Periodic solutions for second order Hamiltonian systems with general superquadratic potential*

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Abstract

In this paper, we study the existence of nontrivial solutions and ground state solutions for the second order Hamiltonian systems:

$$\ddot{u}(t) + A(t)u(t) + \nabla F(t, u(t)) = 0 \quad \text{a.e. } t \in [0, T],$$

where $A(t)$ is a $N \times N$ symmetric matrix, continuous and T-periodic in t. Replacing the classical Ambrosetti-Rabinowitz superquadratic condition by a general superquadratic condition, we prove some existence theorems, which unify and improve some recent results in the literature.

1 Introduction and main results

Consider the second order Hamiltonian systems

$$\ddot{u}(t) + A(t)u(t) + \nabla F(t, u(t)) = 0 \quad a.\,e.\; t \in [0, T], \tag{1.1}$$

where $A(t)$ is a $N \times N$ symmetric matrix, continuous and T-periodic ($T > 0$) in t. $F : [0, T] \times R^N \to R$ satisfies the following assumption:

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(A) $F(t,x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0,T;R^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \qquad |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

Under assumption (A), the energy functional associated to problem (1.1) given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2}\int_0^T (A(t)u(t), u(t))dt - \int_0^T F(t, u(t))dt$$

is of class C^1 on H_T^1, where

$$H_T^1 = \left\{ u : [0,T] \to R^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T;R^N) \end{array} \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right)^{1/2}.$$

It is well known that the critical points of φ are weak solutions of problem (1.1) (see [1]).

The existence of periodic solution of problem (1.1), where $A(t) \equiv 0$, was studied in Rabinowitz [2] under the following superquadratic condition:

(AR) There exist $\mu > 2$ and $L > 0$ such that

$$0 < \mu F(t,x) \leq (\nabla F(t,x), x), \qquad \forall |x| \geq L, \ \text{a.e. } t \in [0,T].$$

Since then, this condition has appeared in most of the studies for superquadratic problems, see [1, 3, 4] and references therein. A more natural condition than condition (AR) is that:

(F_1) $F(t,x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly for a.e. $t \in [0,T]$.

Although the (AR) condition is quite natural and important not only to ensure that the Euler-Lagrange functional φ has a mountain pass geometry, but also to guarantee that the Palais-Smale sequence of φ is bounded, it is somewhat restrictive and eliminates many functions. For example, the function

$$F(t,x) = |x|^2 \ln(1 + |x|^2), \qquad \forall (t,x) \in R \times R^N$$

is superquadratic at infinity, but it does not satisfy condition (AR) for any $\mu > 2$.

For this reason, in recent years, some authors tried to weaken condition (AR). We refer the readers to [5-16]. Fei [5] studied problem (1.1), where $A(t) \equiv 0$, replacing the (AR) condition by

$$\liminf_{|x| \to \infty} \frac{(\nabla F(t,x), x) - 2F(t,x)}{|x|^\beta} > 0 \qquad \text{uniformly in } t,$$

for some $\beta > 1$. This result has been generalized in [6, 9, 10], where the existence of periodic and subharmonic solutions of problem (1.1) was established by using Rabinowitz's generalized mountain pass theorem and the local linking theorem related to it. See also [8, 15, 16]. On the other hand, Schechter [12] assumed the local superquadratic condition: there is a subset $E \subset [0, T]$ with $\text{meas}(E) > 0$ such that

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^2} > 0 \qquad \text{uniformly for a.e. } t \in E,$$

instead of (AR). By means of a Nehari type argument, Szulkin and Weth [13] proved the existence of a ground state solution of problem (1.1), i.e., solutions corresponding to the least energy of the action functional φ of problem (1.1). They made the following assumptions:

(A_1) $F \in C(R \times R^N, R)$, $\nabla F \in C(R \times R^N, R^N)$ and F is 2π-periodic in t.

(A_2) $F(t,x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in t.

(A_3) $|\nabla F(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly in t.

(A_4) $s \longmapsto s^{-1}(\nabla F(t,sx), x)$ is strictly increasing for all $x \neq 0$ and $s > 0$.

Theorem A (see [13, Theorem 29]). *Suppose that F satisfies (A_1)-(A_4). Then problem (1.1), where $A(t) = -I_N$, has a 2π-periodic ground state solution.*

Very recently, Chen and Ma [14] considered the case that 0 lies in a gap of $\sigma(B)$, where $B := -d^2/dt^2 - A(t)$, i.e.,

(L_0) $\underline{\Lambda} := \sup(\sigma(B) \bigcap (-\infty, 0)) < 0 < \overline{\Lambda} := \inf(\sigma(B) \bigcap (0, +\infty))$.

They obtained a ground state solution of problem (1.1) under conditions (A_2), (A_3) and:

(B_1) $(\nabla F(t,x), y) \neq (\nabla F(t,y), x)$ for any $t \in R$, if $|x| \neq |y|$ and $(x,y) \neq 0$.

(B_2) $(\nabla F(t,x), y)(x,y) \geq 0$ uniformly in t.

(B_3) $|\nabla F(t,x)| \leq a(1 + |x|^{\lambda-1})$ for some $a > 0$ and $\lambda > 2$.

(B_4) $F(t,x) \geq 0$ for all $x \in R^N$, $\frac{1}{2}(\nabla F(t,x), x) > F(t,x)$ for all $x \in R^N \setminus \{0\}$.

(B_5) $F(t,x) = F(t,y)$ and $(\nabla F(t,x), y) \leq (\nabla F(t,x), x)$ uniformly in t, if $|x| = |y|$.

