Path space and free loop space

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Abstract

Riemannian geometry on the space of (continuous) paths in a manifold M has been studied by Cruzeiro and Malliavin. I will use concepts in path space analysis to define a Levi-Civita connection on free loop space, using the G^0 metric. A tangent vector X at a loop γ is a vector field along γ such that $X(s) \in T_{\gamma(s)}M$. Following closely the calculations done by Fang, the Riemannian curvature R^{LM} is given by $R^{LM}(X,Y)Z(\cdot) = R^M(X(\cdot),Y(\cdot))Z(\cdot)$.

1 Introduction

Let (M^d, g) be a Riemannian manifold. Fix a point $o \in M$. Let $W(M,\nu)$ be the probability space of continuous paths in M, starting from o and ν is Wiener measure on this space. ν is defined in terms of the heat kernel on M, which is the solution to the heat equation using the Laplace Beltrami operator. I will not give further details, referring the reader to other resources for more information.

Analysis on $W(M,\nu)$ is an active area of research, which began with [Dri92]. The Riemannian geometry of path space is described in detail in [CM02], whereby the Levi-Civita connection is given explicitly.

The authors in [CM02] further introduced a Markovian connection. In [Fan01], Fang computed the curvature of the Markovian connection. In [CM02] or [Fan01], they considered the space of continuous but nowhere differentiable paths supported by Wiener measure, while this article only consider smooth paths or loops.

Although path space analysis uses stochastic calculus, the calculations can be easily adapted to P_oM, the space of C^∞ based paths. This article contains

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2 parts. The first part, is a very quick survey on the essential concepts in path space analysis, without the stochastic analysis.

The second part of this article, applies the analysis in path space to the free loop space LM, the space of smooth loops in M. Given $\gamma \in LM$, $\gamma : S^1 \to M$. LM is an infinite dimensional, paracompact manifold, modeled on the topological vector space $L\mathbb{R}^d$, with the topology of uniform convergence of the functions and all their derivatives. (See Chapter 3 in [PS86].) A major difficulty is that there is no canonical frame along a loop, unlike the case of based paths, whereby given a fixed frame at $o \in M$, there is a horizontal lift of the path in M to a path in the principal $O(M)$ bundle, hence defining a frame along the path.

Define a G^0 metric on LM, in Definition 3.15. The Levi-Civita connection on LM is defined by Definition 3.10 and the curvature is computed following the calculations in [Fan01]. See Theorem 3.20.

2 Analysis on Path Space

I will begin by giving a quick review of the analysis on path space in preparation for the loop space case.

2.1 Principal $O(M)$ Bundle

Let $(M^d, \nabla^M \equiv \nabla, g, o, u_0)$ be a smooth compact d-dimensional Riemannian manifold with the Levi-Civita covariant derivative ∇, a Riemannian metric g, a fixed base point $o \in M$ and a fixed orthogonal frame $u_0 : \mathbb{R}^d \to T_oM$. In $\mathbb{R}^d$, o will be the origin. Consider the principal $O(M)$ bundle, $\pi : O(M) \to M$.

Given a covariant derivative on TM, I now describe how to lift this covariant derivative on the principal $O(M)$-bundle. Write $E \equiv Hom(\mathbb{R}^d, TM)$, where Hom means linear transformations. E is a vector bundle over M with fiber $E_m \equiv Hom(\mathbb{R}^d, T_mM)$ for each m. If u is a differential curve in $O(M)$, define $\nabla u/ds \in \Gamma_m(E)$, by

$$\frac{\nabla u}{ds}(s) \cdot \xi = \frac{\nabla(u(s)\xi)}{ds} \tag{2.1}$$

for all $\xi \in \mathbb{R}^d$. $u(s)^{-1}\frac{\nabla u}{ds}$ maps $\mathbb{R}^d$ onto itself, i.e. it is a linear transformation on $\mathbb{R}^d$. Because the connection is Riemannian, $u^{-1}\nabla u/ds$ is in $\mathfrak{so}(d)$, the Lie algebra of the Lie group $SO(d)$.

Definition 2.1. *(connection 1-form ω) Define the connection 1-form $\omega = \omega^\nabla$ on $O(M)$ with values in $\mathfrak{so}(d)$ by*

$$\omega(u'(s)) = u(s)^{-1}\frac{\nabla u}{ds}(s)$$

where $u(s)$ is any smooth path in $O(M)$.

Definition 2.2. *(canonical 1-form ϑ) The canonical 1-form on $O(M)$ is the 1-form $\vartheta : T_uO(M) \to \mathbb{R}^d$ given by*

$$\vartheta(\xi) = u^{-1}\pi_*\xi$$

for all $\xi \in T_uO(M)$ and $u \in O(M)$.

Definition 2.3. *(Horizontal vector fields) The standard horizontal vector fields $B(a) \in \Gamma(TO(M))$ for $a \in \mathbb{R}^d$, that is, for each $a \in \mathbb{R}^d$, define a section of vector fields in $TO(M)$ by the following: For each $u \in O(M)$, $B(a)(u)$ is the horizontal lift of $ua \in TM$ to $T_uO(M)$, i.e.*

$$\begin{aligned} B(a):\ O(M) &\to T_uO(M) \\ u &\to \text{lift of } ua \in TM \text{ to } T_uO(M). \end{aligned}$$

Note that B has 2 arguments, $a \in \mathbb{R}^d$ and $u \in O(M)$. Usually the u argument will be suppressed. Alternatively, $B(a)(u)$ is the unique element in $T_uO(M)$ such that

1. *$\pi_* B(a)(u) = ua$ or $\vartheta(B(a)(u)) = a$.*
2. *$\omega(B(a)(u)) = 0$.*

Denote the horizontal tangent space to u by $\mathcal{H}_u$.

