in Y^*$ defined by $\xi(\lambda\phi) = \lambda$ for all $\lambda \in \mathbb{C}$. Clearly, ξ is a ϕ-invariant linear functional on Y with $|\xi| \leq p$. Using (ii), we may obtain a ϕ-invariant extension $\widetilde{\xi}$ of ξ to $\mathcal{L}_X(\mathcal{A})$ such that $|\widetilde{\xi}| \leq p$. Hence, $\widetilde{\xi}$ is a ϕ-mean of norm one on $\mathcal{L}_X(\mathcal{A})$. ∎

Corollary 2.19. *Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \Delta(\mathcal{A})$ with $\|\phi\| = 1$. Then $\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one if and only if any Banach right $\mathcal{A}$-module X has the Hahn-Banach ϕ-extension property.*

Lemma 2.20. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Suppose that Y and Z are two invariant subspaces of X with the ϕ-ergodic property. Then $\overline{Y + Z}$ has the ϕ-ergodic property.*

Proof. Let $y_i \in Y$, $z_i \in Z$, and $a_i \in S_\phi$ and let

$$x_i := y_i + z_i - (y_i + z_i) \cdot a_i$$

for $i = 1, 2$. Given $\varepsilon > 0$. Then by hypotheses there exists $u \in S_\phi$ such that $\|u\| \leq 1 + \varepsilon$ and

$$\|(y_1 + y_2 - y_1 \cdot a_1 - y_2 \cdot a_2) \cdot u\| < \varepsilon.$$

Since $(z_1 + z_2 - z_1 \cdot a_1 - z_2 \cdot a_2) \cdot u \in N(Z, \phi)$ we may find $v \in S_\phi$ such that $\|v\| \leq 1 + \varepsilon$ and

$$\|(z_1 + z_2 - z_1 \cdot a_1 - z_2 \cdot a_2) \cdot uv\| < \varepsilon.$$

Thus $\|(x_1 + x_2) \cdot uv\| < \varepsilon(2 + \varepsilon)$. Since $uv \in S_\phi$, $\|uv\| \leq (1 + \varepsilon)^2$ and ε is arbitrary, it follows that $x_1 + x_2 \in N(\overline{Y + Z})$. Therefore, $\overline{Y + Z}$ has the ϕ-ergodic property. ∎

By the above lemma for an arbitrary Banach right $\mathcal{A}$-module X, there is a maximal invariant subspace in X which has the ϕ-ergodic property. We denote by $B_\mathcal{A}(X, \mathcal{A}^*)$ the space of all bounded right $\mathcal{A}$-module maps from X into $\mathcal{A}^*$.

Theorem 2.21. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let X be a Banach right $\mathcal{A}$-module. Then X has the ϕ-ergodic property if and only if $\overline{\mathrm{Im}T}$ has the ϕ-ergodic property for all $T \in B_\mathcal{A}(X, \mathcal{A}^*)$.*

Proof. Suppose that X has the ϕ-ergodic property. Now, let $f_1, f_2 \in \mathrm{Im}T$ for some $T \in B_\mathcal{A}(X, \mathcal{A}^*)$ and $a_1, a_2 \in S_\phi$. Then there exist $x_j \in X$ such that $f_j = Tx_j$, $j = 1, 2$. Given $\varepsilon > 0$, there exists $u \in S_\phi$ such that $\|u\| \leq 1 + \varepsilon$ and

$$\|(x_1 - x_1 \cdot a_1) \cdot u\| < \varepsilon.$$

Since $(x_2 - x_2 \cdot a_2) \cdot u \in N(X, \varphi)$, we may find $v \in S_\phi$ such that $\|v\| \le 1 + \varepsilon$ and

$$\|(x_2 - x_2 \cdot a_2) \cdot uv\| < \varepsilon.$$

Then

$$\|(f_1 + f_2 - f_1 \cdot a_1 - f_2 \cdot a_2) \cdot uv\| < \|T\|\varepsilon(1 + \varepsilon)$$

whence $(f_1 + f_2 - f_1 \cdot a_1 - f_2 \cdot a_2) \in N(\overline{\mathrm{Im}T}, \phi)$. So $\overline{\mathrm{Im}T}$ has the ϕ-ergodic property by Theorem 2.6.

Conversely, Fix $\xi \in X^*$ and define a bounded linear operator T_ξ from X into $\mathcal{A}^*$ via $T_\xi(x) = \xi \circ x$ for all $x \in X$. It is easy to see that $T_\xi \in B_{\mathcal{A}}(X, \mathcal{A}^*)$. From Lemma 2.20 the linear span of the collection $\bigcup\{\mathrm{Im}T_\xi : \ \xi \in X^*\}$, which is equal to $\mathcal{L}_X(\mathcal{A})$, has the ϕ-ergodic property. Thus, X has also the ϕ-ergodic property by Corollary 2.11. ∎

Remark 2.22. Recall that a Lau algebra $\mathcal{A}$ is a Banach algebra which is the predual of von Neumann algebra $\mathfrak{M}$ such that the identity element $\mathfrak{e}$ of $\mathfrak{M}$ is a multiplicative linear functional on $\mathcal{A}$. In this case, the $\mathfrak{e}$-means of norm one are nothing but the *topological left invariant means* on $\mathcal{A}^*$; see [7] and [8], [1] for more details concerning the left amenability of Lau algebras. Following [7], $\mathcal{A}$ is called *left amenable* if there is a topological left invariant mean on $\mathcal{A}^*$. Therefore the following statements are equivalent.