Theorem B (see [14, Theorem 1.1]). *Suppose that $F : R^N \times R \to R$ is continuous, T-periodic in t and continuously differentiable in x. Assume that (L_0), (A_2), (A_3) and (B_1)-(B_5) are satisfied. Then problem (1.1) has at least one ground state T-periodic solution.*

We should mention that in [14], the authors applied a variant generalized weak linking theorem for strongly indefinite functionals developed by Schechter and Zou [17], which was used to investigate the existence of ground solutions for Schrödinger equations, see [18, 19, 20]. This approach is not very satisfactory, because working with a family of perturbed functionals makes things unnecessarily complicated.

In this paper, using an approach different to that of [13, 14], i.e., the generalized mountain pass theorem of Rabinowitz, we can prove the same result under more generic conditions, which unifies and improves Theorems A and B.

Theorem 1.1. *Assume that assumptions (L_0), (A) and (F_1) are satisfied and:*

(F_2) $|\nabla F(t,x)| = o(|x|)$ *as* $|x| \to 0$ *uniformly for a.e.* $t \in [0,T]$.

(F_3) *For each* $(t,x) \in [0,T] \times R^N$,

$$s \longmapsto \frac{(\nabla F(t,sx),x)}{s} \text{ is increasing in } s > 0.$$

Then problem (1.1) has at least one T-periodic ground state solution.

Remark 1.1. It is important to note that condition (F_3) holds for functions F in Theorem B. Indeed, assumption (B_1), together with (A_3) and (B_2), implies that

$$g(s) := \frac{(\nabla F(t,sx),x)}{s} \text{ is strictly increasing in } s > 0.$$

For $s_1, s_2 \in (0,+\infty)$ with $s_1 \neq s_2$, using (B_1), we get

$$(\nabla F(t,s_1x),s_2x) \neq (\nabla F(t,s_2x),s_1x), \qquad \forall t \in [0,T],\ \ x \in R^N \setminus \{0\},$$

so that,

$$g(s_1) = \frac{(\nabla F(t,s_1x),x)}{s_1} \neq \frac{(\nabla F(t,s_2x),x)}{s_2} = g(s_2)$$

for all $t \in [0,T]$ and $x \in R^N \setminus \{0\}$. Hence g must be a strictly monotone mapping on $(0,+\infty)$. It follows from (A_3) that

$$|g(s)| \leq \frac{|\nabla F(t,sx)||x|}{s} = \frac{|\nabla F(t,sx)|}{|sx|}|x|^2 \to 0 \ \ (s \to 0^+),$$

which implies that

$$\lim_{s\to 0^+} g(s) = 0. \tag{1.2}$$

Moreover, using (B_2), we obtain

$$g(s)|sx|^2 = \frac{(\nabla F(t,sx),x)}{s}(sx,sx) = (\nabla F(t,sx),x)(sx,x) \geq 0$$

for all $(t,x) \in [0,T] \times R^N$ and $s > 0$, which yields that

$$g(s) \geq 0, \qquad \forall s > 0.$$

This, together with (1.2) and the monotonicity of g, shows that g is strictly increasing in $s > 0$.

Remark 1.2. Theorem 1.1 unifies and extends Theorems A and B.

- Comparing with Theorem A, the strictly increasing assumption is removed and the range of the matrix $A(t)$ is expanded. Hence our result applies to more general situations. We emphasize that the strictly increasing assumption plays an essential role in their argument. In fact, the starting point of their approach is to show that for each $w \in E\setminus\{0\}$, there exists exactly one point $s_w \in R$ such that $s_w w$ belongs to the Nehari manifold

$$\mathcal{N} = \left\{u \in E\setminus\{0\} : \varphi'(u)u = 0\right\}.$$

  The uniqueness of s_w enables one to define a map $\hat{m} : E\setminus\{0\} \to \mathcal{N}$ with $\hat{m}(w) = s_w w$, which is important in the remaining proof. If $s \to s^{-1}(\nabla F(t,sx),x)$ is not strictly increasing, then s_w may not unique and their argument collapses.

- As stated in Remark 1.1, (B_1), jointly with (A_3) and (B_2), is stronger than (F_3). Besides, the conditions (B_3), (B_4) and (B_5) in Theorem B are completely removed. Therefore, Theorem 1.1 greatly improves Theorem B.

There are functions F satisfying our Theorem 1.1 and not satisfying Theorems A and B. For example, let

$$F(t,x) = \begin{cases} |x|^2 \ln|x| - \frac{1}{2}|x|^2 + \frac{1}{2}, & |x| \geq 1, \\ 0, & |x| < 1. \end{cases}$$

When the global condition (F_3) is replaced by the local one:

(F_4) For every $(t,x) \in [0,T] \times R^N$, there exists $M > 0$ such that

$$s \longmapsto \frac{(\nabla F(t,sx),x)}{s} \text{ is increasing in } s > M.$$

We can establish the existence of nontrivial solution of problem (1.1) by using the local linking theorem due to Luan and Mao (see [7]).

Theorem 1.2. *Assume that assumptions (A), (F_1), (F_2) and (F_4) are satisfied. If 0 is an eigenvalue of $-d^2/dt^2 - A(t)$ (with periodic boundary condition), assume also*

(F_5) $F(t,x) \geq 0$ *(or $F(t,x) \leq 0$),* $\quad \forall |x| \leq r, t \in [0,T]$ *for some $r > 0$.*

Then problem (1.1) has at least one nontrivial T-periodic solution.

Remark 1.3. There are functions $F(t,x)$ satisfying our Theorem 1.2 and not satisfying Theorems A, B and 1.1. For example, let

$$F(t,x) = \begin{cases} |x|^2 \ln|x| - |x|^2 + \frac{2}{3}, & |x| \geq 1, \\ -\frac{1}{3}|x|^3, & |x| < 1. \end{cases}$$

It is easy to verify that $F(t,x)$ satisfies all the conditions of Theorem 1.2. But it does not satisfy Theorems A, B and 1.1, since for $t \in [0,T]$ and $x \in R^N\setminus\{0\}$, $s \longmapsto s^{-1}(\nabla F(t,sx),x)$ is nonincreasing on $(0,1/|x|)$.