Let me now describe the vertical vector fields.

Definition 2.4. *(Vertical vector fields) For each $V \in \mathfrak{so}(d)$, define some kind of lift of this vector to a vector $\tilde{V}$ in $T_uO(M)$ by*

$$\tilde{V}_u = \frac{d}{dt}(u \cdot \exp(tV))|_{t=0}. \tag{2.2}$$

This map, $V \to \tilde{V}$ gives an isomorphism

$$\mathfrak{so}(d) \equiv \mathcal{V}_u$$

where $\mathcal{V}_u$ is the vertical tangent space to u. Note that $\pi_* \tilde{V} = 0$ by definition.

Given a tangent vector $\xi_u \in T_uO(M)$, I can decompose the vector into horizontal and vertical components, i.e.

$$\xi_u = \widetilde{\omega(\xi_u)} + B(\vartheta(\xi_u)).$$

Recall that given $V \in \mathfrak{so}(d)$, $\tilde{V} \in T_uO(M)$ is given by Equation (2.2).

Notation 2.5. *Given $X \in TO(M)$, denote the horizontal component by HX, i.e. $HX = B(\vartheta(X_u))$ and the vertical component by VX, i.e. $VX = \widetilde{\omega(X_u)}$.*

The following formulas are stated without proof.

Proposition 2.6.
1. *$\vartheta(B(a)) = a$ (By definition).*
2. *$\omega(\tilde{V}) = V$.*
3. *$([\tilde{V}, B(a)]) = B(Va)$.*
4. *$\vartheta(\tilde{V}) = 0$. (By definition of $\tilde{V}$, u is fixed for all t. Hence the projection is always πu for all t.)*
5. *$\omega([\tilde{V}, \tilde{W}]) = [V, W]$.*

Since π_* is an isomorphism between $\mathcal{H}_u$ and $T_{\pi u}M$, I can identify

$$(u, B(a)) \in \mathcal{H}_u \longleftrightarrow ua \in T_{\pi u}M.$$

Under this identification, I will write $\omega(ua) := \omega(B(a)(u))$. (I am abusing the notation here, since ω is a 1-form in $TO(M)$. Here, $ua \in T_{\pi u}M$.)

Proposition 2.7. *Identify $\mathcal{H}_u$ with $T_{\pi u}M$. Alternatively, one can replace ω by $\omega(\pi_*^{-1})$. Then*

$$u^{-1}R(ua, ub)uc = (d\omega + \omega \wedge \omega)(ua, ub)c.$$

The proof is omitted.

2.2 Structure Equations

Definition 2.8. (i) *The curvature tensor of ∇ is defined by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, $Z \in \Gamma(TM)$.

(ii) *The torsion tensor of ∇ is defined by*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where X, $Y \in \Gamma(TM)$.

(iii) *The curvature form Ω of ω is the $\mathfrak{so}(d)$-valued 2-form on $O(M)$ defined by*

$$\Omega(X, Y) = d\omega(HX, HY) \equiv (d\omega)^H(X, Y),$$

where X, $Y \in TO(M)$ and HX and HY are the horizontal components of X and Y.

(iii′) *For all $u \in O(M)$ and $a, b \in \mathbb{R}^d$, set*

$$\Omega_u(a, b) = \Omega(B(a)(u), B(b)(u)) \in \mathfrak{so}(d).$$

(iv) *The torsion form Θ of ω is the $\mathbb{R}^d$-valued 2-form on $O(M)$ defined by*

$$\Theta(X, Y) = d\theta^H(X, Y) \equiv d\theta(HX, HY)$$

for all $X, Y \in T_uO(M)$ and $u \in O(M)$.

(iv′) *For all $u \in O(M)$ and $a, b \in \mathbb{R}^d$, set*

$$\Theta_u(a, b) = \Theta(B(a)(u), B(b)(u)) \in \mathbb{R}^d.$$

Lemma 2.9. *(Structure Equations)*

i $\Theta = d\vartheta + \omega \wedge \vartheta$. *(first structure equation);*

ii $\Omega = d\omega + \omega \wedge \omega$. *(second structure equation);*

iii $\Omega_u(a, b) = u^{-1}R(ua, ub)u$ *for all $u \in O(M)$ and $a, b \in \mathbb{R}^d$.*

iv $\Theta_u(a, b) = u^{-1}T(ua, ub)$ *for all $u \in O(M)$ and $a, b \in \mathbb{R}^d$.*

The proof is omitted and can be easily found in texts. For example see Section III, Theorem 2.4 and Section III.5 in [KN96].

2.3 Horizontal Lift

First, I will set the following convention. Derivatives with prime will denote differentiation with respect to s and dot will denote differentiation with respect to t. Note that s will be reserved for the argument of a path, i.e. $\sigma(s)$. Most of the time, I will omit the argument s. In summary,

$$\frac{d}{ds}X(s,t) = X'(s,t)$$
$$\frac{d}{dt}X(s,t) = \dot{X}(s,t).$$

Definition 2.10. *An absolutely continuous path σ has finite energy if*

$$G^1(\sigma',\sigma') := \int_0^1 g\left(\sigma',\sigma'\right) ds < \infty.$$

The following notations are put together for convenience.

Notation 2.11. 1. *Let $H(M)$ be the space of absolutely continuous paths in M with finite energy, starting from o. I will reserve σ for a path in M.*

2. *Let $H(O(M))$ be the space of absolutely continuous paths in $O(M)$ with finite energy, with initial frame u_0. I will reserve u for a path in $O(M)$.*

3. *Let $H(\mathbb{R}^d)$ be the space of absolutely continuous paths in $\mathbb{R}^d$ with finite energy, starting from the origin. I will reserve w and h for a path in $\mathbb{R}^d$. h will denote a vector field in $H(T_oM)$ which I will of course identify with $H(\mathbb{R}^d)$ using u_0.*

Some more definitions.