(i) $\mathcal{A}$ is left amenable.

(ii) Every Banach right $\mathcal{A}$-module X has the $\mathfrak{e}$-ergodic property.

(iii) Every Banach right $\mathcal{A}$-module X has the Hahn-Banach $\mathfrak{e}$-extension property.

(iv) $\overline{\mathrm{Im}T}$ has the $\mathfrak{e}$-ergodic property for all $T \in B_{\mathcal{A}}(X, \mathcal{A}^*)$ and all Banach right $\mathcal{A}$-module X.

The next result describes the interaction between ϕ-amenability of a Banach algebra and its closed left ideals.

Theorem 2.23. *Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \sigma(\mathcal{A})$. Suppose that I is a closed left ideal of $\mathcal{A}$ such that $\|\phi|_I\| = 1$. Then the following statements are equivalent.*

(i) *I is $\phi|_I$-amenable with a $\phi|_I$-mean of norm one.*

(ii) *$\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one.*

Proof. (i)⇒(ii). Suppose that $a \in \ker\phi$ and $\varepsilon > 0$. Given $v_0 \in S_{\phi|_I}$ and $\delta > 0$ such that $(1+\delta)^2 \le 1 + \varepsilon$ and $\|v_0\| \le 1 + \delta$. Since $av_0 \in \ker\phi|_I$, it follows from [5, Theorem 2.4] that there exists $v \in S_{\phi|_I}$ such that $\|v\| \le 1 + \delta$ and $\|av_0v\| \le \delta$. If we set $u := v_0v$, then $u \in S_\phi$. Moreover, $\|u\| \le (1+\delta)^2 \le 1 + \varepsilon$ and $\|au\| \le \varepsilon$. Again by [5, Theorem 2.4] we conclude that (ii) holds.

(ii)⇒(i). Fix $a \in \ker\phi|_I$ and $\varepsilon > 0$. Let v_0 and $\delta > 0$ be as in the proof of previous implication. Thus there exists $u \in S_\phi$ such that $\|u\| \le 1 + \delta$ and $\|au\| \le \delta$. By setting $v := uv_0$, we have $v \in S_{\phi|_I}$. Moreover, $\|v\| \le 1 + \varepsilon$ and $\|av\| \le \|au\|\|v_0\| \le \varepsilon$. These imply that (i) holds. ∎

Example 2.24. Let H be a locally compact hypergroup. Then the hypergroup algebra $L^1(H)$ is an ideal in the measure algebra $M(H)$. Consider $\widetilde{\mathfrak{e}} \in \Delta(M(H))$ defined by $\widetilde{\mathfrak{e}}(\mu) = \mu(H)$. Moreover, $\widetilde{\mathfrak{e}}|_{L^1(H)} = \mathfrak{e}$ which is defined in Example 2.12. It is well-known that H is amenable if and only if $L^1(H)$ is $\mathfrak{e}$-amenable with a $\mathfrak{e}$-mean of norm one; see [14]. Thus, it follows from Theorem 2.23 that H is amenable if and only if $M(H)$ is $\widetilde{\mathfrak{e}}$-amenable with a $\widetilde{\mathfrak{e}}$-mean of norm one.

Example 2.25. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \Delta(\mathcal{B})$. Then the *$\theta$-Lau product* of two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times_\theta \mathcal{B}$, is defined as the space $\mathcal{A} \times \mathcal{B}$ endowed with the norm $\|(a,b)\| = \|a\| + \|b\|$ and the product

$$(a,b)(a',b') = (aa' + \theta(b)a' + \theta(b')a, bb'), \quad (a,a' \in \mathcal{A}, b,b' \in \mathcal{B}).$$

It is clear that with this norm and product, $\mathcal{A} \times_\theta \mathcal{B}$ is a Banach algebra and $\mathcal{A}$ is a closed two-sided ideal of $\mathcal{A} \times_\theta \mathcal{B}$. Also, recall from [9, Proposition 2.4] that

$$\Delta(\mathcal{A} \times_\theta \mathcal{B}) = \Delta(\mathcal{A}) \times \{\theta\} \cup \{0\} \times \Delta(\mathcal{B}).$$

Since $(\phi,\theta)|_{\mathcal{A}} = \phi$ for all $\phi \in \Delta(\mathcal{A})$, it follows from above theorem that if $\|\phi\| = 1$, then $\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one if and only if $\mathcal{A} \times_\theta \mathcal{B}$ is (ϕ,θ)-amenable with a (ϕ,θ)-mean of norm one. This result was originally obtained in [10, Proposition 2.8].

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