We shall prove more general results than Theorems 1.1 and 1.2.

Theorem 1.3. *Assume that assumptions (L_0), (A), (F_1) and (F_2) are satisfied and:*

(F_3') *There exists $\theta \geq 1$ such that*

$$\mathcal{F}(t,sx) \leq \theta\mathcal{F}(t,x), \qquad \forall(t,x) \in [0,T] \times R^N,\ \ s \in [0,1].$$

Then problem (1.1) has at least one ground state T-periodic solution.

Remark 1.4. Condition (F_3') is originally due to Jeanjean [21] for a semilinear problem on R^N. Later, it was used in Liu and Li [22] to obtain infinitely many solutions for p-Laplacian equations setting on a bounded domain. To the best of our knowledge, there are few works concerning Hamiltonian systems with this assumption.

Theorem 1.4. *Assume that assumptions (A), (F_1) and (F_2) are satisfied and:*

(F_4') *There exists $\theta \geq 1$ and $C_* \in L^1(0,T;R^+)$ such that*

$$\mathcal{F}(t,sx) \leq \theta\mathcal{F}(t,x) + C_*(t)$$

for all $(t,x) \in [0,T] \times R^N$ and $s \in [0,1]$.

If 0 is an eigenvalue of $-d^2/dt^2 - A(t)$ (with periodic boundary condition), assume also (F_5). Then problem (1.1) has at least one nontrivial T-periodic solution.

Remark 1.5. If (F_4) holds, then for each $(t,x) \in [0,T] \times R^N$, $s \longmapsto \mathcal{F}(t,sx)$ is increasing in $s > M$. Indeed, suppose that $M \leq a \leq b$, we have

$$\begin{aligned}
&\mathcal{F}(t,bx) - \mathcal{F}(t,ax)\\
&\quad= 2\left[\frac{1}{2}\left((\nabla F(t,bx),bx) - (\nabla F(t,ax),ax)\right) - (F(t,bx) - F(t,ax))\right]\\
&\quad= 2\left[\int_M^b \frac{(\nabla F(t,bx),x)}{b}\tau d\tau - \int_M^a \frac{(\nabla F(t,ax),x)}{a}\tau d\tau\right.\\
&\qquad \left. - \int_a^b \frac{(\nabla F(t,\tau x),x)}{\tau}\tau d\tau\right] + M^2\left(\frac{(\nabla F(t,bx),x)}{b} - \frac{(\nabla F(t,ax),x)}{a}\right)\\
&\quad= 2\int_M^a \left(\frac{(\nabla F(t,bx),x)}{b} - \frac{(\nabla F(t,ax),x)}{a}\right)\tau d\tau\\
&\qquad +2\int_a^b \left(\frac{(\nabla F(t,bx),x)}{b} - \frac{(\nabla F(t,\tau x),x)}{\tau}\right)\tau d\tau\\
&\qquad +M^2\left(\frac{(\nabla F(t,bx),x)}{b} - \frac{(\nabla F(t,ax),x)}{a}\right)\\
&\quad\geq 0.
\end{aligned}$$

Particularly, using assumption (A), we see that

$$C_*(t) = 1 + \sup_{|y|\leq M} \mathcal{F}(t,y) - \inf_{|y|\leq M} \mathcal{F}(t,y) \in L^1(0,T;R^+).$$

With this $C_*(t)$ and $\theta = 1$, it is easy to check that (F_4') holds. Similarly, (F_3) implies (F_3'). Therefore, Theorems 1.3 and 1.4 generalize Theorems 1.1 and 1.2, respectively.

The paper is organized as follows. In Section 2, we state some preliminaries and discuss the $(C)^*$ condition. In Section 3, we prove the main theorems.

2 Preliminaries

Since the embedding of H_T^1 into $C(0,T;R^N)$ is compact, there exists a constant $C>0$ such that

$$\|u\|_\infty \le C\|u\|, \qquad \forall u \in H_T^1, \tag{2.1}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

By the spectral theorem for compact self-adjoint operators on a Hilbert space, the differential operator $u \to -\ddot{u} - A(t)u$ has a sequence of eigenvalues (counted in their multiplicities)

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

with $\lambda_n \to +\infty$ as $n \to \infty$, and the corresponding system of eigenfunctions $\{e_n : n \in N\}$ $(-\ddot{e}_n - A(t)e_n = \lambda_n e_n)$ which forms an orthogonal basis of H_T^1. Set

$$n^- = \#\{i\,|\lambda_i < 0\}, \qquad n^0 = \#\{i\,|\lambda_i = 0\}, \qquad \bar{n} = n^- + n^0,$$

and

$$H^- = \operatorname{span}\{e_1,\cdots,e_{n^-}\}, \quad H^0 = \operatorname{span}\{e_{n^-+1},\cdots,e_{\bar{n}}\}, \quad H^+ = \overline{\operatorname{span}\{e_{\bar{n}+1},\cdots\}}.$$

Then one has

$$H_T^1 = H^+ \bigoplus H^- \bigoplus H^0,$$

and there exists $\delta > 0$ such that

$$\int_0^T |\dot{u}|^2 dt - \int_0^T (A(t)u,u)dt \ge \delta\|u\|^2 \tag{2.2}$$

for $u \in H^+$ and

$$\int_0^T |\dot{u}|^2 dt - \int_0^T (A(t)u,u)dt \le -\delta\|u\|^2 \tag{2.3}$$

for $u \in H^-$ (see [1, p. 89]). For $u \in H_T^1$, we always write $u = u^+ + u^- + u^0$, where $u^\pm \in H^\pm$ and $u^0 \in H^0$.