Definition 2.12. 1. *A path u in $H(O(M))$ is said to be horizontal if $\nabla u(s)/ds = 0$ or equivalently, $\omega(u'(s)) = 0$. Denote the space of absolutely continuous horizontal paths in $O(M)$ by $Hor(O(M))$.*

2. *For a path $\sigma \in H(M)$, define $H(\sigma)$, called the horizontal lift of $\sigma \in H(M)$ to $Hor(O(M))$ by*
$$H : \sigma \longrightarrow H(\sigma) = u \in Hor(O(M))$$
such that
$$\pi u = \sigma.$$

Definition 2.13. *Define a map $\Phi : H(\mathbb{R}^d) \to Hor(O(M))$ as follows: Given a path $w \in H(\mathbb{R}^d)$, define $\Phi(w) = u \in Hor(O(M))$ as the unique solution to the differential equation*
$$u'(s) = B(w'(s))(u(s)),\ u(0) = u_0. \tag{2.3}$$

The following theorem, taken from pages 281 and 282, Theorem 2.1 in [Dri92], will be stated without proof.

Theorem 2.14. *The sets $Hor(O(M))$, $H(M)$ and $H(\mathbb{R}^d)$ are in one to one correspondence. In particular, the map*

$$\Phi : H(\mathbb{R}^d) \to Hor(O(M))$$

and the projection

$$\pi : Hor(O(M)) \to H(M)$$

are bijections. Furthermore, the inverse of π is the horizontal lift map H and $w = \Phi^{-1}(u)$ is given by

$$w(s) = \int_0^s \vartheta(u'(r))dr.$$

Definition 2.15. *(Cartan's Development map) The map $I = \pi \circ \Phi$ is known as the Cartan's Development map.*

Now onto the geometry of the spaces. Pick a path $h \in H(\mathbb{R}^d)$. Note that one should also think of h as a vector field in $H(\mathbb{R}^d)$. For $\sigma \in H(M)$, define a vector field along σ, labelled $X^h(\sigma)$ by

$$X^h(\sigma)(s) = H(\sigma)(s) \cdot h(s),\ s \in [0,1]. \tag{2.4}$$

Now define a flow of this vector field $X^h(\cdot)$ along σ. So this flow will have 2 variables s and t. I will reserve s for the path and t for the flow. Define the flow along X^h, starting at σ to be the solution $\alpha : \mathbb{R} \to H(M)$ by

$$\dot{\alpha}(t) = X^h(\alpha(t)) = H(\alpha(t)) \cdot h,\ \alpha(0) = \sigma. \tag{2.5}$$

Again, I suppress the variable s, which is the variable reserved for a path in $H(M)$. The second condition says that $\alpha(s,0) = \sigma(s)$.

Remark 2.16. *Such a solution α to the functional differential Equation (2.5) exists and is unique. See Remark 2.2 on page 282 in [Dri92].*

The next theorem is very important, because it says how to differentiate a horizontal lift u by X^h. Before I begin, here is another definition.

Notation 2.17. *For a in $H(\mathbb{R}^d)$, define a functional on $H(M)$ by*

$$q_a(s) = \int_0^s \Omega_{u(r)}(a(r), w'(r))dr.$$

The frame u in question here is $u = u_w = \Phi(w)$ and I use the 1-1 correspondence, $\sigma = \pi \circ \Phi(w)$.

Theorem 2.18. *Assume all the notation as above and let $\alpha(t)$ be a solution to the flow equation (2.5). Let $u(t) = H(\alpha(t)) \in Hor(O(M))$ be a horizontal lift of $\alpha(t)$ to the space of horizontal paths in $H(O(M))$. Also define*

$$w(t) = \Phi^{-1}(u(t)) \in H(\mathbb{R}^d).$$

Then $u(t)$ and $w(t)$ both satisfy

$$\dot{u(t)} = -\widetilde{q_h(t)} + B(h)(u(t)) \tag{2.6}$$

and

$$\dot{w}' = q_h w' + \Theta_u(h, w') + h'. \tag{2.7}$$

In terms of all the parameters, Equation (2.6) is written as

$$\frac{\partial u}{\partial t}(s,t) = -u(s,t) \int_0^s \left(\Omega_{u_t}(h, w_t')\right)(r) dr + B(h)(u_t)(s)$$

and Equation (2.7) can be written as

$$\dot{w}'(s,t) = \int_0^s \left(\Omega_{u_t}(h, w_t')\right)(r) dr \cdot w'(s,t) + \left(\Theta_{u_t}(h, w_t')\right)(s) + h'(s).$$

Here, $u_t = u(\cdot, t)$ and $w_t = w(\cdot, t)$.