In order to find the critical points of φ, we shall show that φ satisfies the $(C)^*$ condition. Let X be a real Banach space with $X = X^1 \bigoplus X^2$ and $X_0^j \subset X_1^j \subset \cdots \subset X^j$ such that $X^j = \overline{\bigcup_{n\in N} X_n^j}$, $j = 1,2$. For every multi-index $\alpha = (\alpha_1,\alpha_2) \in N^2$, denote $X_\alpha = X_{\alpha_1}^1 \bigoplus X_{\alpha_2}^2$. We say $\alpha \le \beta \Longleftrightarrow \alpha_1 \le \beta_1, \alpha_2 \le \beta_2$. A sequence $(\alpha_n) \subset N^2$ is admissible if, for every $\alpha \in N^2$ there is $m \in N$ such that $n \ge m \Rightarrow \alpha_n \ge \alpha$. We say that $\varphi \in C^1(X,R)$ satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_n \varphi(u_{\alpha_n}) < \infty, \quad (1+\|u_{\alpha_n}\|)\|\varphi'(u_{\alpha_n})\| \to 0$$

contains a subsequence which converges to a critical point of φ.

Lemma 2.1. *Suppose that assumptions (A), (F_1) and (F_4') hold, then φ satisfies the $(C)^*$ condition.*

Proof. Let $X = H_T^1$, $X^1 = H^+$, $X^2 = H^- \bigoplus H^0$ and define

$$X_n^1 = \text{span}\{e_{\bar{n}+1}, \cdots, e_{\bar{n}+n}\}, \quad \forall n \in N,$$
$$X_n^2 = X^2, \quad \forall n \in N.$$

Then

$$X^j = \overline{\bigcup_{n\in N} X_n^j}, \quad j = 1, 2.$$

Set (u_{α_n}) be a sequence in H_T^1 such that (α_n) is admissible and satisfying

$$u_{\alpha_n} \in X_{\alpha_n}, \ \ c_1 := \sup_n \varphi(u_{\alpha_n}) < \infty, \quad (1 + \|u_{\alpha_n}\|)\|\varphi'(u_{\alpha_n})\| \to 0.$$

Hence, with $u_n := u_{\alpha_n}$, we have

$$\limsup_{n\to\infty} \int_0^T \left(\frac{1}{2}(\nabla F(t, u_n), u_n) - F(t, u_n)\right) dt = \limsup_{n\to\infty} \left(\varphi(u_n) - \frac{1}{2}\langle\varphi'(u_n), u_n\rangle\right) \leq c_1. \quad (2.4)$$

Arguing indirectly, assume $\|u_n\| \to \infty$. Take $w_n = u_n/\|u_n\|$, going if necessary to a subsequence, we get

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{weakly in } H_T^1, \\ w_n &\to w \quad \text{strongly in } C(0, T; R^N). \end{aligned} \quad (2.5)$$

If $w = 0$, inspired by [21, 23], we choose a sequence $(s_n) \subset R$ such that

$$\varphi(s_n u_n) = \max_{s\in[0,1]} \varphi(s u_n).$$

For any $m > 0$, taking $v_n = \sqrt{2m} w_n$, one has

$$v_n \to 0 \quad \text{in } C(0, T; R^N) \quad (2.6)$$

by (2.5). Now for n large enough, $\sqrt{2m}\|u_n\|^{-1} \in (0, 1)$, we deduce

$$\begin{aligned} \varphi(s_n u_n) &\geq \varphi(v_n) \\ &= m - \frac{1}{2}\int_0^T |v_n|^2 dt - \frac{1}{2}\int_0^T (A(t)v_n, v_n)dt - \int_0^T F(t, v_n)dt, \end{aligned}$$

which implies that

$$\liminf_{n\to\infty} \varphi(s_n u_n) \geq m - \int_0^T F(t, 0)dt \geq m - a(0)\int_0^T b(t)dt$$

by (2.6) and assumption (A). Since m is arbitrary, we have

$$\lim_{n\to\infty} \varphi(s_n u_n) = +\infty.$$

Noticing $\varphi(0) < +\infty$ and $\sup_n \varphi(u_n) \le c_1$, we see that, for n sufficiently large, $s_n \in (0,1)$, and

$$\begin{aligned}\int_0^T |s_n\dot{u}_n|^2 dt - \int_0^T (A(t)s_n u_n, s_n u_n)dt - \int_0^T (\nabla F(t, s_n u_n), s_n u_n)dt \\ &= \langle \varphi'(s_n u_n), s_n u_n\rangle \\ &= s_n \frac{d}{ds}\bigg|_{s=s_n} \varphi(su_n) \\ &= 0.\end{aligned}$$

Therefore, using (F_4'),

$$\begin{aligned}&\int_0^T \left(\frac{1}{2}(\nabla F(t,u_n),u_n) - F(t,u_n)\right) dt \\ &\quad\ge \frac{1}{2\theta}\int_0^T \mathcal{F}(t, s_n u_n)dt - \frac{1}{2\theta}\int_0^T C_*(t)dt \\ &\quad= \frac{1}{\theta}\int_0^T \left(\frac{1}{2}(\nabla F(t, s_n u_n), s_n u_n) - F(t, s_n u_n)\right) dt \\ &\qquad -\frac{1}{2\theta}\int_0^T C_*(t)dt \\ &\quad= \frac{1}{\theta}\int_0^T \left(\frac{1}{2}|s_n\dot{u}_n|^2 - \frac{1}{2}(A(t)s_n u_n, s_n u_n) - F(t, s_n u_n)\right) dt \\ &\qquad -\frac{1}{2\theta}\int_0^T C_*(t)dt \\ &\quad= \frac{1}{\theta}\varphi(s_n u_n) - \frac{1}{2\theta}\int_0^T C_*(t)dt \\ &\quad\to +\infty,\end{aligned}$$

a contradiction with (2.4).

If $w \neq 0$, the set $\Omega_1 = \{t \in [0,T] : w(t) \neq 0\}$ has positive Lebesgue measure. For $t \in \Omega_1$, one has $|u_n(t)| \to \infty$ as $n \to \infty$, so that, using (F_1),

$$\frac{F(t,u_n(t))}{|u_n(t)|^2}|w_n(t)|^2 \to +\infty \quad \text{as } n \to \infty.$$

It follows from Lebesgue-Fatou Lemma (see [24]) that

$$\int_{w\neq 0} \frac{F(t,u_n)}{\|u_n\|^2}dt = \int_{w\neq 0} \frac{F(t,u_n)}{|u_n|^2}|w_n|^2 dt \to +\infty \quad \text{as } n \to \infty. \tag{2.7}$$

On the other hand, assumptions (A) and (F_1) imply that there exists $L > 0$ such that

$$F(t,x) \ge -\max_{s\in[0,L]} a(s)b(t), \qquad \forall x \in R^N \text{ and a.e. } t \in [0,T]. \tag{2.8}$$

Hence we obtain

$$\int_{w=0} \frac{F(t,u_n)}{\|u_n\|^2}dt \ge -\frac{\max_{s\in[0,L]} a(s)\int_0^T b(t)dt}{\|u_n\|^2},$$

which yields that

$$\liminf_{n\to\infty}\int_{w=0}\frac{F(t,u_n)}{\|u_n\|^2}dt \geq 0. \tag{2.9}$$

By the continuity of $A(\cdot)$, we deduce that there exists a constant $G \geq 1$ such that

$$\left|\int_0^T (A(t)u,u)dt\right| \leq G\int_0^T |u|^2dt, \qquad \forall u \in H_T^1. \tag{2.10}$$

Note

$$\varphi(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_0^T |u_n|^2dt - \frac{1}{2}\int_0^T (A(t)u_n,u_n)dt - \int_0^T F(t,u_n)dt, \quad \forall n \in N.$$

Dividing both sides by $\|u_n\|^2$ and letting $n \to \infty$, we obtain

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2}\int_0^T |w|^2dt + \frac{1}{2}\int_0^T (A(t)w,w)dt + \lim_{n\to\infty}\int_0^T \frac{F(t,u_n)}{\|u_n\|^2}dt \\
&\geq \frac{1}{2}(1-G)\int_0^T |w|^2dt + \lim_{n\to\infty}\left(\int_{w=0} + \int_{w\neq 0}\right)\frac{F(t,u_n)}{\|u_n\|^2}dt \\
&\geq \frac{1}{2}(1-G)\int_0^T |w|^2dt + \lim_{n\to\infty}\int_{w\neq 0}\frac{F(t,u_n)}{\|u_n\|^2}dt \\
&= +\infty
\end{aligned}$$

by (2.10), (2.9) and (2.7). This is impossible.

In any case, we deduce a contradiction. Hence (u_n) is a bounded sequence in H_T^1. Arguing then as in [1, Proposition 4.1], we conclude that the $(C)^*$ condition is satisfied. ∎

To end this section, we state the local linking theorem due to Luan and Mao.

Proposition 2.1. *([7, Theorem 2.2]). Suppose that $\varphi \in C^1(X,R)$ satisfies the following assumptions:*

(φ_1) *$X^1 \neq \{0\}$ and φ has a local linking at 0 with respect to (X^1,X^2), i.e., for some $r_0 > 0$,*

$$\begin{aligned}
\varphi(u) &\geq 0, \quad \forall\, u \in X^1 \text{ with } \|u\| \leq r_0, \\
\varphi(u) &\leq 0, \quad \forall\, u \in X^2 \text{ with } \|u\| \leq r_0.
\end{aligned}$$

(φ_2) *φ satisfies $(C)^*$ condition.*

(φ_3) *φ maps bounded sets into bounded sets.*

(φ_4) *For every $m \in N$, $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ on $X_m^1 \bigoplus X^2$.*

Then φ has at least one nonzero critical point.

3 Proofs of the theorems

Since $F(t,x)$ may be replaced by $F(t,x) - F(t,0)$, without loss of generality, we may assume that $F(t,0) = 0$ for all $t \in R$.

Proof of Theorem 1.3. The proof will be divided into several steps.

Step 1. Note that (F_3') corresponds to the special case of (F_4') with $C_*(t) \equiv 0$. As in the proof of Lemma 2.1, we conclude that, under assumptions (A), (F_1) and (F_3'), φ satisfies the (C) condition, i.e., (u_n) has a convergent subsequence in H_T^1 whenever $\{\varphi(u_n)\}$ is bounded and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.

Step 2. There exist constants $\alpha, \rho > 0$ such that

$$\varphi(u) \geq \alpha > 0, \qquad \forall u \in H^+ \bigcap \partial B_\rho. \tag{3.1}$$

Applying (F_2), for $0 < \varepsilon < \delta/4$, there exists $L_1 > 0$ such that

$$|\nabla F(t,x)| \leq \varepsilon |x|, \qquad \forall |x| \leq L_1 \text{ and a.e. } t \in [0,T]. \tag{3.2}$$

Hence

$$|F(t,x)| \leq \varepsilon |x|^2, \qquad \forall |x| \leq L_1 \text{ and a.e. } t \in [0,T]. \tag{3.3}$$

Now for $u \in H^+$ with $\|u\| \leq L_1/C$, we have, using (3.3), (2.2) and (2.1),

$$\begin{aligned} \varphi(u) &\geq \frac{\delta}{2}\|u\|^2 - \int_0^T F(t,u)dt \\ &\geq \frac{\delta}{2}\|u\|^2 - \varepsilon \int_0^T |u|^2 dt \\ &\geq \frac{\delta}{4}\|u\|^2. \end{aligned}$$

So, choosing $\rho = L_1/C$ and $\alpha = \delta\rho^2/4$, it follows that (3.1) holds.