Remark 2.19. *Only h is independent of t. u, q_h, w are all dependent on t.*

Proof. Note that $\dot{u}$ is a tangent vector in $H(O(M))$. So recall I can split a vector into horizontal and vertical components as follows,

$$\dot{u}(t) = \widetilde{\omega(\dot{u}(t))} + B(\vartheta(\dot{u}(t))).$$

So the proof is just computing $\widetilde{\omega(\dot{u}(t))}$ and $\vartheta(\dot{u}(t))$. Let me first compute $\vartheta(\dot{u}(t))$. By definition, the horizontal lift of $u(t)$ is given by $u(t) = H(\alpha(t))$. By definition, $\pi(u(t)) = \alpha(t)$ and thus

$$\begin{aligned} \vartheta(\dot{u}(t)) &= u(t)^{-1}\pi_*(\dot{u}(t)) = u(t)^{-1}\dot{\alpha}(t) \\ &= u(t)^{-1}H(\alpha(t)) \cdot h = h. \end{aligned}$$

The second equality follows because the push forward of $\dot{u}(t)$ by π is just $\dot{\alpha}(t)$. Now lets compute $\omega(\dot{u}(t))$. Pick any fixed vector $a \in \mathbb{R}^d$, independent of s. Then by definition of ω,

$$\omega(\dot{u}(t))a = u(t)^{-1}\frac{\nabla}{dt}u(t)a,$$

or

$$u(t)\omega(\dot{u}(t))a = \frac{\nabla}{dt}u(t)a.$$

Take covariant derivatives on both sides and since $\nabla u(s,t)/ds = 0$,

$$\begin{aligned} &\frac{\nabla}{ds}\left[u(t)\omega(\dot{u}(t))a\right] = \frac{\nabla}{ds}\frac{\nabla}{dt}u(t)a, \\ \Longrightarrow &\frac{\nabla u(t)}{ds}\omega(\dot{u}(t))a + u(t)\frac{d}{ds}\omega(\dot{u}(t))a = \frac{\nabla}{dt}\frac{\nabla}{ds}u(t)a + \left[\frac{\nabla}{ds}, \frac{\nabla}{dt}\right] u(t)a \\ \Longrightarrow &u(t)\frac{d}{ds}\omega(\dot{u}(t))a = R(\sigma', \dot{\sigma})u(t)a \\ \Longrightarrow &\frac{d}{ds}\omega(\dot{u}(t))a = u(t)^{-1}R(\sigma', \dot{\sigma})u(t)a. \end{aligned}$$

But $\dot{\alpha} = uh$ and by Definition 2.13 and Theorem 2.14, since $u' = B(w')(u)$ and by taking π_*, I have $\alpha' = \pi_* u' = \pi_* B(w')(u) = uw'$. Hence

$$\frac{d}{ds}\omega(\dot{u})a = u^{-1}R(uw', uh)ua = \Omega_u(w', h)a$$

by Lemma 2.9. Integrate with respect to s,

$$\omega(\dot{u})a = \int_0^{\cdot} \Omega_u(w'(r), h(r))a dr = -q_h a$$

or

$$\omega(\dot{u})(s) = \int_0^{s} \Omega_u(w'(r), h(r)) dr = -q_h(s).$$

This completes the proof of Equation (2.6). To prove Equation (2.7), note that by definition of the map Φ, $u' = B(w')(u)$ and hence $w' = \vartheta(u')$. Thus to compute $\dot{w}'$ is the same as computing $d\vartheta(u'(t))/dt$. First note that $u(s)$ is horizontal for each s and hence $\omega(u') = 0$. Secondly, $u(t) = H(\alpha(t))$ and thus $\pi_* \dot{u}(t) = \dot{\alpha}(t) = u(t)h$. Therefore, $\vartheta(\dot{u}) = h$. Finally, since $u' = B(w')(u)$ and hence $\vartheta(u') = w'$. Using structure equations, I have

$$\begin{aligned}
\frac{d}{dt}\vartheta(u') &= d\vartheta(\dot{u}, u') + \frac{d}{ds}\vartheta(\dot{u}) \\
&= \Theta(\dot{u}, u') - \omega \wedge \vartheta(\widetilde{\omega(\dot{u})}, u') + h' \\
&= \Theta(B(h), B(w')) - \omega(\widetilde{\omega(\dot{u})})\vartheta(u') + h' \\
&= \Theta(B(h), B(w')) - \omega(\dot{u})\vartheta(u') + h' \\
&= \Theta_u(h, w') + q_h w' + h'.
\end{aligned}$$

■

3 Geometry of Loop space

To begin with, I first specialize to the based loop space and fix a frame u_0 at some point $o \in M$, i.e. consider only $\{\gamma : [0,1] \to M,\ \gamma(0) = \gamma(1) = o\}$. Denote this space of based loops by $L_o M$. Furthermore, I will only consider oriented manifolds M. Since the calculus on based path space is so well understood, I will capitalize on it and view the space of based loops as a submanifold in based path space. The advantage of this point of view is obvious. Some of the results in based path space carry over with minor modifications. Last but not least, all the loops are considered to be smooth.

Unfortunately, there is a price to pay. When I consider based loops at o, $L_o M$, I am in fact fixing a parametrization of the loop, identifying a 'starting' point and frame, denoted by u_0. To be more precise, a parametrized loop γ maps $\{e^{2\pi i s} : s \in [0,1]\}$ to M. As such, I can label the argument in γ as s and $s = 0$ will be the starting point. At the end of the day I have to make sure that any definition made, is independent of the parametrization used.

I will now show how to locally trivialize TLM. Note that here, I am considering free loop space LM, not based loop space. Fix a loop $\gamma_0 \in LM$ and pick any point on it, call it $o := \gamma_0(0)$, hence parametrizing the loop. I will abuse notation and write $\{\gamma(s) : 0 \le s \le 1,\ \gamma(0) = \gamma(1)\}$ to mean a parametrized loop. Assign a local frame u_0 in an open neighborhood B in M containing $\gamma_0(0) = o$. Specifically, choose a local orthonormal frame field $\{f_1, \ldots, f_d\}$ in the neighborhood B with the identification $f_i(x) = u_0(x)e_i$, $x \in B$. Here, $\{e_1, \ldots, e_d\}$ is the canonical orthonormal basis in $\mathbb{R}^d$.