Step 3. Let $e \in H^+$ with $\|e\| = 1$ and $\widetilde{H} = H^- \bigoplus H^0 \bigoplus \text{span}\{e\}$. Then there exists $\varepsilon_1 > 0$ such that

$$\text{meas}\,\{t \in [0,T] : |u(t)| \geq \varepsilon_1 \|u\|\} \geq \varepsilon_1, \qquad \forall u \in \widetilde{H} \backslash \{0\}\,. \tag{3.4}$$

Indeed, if this does not hold, we have, for any positive integer n, there exists $u_n \in \widetilde{H} \backslash \{0\}$ such that

$$\text{meas}\left\{t \in [0,T] : |u_n(t)| \geq \frac{1}{n}\|u_n\|\right\} < \frac{1}{n}.$$

By the homogeneity of the above inequality, we may assume that $\|u_n\| = 1$ and

$$\text{meas}\left\{t \in [0,T] : |u_n(t)| \geq \frac{1}{n}\right\} < \frac{1}{n}, \qquad \forall n \in N. \tag{3.5}$$

Noting $\dim \widetilde{H} < +\infty$, it follows from the compactness of the unit sphere of $\widetilde{H}$ that there exists a subsequence, say (u_n), such that $u_n \to u_0$ for some $u_0 \in \widetilde{H}$. Hence, using the equivalence of the norms on $\widetilde{H}$, we have $u_n \to u_0$ in L^2, i.e.,

$$\int_0^T |u_n - u_0|^2 dt \to 0 \qquad \text{as } n \to \infty. \tag{3.6}$$

Evidently, $\|u_0\| = 1$. So there exist constants $d_1, d_2 > 0$ such that

$$\text{meas}\,\{t \in [0,T] : |u_0(t)| \geq d_1\} \geq d_2. \tag{3.7}$$

Otherwise, one has

$$\text{meas}\left\{t \in [0,T] : |u_0(t)| \geq \frac{1}{n}\right\} = 0, \qquad \forall n \in N,$$

which yields that

$$0 \leq \int_0^T |u_0|^2 dt \leq T\|u_0\|_\infty^2 \leq \frac{T}{n^2} \to 0$$

as $n \to \infty$. Thus, $u_0 = 0$, a contradiction with the fact $\|u_0\| = 1$.

Now for $n \in N$, let

$$\Lambda_0 = \{t \in [0,T] : |u_0(t)| \geq d_1\}, \qquad \Lambda_n = \left\{t \in [0,T] : |u_n(t)| < \frac{1}{n}\right\},$$

and $\Lambda_n^c = [0,T]\backslash\Lambda_n$. By (3.7) and (3.5), we obtain, for n sufficiently large,

$$\text{meas}(\Lambda_0 \bigcap \Lambda_n) \geq \text{meas}(\Lambda_0) - \text{meas}(\Lambda_n^c) \geq d_2 - \frac{1}{n} \geq \frac{d_2}{2}.$$

Therefore, for n large enough,

$$\begin{aligned}
\int_0^T |u_n - u_0|^2 dt &\geq \int_{\Lambda_0 \bigcap \Lambda_n} |u_n - u_0|^2 dt \\
&\geq \int_{\Lambda_0 \bigcap \Lambda_n} \left(d_1 - \frac{1}{n}\right)^2 dt \\
&\geq \frac{d_1^2}{4} \cdot \text{meas}(\Lambda_0 \bigcap \Lambda_n) \\
&\geq \frac{d_1^2 d_2}{8},
\end{aligned}$$

which contradicts (3.6). Hence (3.4) holds.

It follows from (F_3') that

$$\mathcal{F}(t,x) \geq \frac{1}{\theta}\mathcal{F}(t,0) = 0, \qquad \forall (t,x) \in [0,T] \times R^N,$$

that is,

$$(\nabla F(t,x), x) - 2F(t,x) \geq 0, \qquad \forall (t,x) \in [0,T] \times R^N. \tag{3.8}$$

For $(t,x) \in [0,T] \times R^N$ and $s > 0$, we have

$$\frac{d}{ds}\left(\frac{F(t,sx)}{s^2}\right) = \frac{(\nabla F(t,sx), sx) - 2F(t,sx)}{s^3} \geq 0. \tag{3.9}$$

By (F_2),

$$\lim_{s \to 0^+} \frac{F(t,sx)}{s^2} = 0. \tag{3.10}$$

From (3.10) and (3.9), we deduce that

$$F(t,x) \geq 0, \qquad \forall (t,x) \in [0,T] \times R^N. \tag{3.11}$$

Now, for $u \in \widetilde{H}$, take

$$\Omega_u = \{t \in [0,T] : |u(t)| \geq \varepsilon_1 \|u\|\}.$$

By (F_1), for $M = G\varepsilon_1^{-3}$, there exists $L_2 > 0$ such that

$$F(t,x) \geq M|x|^2, \quad \forall |x| \geq L_2 \text{ and a.e. } t \in [0,T]. \tag{3.12}$$

Hence, for $u \in \widetilde{H}$ with $\|u\| \geq L_2/\varepsilon_1$, we obtain

$$\begin{aligned}
\varphi(u) &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{1}{2}\int_0^T |\dot{u}^+|^2 dt - \frac{1}{2}\int_0^T (A(t)u^+, u^+)dt - \int_0^T F(t,u)dt \\
&\leq \frac{1}{2}\int_0^T |\dot{u}^+|^2 dt + \frac{G}{2}\int_0^T |u^+|^2 dt - \int_{\Omega_u} F(t,u)dt \\
&\leq \frac{G}{2}\|u^+\|^2 - M\int_{\Omega_u} |u|^2 dt \\
&\leq \frac{G}{2}\|u^+\|^2 - M \cdot \varepsilon_1^2 \|u\|^2 \cdot \text{meas}\Omega_u \\
&\leq \frac{G}{2}\|u^+\|^2 - M \cdot \varepsilon_1^3 \|u\|^2 \\
&\leq -\frac{G}{2}\|u\|^2
\end{aligned} \tag{3.13}$$

by (3.12), (3.11), (2.10) and (2.3). Let

$$Q = \{se : 0 \leq s \leq s_1\} \bigoplus \left\{u \in H^- \bigoplus H^0 : \|u\| \leq s_1\right\}.$$

Then we have

$$\partial Q = Q_1 \bigcup Q_2 \bigcup Q_3,$$

where

$$\begin{aligned}
Q_1 &= \left\{u \in H^- \bigoplus H^0 : \|u\| \leq s_1\right\}, \\
Q_2 &= s_1 e \bigoplus \left\{u \in H^- \bigoplus H^0 : \|u\| \leq s_1\right\}, \\
Q_3 &= \{se : 0 \leq s \leq s_1\} \bigoplus \left\{u \in H^- \bigoplus H^0 : \|u\| = s_1\right\}.
\end{aligned}$$

By (3.13), one has

$$\varphi(u) \leq 0, \quad \forall\, u \in Q_2 \bigcup Q_3$$

for $s_1 \geq L_2/\varepsilon_1$. It follows from (3.11) that

$$\varphi(u) \leq -\frac{\delta}{2}\|u^-\|^2 - \int_0^T F(t,u)dt \leq 0, \quad \forall\, u \in H^- \bigoplus H^0,$$

which implies that

$$\varphi(u) \leq 0, \quad \forall\, u \in Q_1.$$

Thus we obtain

$$\varphi(u) \le 0, \quad \forall\, u \in \partial Q$$

for $s_1 > \max\{\rho, L_2/\varepsilon_1\}$.

As shown in [25], a deformation lemma can be proved with the weaker condition (C) replacing the usual Palais-Smale condition, and it turns out that the generalized mountain pass theorem (see [26, Theorem 5.29]) holds true under condition (C). Hence, by the generalized mountain pass theorem, there exists a critical point $u^* \in H_T^1$ such that $\varphi(u^*) \ge \alpha > 0$.

Step 4. Now suppose that $0 \notin \sigma(B)$, then $H^0 = \{0\}$. To get ground state solution, we denote by K the critical set of φ, i.e., $K = \left\{u \in H_T^1 : \varphi'(u) = 0, u \neq 0\right\}$, and adapt the argument of Jeanjean and Tanaka [27], where an asymptotically linear problem in definite case is considered. Set

$$m = \inf\{\varphi(u) : u \in K\}.$$

For any $u \in K$, using (3.8), we obtain

$$\varphi(u) = \varphi(u) - \frac{1}{2}\langle \varphi'(u), u\rangle = \int_0^T \left[\frac{1}{2}(\nabla F(t,u), u) - F(t,u)\right] dt \ge 0.$$

Hence, $0 \le m \le \varphi(u^*)$, where u^* is the nontrivial critical point found before.

Suppose that $(u_n) \subset K$ such that $\varphi(u_n) \to m$. Then (u_n) is a Cerami sequence. By *Step 1*, (u_n) has a convergent subsequence. Without loss of generality, we can assume that

$$\begin{aligned} u_n &\to u \qquad \text{in } H_T^1,\\ u_n &\to u \qquad \text{in } C(0,T;R^N). \end{aligned}$$

If $u = 0$, one has $\|u_n\|_\infty \to 0$ as $n \to \infty$, so that, there exists $N_1 > 0$ such that

$$\|u_n\|_\infty \le L_1, \qquad \forall n \ge N_1.$$

Note

$$0 = \langle \varphi'(u_n), u_n^+\rangle \ge \frac{\delta}{2}\|u_n^+\|^2 - \int_0^T (\nabla F(t,u_n), u_n^+)dt, \qquad \forall n \in N.$$

Using (3.2) and (2.1), we obtain

$$\begin{aligned} \|u_n^+\|^2 &\le \frac{2}{\delta}\int_0^T |\nabla F(t,u_n)||u_n^+|dt\\ &\le \frac{2\varepsilon}{\delta}\int_0^T |u_n||u_n^+|dt\\ &\le \frac{2\varepsilon}{\delta}T\|u_n\|_\infty\|u_n^+\|_\infty\\ &\le \frac{2\varepsilon}{\delta}TC^2\|u_n\|\|u_n^+\|, \qquad \forall n \ge N_1. \end{aligned}$$

Similarly, one has

$$\|u_n^-\|^2 \le \frac{2\varepsilon}{\delta}TC^2\|u_n\|\|u_n^-\|, \qquad \forall n \ge N_1.$$

Hence, we obtain

$$\|u_n\|^2 = \|u_n^-\|^2 + \|u_n^+\|^2 \leq \frac{4\varepsilon}{\delta} T C^2 \|u_n\|^2, \qquad \forall n \geq N_1.$$