For any parametrized loop $\{\gamma(s) : 0 \le s \le 1,\ \gamma(0) = \gamma(1) \in B\} \in \bigcup_{x\in B} L_x M$, there is a unique horizontal lift of γ in $H(SO(M))$, that traverse along γ once. Call this lift $u_\gamma(\cdot) \equiv \{u_\gamma(s) : 0 \le s \le 1\}$, with $u_\gamma(0) = u_0$. Another description of the horizontal lift is that it is the parallel translation operator along γ. As the path traverse one round, $u_\gamma(1) : \mathbb{R}^d \to T_o M$ may not be equal to u_0.

Definition 3.1. *For each $\{\gamma(s) : 0 \le s \le 1,\ \gamma(0) = \gamma(1) \in B\} \in \bigcup_{x\in B} L_x M$, define the holonomy operator $h_\gamma \in SO(d)$ such that $u_\gamma(1) = u_\gamma(0)h_\gamma$, where u_γ is the unique horizontal lift of γ, with an initial frame $u_0 \equiv u_\gamma(0) : \mathbb{R}^d \to T_o M$.*

Remark 3.2. *This definition is dependent on the initial frame u_0 used, up to conjugacy by $SO(d)$.*

Now this holonomy operator $h : \bigcup_{x\in B} L_x M \to SO(d)$ defines a continuous smooth map on the compact-open topology. Set $g_0 := h_{\gamma_0}$ and note that $\gamma_0 \in L_o M$. As $SO(d)$ is a compact connected Lie group, there is a $\xi_0 \in \mathfrak{so}(d)$ such that $\exp(\xi_0) = g_0$.

Now the exponential map may not be a local diffeomorphism at ξ_0. However, it is a local diffeomorphism at the origin. Choose an open set U containing 0 in the Lie algebra $\mathfrak{so}(d)$ such that the exponential map is a diffeomorphism onto an open set $V = \exp(U)$ containing the identity e of $SO(d)$.

Consider the path $\alpha : s \in [0,1] \mapsto \exp(s\xi_0)$, a path joining e to g_0. Let L_g denote left multiplication by g. For each $\xi \in U \subseteq \mathfrak{so}(d)$, define a left invariant vector field $L_{g,*}\xi$ on $SO(d)$. Solve the flow equation for $\beta_s(\xi, \cdot) \equiv \beta_s(\cdot)$,

$$\dot{\beta}_s(t) = sL_{\beta_s(t),*}\xi,\ \beta_s(0) = \alpha_s. \tag{3.1}$$

The solution is given by

$$\beta_s(\xi, t) = \exp(s\xi_0)\exp(st\xi),\ s, t \in [0,1]. \tag{3.2}$$

Each $\xi \in U \subseteq \mathfrak{so}(d)$ defines a path $0 \le s \le 1 \mapsto \beta_s(\xi, 1)$ which joins the identity e in $SO(d)$ to each $g \in g_0 V \subseteq SO(d)$. Here, $\xi \in U \subseteq \mathfrak{so}(d)$ such that $g = g_0 \exp(\xi)$. h is a continuous map from $\bigcup_{x\in B} L_x M$ to $SO(d)$, O containing γ_0. Without loss of generality, assume that $O \subseteq h^{-1}(g_0 V)$.

Notation 3.3. *Define for each loop $\gamma \in O$, a skew symmetric matrix $\xi(\gamma) \in U \subseteq \mathfrak{so}(d)$ such that*

$$\gamma \in O \subseteq \bigcup_{x\in B} L_x M \to \{\beta_s(\xi(\gamma), 1) \in SO(d) : s \ge 0\},$$

where $\beta_1(\xi(\gamma), 1) = h_\gamma = g_0 \exp(\xi(\gamma))$ for $\gamma \in O$ and $g_0 = h_{\gamma_0}$. β solves Equation (3.1) and is given explicitly by (3.2). This choice of β is not canonical.

Write for each γ in a neighborhood $O \subseteq \bigcup_{x\in B} L_x M$,

$$\eta_{\xi(\gamma)}(s) := \beta_s(\xi(\gamma), 1)^{-1},\ s \ge 0.$$

In future, I will drop the variable γ and the s variable. It should be understood that η_ξ is a path.

Definition 3.4. *Refer to Notation 3.3. Choose a small neighborhood $B \subseteq M$ such that an orthonormal frame field u_0 can be defined over B. Fix a loop γ_0 such that $\gamma_0(0) \in B$. Now define a local frame r for some neighborhood $O \subseteq \bigcup_{x\in B} L_x M$, for $0 \le s \le 1$,*

$$\gamma \in O \mapsto r(\gamma, s) \equiv r^{\xi}(\gamma, s) := u_\gamma(s)\eta_{\xi(\gamma)}(s). \tag{3.3}$$

Note that ξ depends on γ and the initial local frame u_0 over a neighborhood $B \subseteq M$. u_γ is the unique horizontal lift of γ, with an initial frame $u_0 \equiv u_\gamma(0) : \mathbb{R}^d \to T_{\pi(u_0)}M$, $\pi : SO(M) \to M$. Thus, I have a local trivialization of $T\bigcup_{x\in B} L_x M$ using

$$(\gamma, v) \in O \times L\mathbb{R}^d \to \{(\gamma, r^{\xi}v)(s) : 0 \le s \le 1,\ \gamma(0) = \gamma(1) \in B\} \in T_\gamma \bigcup_{x\in B} L_x M.$$

This local frame r^ξ is defined only in a neighborhood O containing some chosen and fixed loop γ_0. It depends on the choice of $\xi_0 \in \mathfrak{so}(d)$, which as a reminder, is defined by $h_{\gamma_0} = \exp(\xi_0)$.