As $\|u_n\| \neq 0$ and ε is arbitrary, this is a contradiction. Hence, $u \neq 0$, a nontrivial critical point of φ. By (3.8) and Fatou's lemma, we deduce

$$\begin{aligned} m &= \liminf_{n\to\infty} \varphi(u_n) \\ &= \liminf_{n\to\infty} \left(\varphi(u_n) - \frac{1}{2}\langle \varphi'(u_n), u_n \rangle \right) \\ &= \liminf_{n\to\infty} \int_0^T \left[\frac{1}{2}(\nabla F(t,u_n), u_n) - F(t,u_n) \right] dt \\ &\geq \int_0^T \left[\frac{1}{2}(\nabla F(t,u), u) - F(t,u) \right] dt \\ &= \varphi(u) \\ &\geq m. \end{aligned}$$

Therefore, u is a nontrivial critical point of φ with $\varphi(u) = m$. This completes the proof. ■

Proof of Theorem 1.4. We shall apply Proposition 2.1 to the functional φ associated to problem (1.1). We only consider the case where 0 is an eigenvalue of $-d^2/dt^2 - A(t)$ and

$$F(t,x) \geq 0, \qquad \forall |x| \leq r \text{ and } t \in [0,T]. \tag{3.14}$$

The other cases are similar.

(1) φ maps bounded sets into bounded sets.
It follows from (2.10) and (2.8) that

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (A(t)u,u)dt - \int_0^T F(t,u)dt \\ &\leq \frac{1}{2}\int_0^T |\dot{u}|^2 dt + \frac{G}{2}\int_0^T |u|^2 dt + \max_{s\in[0,L]} a(s) \int_0^T b(t)dt \\ &\leq \frac{G}{2}\|u\|^2 + \max_{s\in[0,L]} a(s) \int_0^T b(t)dt \end{aligned}$$

for $u \in H_T^1$, so (φ_3) holds.

(2) φ has a local linking at 0 with respect to (X^1, X^2).
Combining (3.3) with (2.2) and (2.1), we have, for $u \in X^1$ with $\|u\| \leq r_1 := L_1/C$,

$$\begin{aligned} \varphi(u) &\geq \frac{\delta}{2}\|u\|^2 - \int_0^T F(t,u)dt \\ &\geq \frac{\delta}{2}\|u\|^2 - \varepsilon \int_0^T |u|^2 dt \\ &\geq \frac{\delta}{4}\|u\|^2, \end{aligned}$$

which implies that

$$\varphi(u) \geq 0, \qquad u \in X^1, \ \|u\| \leq r_1.$$

For $u = u^- + u^0 \in X^2 = H^- \bigoplus H^0$ satisfying $\|u\| \leq r_2 := r/C$, using (3.14), (2.3) and (2.1), we obtain

$$\begin{aligned} \varphi(u) &\leq -\frac{\delta}{2}\|u^-\|^2 - \int_0^T F(t,u)dt \\ &\leq -\frac{\delta}{2}\|u^-\|^2, \end{aligned}$$

which implies that

$$\varphi(u) \leq 0, \qquad u \in X^2, \ \|u\| \leq r_2.$$

Hence, (φ_1) holds with $r_0 = \min\{r_1, r_2\}$.

(3) Finally, we claim that, for every $m \in N$,

$$\varphi(u) \to -\infty \quad \text{as} \quad \|u\| \to \infty \text{ on } X_m^1 \bigoplus X^2.$$

Indeed, it follows from the equivalence of norms on finite-dimensional space $X_m^1 \bigoplus H^0$, there exists $c_2 > 0$ such that

$$\|u\| \leq c_2|u|_2, \qquad \forall u \in X_m^1 \bigoplus H^0.$$

Applying (F_1), there exists $L_3 > 0$ such that

$$F(t,x) \geq \frac{1}{2}c_2^2(G+\delta)|x|^2, \quad \forall |x| \geq L_3 \text{ and a.e. } t \in [0,T].$$

By assumption (A), one has

$$|F(t,x)| \leq \max_{s\in[0,L_3]} a(s)b(t), \qquad \forall |x| \leq L_3 \text{ and a.e. } t \in [0,T],$$

which implies that

$$F(t,x) \geq \frac{1}{2}c_2^2(G+\delta)|x|^2 - \frac{1}{2}c_2^2(G+\delta)L_3^2 - \max_{s\in[0,L_3]} a(s)b(t),$$
$$\forall x \in R^N \text{ and a.e. } t \in [0,T].$$

Combining this with (2.10) and (2.3), we obtain, for $u \in X_m^1 \bigoplus X^2$,

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (A(t)u,u)dt - \int_0^T F(t,u)dt \\ &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{1}{2}\int_0^T |\dot{u}^+|^2 dt - \frac{1}{2}\int_0^T (A(t)u^+,u^+)dt - \int_0^T F(t,u)dt \\ &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{G}{2}\|u^+\|^2 - \frac{1}{2}c_2^2(G+\delta)|u|_2^2 + c_3 \\ &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{G}{2}\|u^+\|^2 - \frac{1}{2}c_2^2(G+\delta)(|u^+|_2^2 + |u^0|_2^2) + c_3 \\ &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{G}{2}\|u^+\|^2 - \frac{1}{2}(G+\delta)\|u^+\|^2 - \frac{1}{2}(G+\delta)\|u^0\|^2 + c_3 \\ &\leq -\frac{\delta}{2}\|u^-\|^2 - \frac{\delta}{2}\|u^+\|^2 - \frac{1}{2}(G+\delta)\|u^0\|^2 + c_3 \\ &\leq -\frac{\delta}{2}\|u\|^2 + c_3, \end{aligned}$$

where $c_3 = c_2^2(G+\delta)L_3^2T/2 + \max_{s\in[0,L_3]} a(s)\int_0^T b(t)dt$. This implies that

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty \text{ on } X_m^1 \bigoplus X^2.$$

Therefore, Theorem 1.4 follows from (1)-(3), Lemma 2.1 and Proposition 2.1. ∎

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