Definition 3.5. *In future, drop ξ from r^ξ and write it as $r = u\eta_\xi$. Define $\Psi(\gamma) = (H\eta_\xi)(\gamma)$. Recall $H(\gamma)$ is the horizontal lift of γ, with starting frame u_0. Thus, Ψ maps a loop in $L_o M$ to a loop in $L_{u_o} SO(M)$ using a local trivialization. Of course, this map depends on the choice of ξ.*

Definition 3.6. *Let $\rho(\cdot, t)$ be a family of loops in the free loop space LM such that $\rho(\cdot, 0) = \gamma(\cdot)$ and $\partial\rho(\cdot, t)/\partial t|_{t=0} = \nu(\cdot)$, $\nu(\cdot) \in T_\gamma LM$. Define for a smooth function $F : LM \to S$, S a manifold,*

$$D_\nu F(\gamma) := \frac{\partial F(\rho(\cdot, t))}{\partial t}\Big|_{t=0} \in T_{F(\gamma)}S.$$

Now suppose I fix a point $o \in M$ and a frame u_o at o. Given a based loop $\gamma \in L_o M$ and a tangent field $\nu \in T_\gamma L_o M$, I want to compute $D_\nu r \in TLSO(M)$, and for each $0 \le s \le 1$, $D_\nu r(s) \in TSO(M)$ has horizontal and vertical components.

Definition 3.7. *Let v be a loop in $H(\mathbb{R}^d)$ and $\nu = rv$, r as defined in Definition 3.4. Define a skew symmetric matrix path A, as*

$$A(v)(\cdot) \equiv A_v(\cdot) = \beta(\xi(\gamma), 1) D_\nu \beta^{-1}(\xi(\gamma), 1).$$

Do note that $A_v(\cdot) \equiv A_v(\gamma, \cdot)$ is dependent on γ for $\gamma \in O \subseteq LM$, since ξ is dependent on γ.

The vertical vector field is given by $r^{-1}\nabla^M_\nu r(\cdot)$ (See Equation (2.1).) and using Equation (2.6), is

$$\eta_\xi^{-1} u^{-1}(\nabla_\nu u)\eta_\xi + \eta_\xi^{-1} D_\nu \eta_\xi = A_v - \eta_\xi^{-1} q_{\eta_\xi v}\eta_\xi.$$

The horizontal vector field, however is the unique vector field such that

$$r\vartheta(D_\nu r) = \pi_* D_\nu r = rv,$$

and thus $\vartheta(D_\nu r) = v$. Let me summarize it as an analog of Theorem 2.18.

Corollary 3.8. *Fix a point $o \in M$ and a frame u_0 at o and consider the based loop space L_oM. Let $\gamma \in L_oM$ and $\nu(\gamma,\cdot) \in T_\gamma L_oM$ with $\nu(\gamma,\cdot) = (rv)(\gamma,\cdot) = (u\eta_\xi v)(\gamma,\cdot)$, $v(\gamma,s) \in L\mathbb{R}^d$ for $0 \le s \le 1$ with $v(0) = 0$. Refer to Definition 3.4 for the definition of r. Then*

$$\vartheta(D_\nu r)(\gamma,\cdot) = v(\gamma,\cdot),$$
$$\omega(D_\nu r)(\gamma,\cdot) = A_v(\gamma,\cdot) - \eta^{-1}_{\xi(\gamma)}(\cdot)\int_0^\cdot u_\gamma^{-1}(r)R(\nu(\gamma,r),\gamma'(r))u_\gamma(r)dr\cdot\eta_{\xi(\gamma)}(\cdot).$$

3.1 Covariant Derivative on Free Loop Space

I am now ready to do some analysis on free loop space. The obvious difference from the analysis on based loop space is that the starting point $\gamma(0)$ is allowed to vary. Although not stated explicitly, in the case of based paths (loops), the vector fields along the path (loop) X are such that $X(0) = 0$. When considering free path (loop) space, there is no such restriction.

In the case of based paths (loops), the starting frame u_0 at $\gamma(0)$ is fixed. On free paths (loops), the starting frame u_0 is allowed to vary. This does not change much of the analysis done earlier on, except that now I have to include the derivative of u_0 using the Levi-Civita connection in the earlier computations.

Fix a $\gamma_0 \in LM$. Recall I fix a point $o = \gamma_0(0)$ and choose a local frame u_0 over some open neighborhood $o \in U \subseteq M$. Then, there exists an open neighborhood $\gamma_0 \in O \subseteq \bigcup_{x\in M} L_xM$, such that there is a local trivialization of $T\bigcup_{x\in U} L_xM$ using

$$(\gamma,v) \in O \times L\mathbb{R}^d \to (\gamma, rv) \in T_\gamma \bigcup_{x\in U} L_xM,\ r = u\eta_\xi.$$

$u(\cdot) \equiv u_\gamma(\cdot)$ is the unique horizontal lift of γ, with initial frame $u_0(\gamma(0)) = u_\gamma(0)$ and $\gamma(0) \in U$. Refer to Equation (3.3) in Definition 3.4.

Definition 3.9. *Let $\gamma_0 \in O$ for some open neighborhood $O \subseteq \bigcup_{x\in U} L_xM$ and suppose $\{r(s) = u(s)\eta_\xi(s) : 0 \le s \le 1\}$ is a local trivialization frame over O with u_0 a local orthonormal frame field over $U \subseteq M$. Let $\{z(s) : 0 \le s \le 1,\ z(0) = z(1)\} \in L\mathbb{R}^d$ and $Z(\cdot) := (rz)(\cdot)$. Define a connection 1-form ω for $0 \le s \le 1$ and $\gamma \in O$,*

$$\begin{aligned}\omega_z(s) \equiv{}&\omega_z(\gamma,s)\\ :={}&\eta^{-1}_{\xi(\gamma)}(s)\lambda_z\eta_{\xi(\gamma)}(s) + A_z(\gamma,s)\\ &- \eta^{-1}_{\xi(\gamma)}(s)\int_0^s u_\gamma^{-1}(r)R(Z(\gamma,r),\gamma'(r))u_\gamma(r)dr\cdot\eta_{\xi(\gamma)}(s),\end{aligned} \tag{3.4}$$

whereby $\lambda_z := u_0^{-1}\nabla^M_{u_0z(0)}u_0$.

Note that $\lambda_z = u_0^{-1}\nabla^M_{u_0z(0)}u_0$ is the connection 1-form in the direction $u_0z(0)$. Simply put, it is the connection 1-form for the base point, at time 0. Some results done earlier transfer over to free paths (loops) by adding this additional term λ. For example, $\eta_\xi^{-1}\lambda_v\eta_\xi$ should be added to the vertical component of the tangent field in Corollary 3.8, i.e.

$$\omega(D_Zr) = \eta_\xi^{-1}\lambda_z\eta_\xi + A_z - \eta_\xi^{-1}\int_0^\cdot u^{-1}(r)R(Z(r),\gamma'(r))u(r)dr\cdot\eta_\xi.$$

As a result, the connection 1-form $\omega_z(s) = r^{-1}(s)\nabla^M_{Z(s)} r(s)$, the vertical component of $(D_Z r)(s)$, for $0 \leq s \leq 1$. Here, $\nabla^M_{Z(s)} r(s)$ is given by Equation (2.1).

Continue the above set up, define a covariant derivative on free loop space as follows.

Definition 3.10. *(Covariant derivative ∇^{LM} on free loop space) Let Z_i, $i = 1,2$ be smooth sections of TLM. Define a local trivialization over $O \subseteq \bigcup_{x \in U} L_x M$, $\{r(\gamma, s) : 0 \leq s \leq 1\}$ as in Definition 3.4. Thus the vector fields Z_i can be written as $\{Z_i(\gamma, s) = r(\gamma, s) z_i(\gamma, s) : 0 \leq s \leq 1\}$ for $\gamma \in O$. Here, $z_i : (\gamma, s) \in O \times [0,1] \mapsto z_i(\gamma, s) \in T_{\gamma(s)} M$ is smooth and $z_i(\gamma, 0) = z_i(\gamma, 1)$ for any $\gamma \in O$.*

Define the covariant derivative ∇^{LM} as, for each $0 \leq s \leq 1$,

$$(\nabla^{LM}_{Z_1} Z_2)(\gamma, s) = r(\gamma, s) \cdot [D_{Z_1} z_2(\gamma, s) + \omega_{z_1}(\gamma, s) z_2(\gamma, s)],$$

whereby $\omega_{z_1}(\cdot) \equiv \omega_{z_1}(\gamma, \cdot)$ is given by Equation (3.4). To clarify,

$$D_{Z_1} z_2(\gamma, s) := \frac{\partial}{\partial t} z_2(\alpha(t, s))\Big|_{t=0}$$

such that $\{\alpha(t, \cdot) : -\epsilon \leq t \leq \epsilon,\ \epsilon > 0\}$ is any family of loops in O with $\partial \alpha(t, s)/\partial t|_{t=0} = Z_1(\gamma, s)$ for $0 \leq s \leq 1$.

The covariant derivative is defined, based on the choice of r, which fixes a starting point $\gamma(0)$ for γ, hence parametrizing the field Z_i, $i = 1,2$. The next lemma says that the covariant derivative is well-defined.

Lemma 3.11. *∇^{LM} is well-defined, independent of the frame r used.*

Proof. Let O be open in $\bigcup_{x \in U} L_x M$ and $\{r(\gamma, s) = u_\gamma(s) \eta_{\xi(\gamma)}(s) : 0 \leq s \leq 1\}$ be a frame over any $\gamma \in O$, with a starting frame $u_\gamma(0)$ at $\pi(u_\gamma(0)) \in M$, $\pi : SO(M) \rightarrow M$. Let $0 \leq \tau \leq 1$ and suppose a new frame $\hat{r}(\gamma, s) = \hat{u}_\gamma(s) \hat{\eta}_{\xi(\gamma)}(s)$ over $\gamma \in \hat{O} \subseteq \bigcup_{x \in U} L_x M$ is chosen, such that at $s = 0$, I begin with a frame $\hat{u}_\gamma(0)$ at $\pi(\hat{u}_\gamma(0)) \in M$. Without loss of generality, by taking the intersection of O and $\hat{O}$ if necessary, I will assume $O = \hat{O}$. The frame $\hat{r}$ can be written for $\gamma \in O$ and $0 \leq s \leq 1$,

$$\hat{r}(\gamma, s) = u_\gamma(s + \tau) \eta_{\xi(\gamma)}(s + \tau) \rho(\gamma, s) = r(\gamma, s + \tau) \rho(\gamma, s),\ \rho(\gamma, s) \in SO(d).$$

Let Z_i, $i = 1,2$ be 2 smooth sections in TLM. For $0 \leq s \leq 1$, in terms of the frame r, $Z_i(\gamma, s) = r(\gamma, s) z_i(\gamma, s)$ for $\gamma \in O$, but in terms of $\hat{r}$,

$$\hat{Z}_i(\gamma, s) = (\hat{r}\rho^{-1})(\gamma, s) z_i(\gamma, s + \tau) = r(\gamma, s + \tau) z_i(s + \tau) = Z_i(\gamma, s + \tau).$$

Write $z_{i,\tau}(\gamma, s) = z_i(\gamma, s + \tau)$, $r_\tau(\gamma, s) = r(\gamma, s + \tau)$ for $0 \leq s \leq 1$. According to Definition 3.10, for $0 \leq s \leq 1$,

$$\begin{aligned}(\nabla^{LM}_{\hat{Z}_1} \hat{Z}_2)(\gamma, s) =& \hat{r}(\gamma, s)[\rho^{-1}(s) D_{\hat{Z}_1} z_{2,\tau}(s) + (D_{\hat{Z}_1} \rho^{-1})(s) \cdot z_{2,\tau}(s) \\ &+ (\hat{r}^{-1} \nabla^M_{\hat{Z}_1} \hat{r})(s) \cdot \rho^{-1}(s) z_{2,\tau}(s)](\gamma) \\ =& r_\tau(\gamma, s)[D_{\hat{Z}_1} z_{2,\tau}(s) + (r_\tau^{-1} \nabla^M_{\hat{Z}_1} r_\tau)(s) \cdot z_{2,\tau}(s)](\gamma) \\ =& (\nabla^{LM}_{Z_1} Z_2)(\gamma, s + \tau).\end{aligned}$$

∎

This quantity $A_x - \eta_\xi^{-1} q_{\eta_\xi x} \eta_\xi$ appears often in the analysis such that it deserves a separate symbol. This quantity appears because of the derivative of the holonomy operator, which happens only for loop space.

Notation 3.12. *Let $O \subseteq \bigcup_{x\in U} L_x M$ as in Definition 3.4. Recall $r = u\eta_\xi$ and write $\Gamma_u := \Omega_u(\eta_\xi\cdot, \eta_\xi\cdot)$. I will write the vertical component as*

$$\Lambda_z = A_z - \eta_\xi^{-1} \int_0^{\cdot} \Gamma_u(z(r), b'(r))dr \cdot \eta_\xi.$$

Here, $rb' = u\eta_\xi b' = \gamma'$, $\gamma = \pi r \in O$. Hence, $\omega_z = \eta_\xi^{-1}\lambda_z \eta_\xi + \Lambda_z$ in Equation (3.4).

3.2 Lie Bracket

To compute the Lie Bracket amounts to compute the Lie Bracket over $\bigcup_{x\in U} L_x M$, $U \subseteq M$. Let $X, Y \in TLM$.

The reference is taken from Pages 148 to 150 in [CM02]. Throughout this subsection, fix a γ_0 and a starting point $o = \gamma_0(0)$. Choose an open neighborhood $o \in U \subseteq M$, fix an orthonormal frame u_0 over $U \subseteq M$ and hence define a neighborhood $O \subseteq \bigcup_{x\in U} L_x M$ and a frame $r = u\eta_\xi$ for $\gamma \in O$, as in Definition 3.4.

Suppose $X = rx$ and $Y = ry$. Assume that x and y, tangent fields in $H(\mathbb{R}^d)$, are independent of γ. Since I have a choice of the starting frame u_0, assume that $\lambda_x = \lambda_y = 0$.

Definition 3.13. *(Cylinder functions) A function f is a cylinder function on $H(M)$ if $f(\sigma) = \tilde{f}(\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_k))$, where $\tilde{f} : M \times \cdots \times M \to \mathbb{R}$. Now by projection π, consider f as a cylinder function on $H(O(M))$*

$$f(u) = \tilde{f}(\pi u(s_1), \pi u(s_2), \ldots \pi u(s_k)).$$

Thus $f(u.)$ is a function from $H(\mathbb{R}^d) \to H(O(M))$. Therefore, only consider in general, cylinder functions f on $H(O(M))$.

For a cylinder function on $\bigcup_{x\in U} L_x M$, $f(\gamma) = F(\gamma(s_1), \ldots, \gamma(s_n))$, lift it to a function on $\bigcup_{x\in U} L_{u_0(x)} SO(M)$ using $\pi : L_{u_0(x)} SO(M) \to L_x M$ if $x = \pi u_0(x)$, i.e.

$$(f \circ \pi)(r_\gamma) = (F \circ \pi)(r_\gamma(s_1), \ldots, r_\gamma(s_n)) := \tilde{F}(r_\gamma(s_1), \ldots, r_\gamma(s_n)).$$

Note that $\tilde{F}$ is a function on $\bigcup_{x\in U} L_{u_0(x)} SO(M)$. Since cylinder functions are dense, it suffices to compute the Lie bracket on cylinder functions.

Recall $\Psi = H\eta_\xi$, H is horizontal lift. Since $F = F \circ \pi \circ \Psi = \tilde{F} \circ \Psi$, using Corollary 3.8,

$$D_X(\tilde{F} \circ \Psi)(\gamma) = x^\alpha(s_k)(\partial^k_{B_\alpha} \tilde{F})(r_\gamma(s_1), \ldots, r_\gamma(s_n)),$$

where I adapt Einstein's summation notation and $B^k_\alpha = B(e_\alpha)(r(s_k))$ is the canonical horizontal vector field at $r(s_k) = u\eta_\xi(s_k)$. Note that there is no derivation by a vertical tangent vector since the function is an honest cylinder function on $H(M)$. However, note that the derivation $\partial^k_{B_\alpha} \tilde{F}$ is a cylinder function on $\bigcup_{x\in U} L_{u_0(x)} SO(M)$.

Now differentiate a second time, using D_Y, again using Corollary 3.8. Note that I have to differentiate using the vertical tangent vector,

$$\begin{aligned} D_Y D_X(\tilde{F}\circ\Psi)(\gamma) =& y^{\beta}(s_j)x^{\alpha}(s_k)\partial^k_{B_\beta}\partial^i_{B_\alpha}\tilde{F}(r_\gamma(s_1),\ldots,r_\gamma(s_n)) \\ &+ x^{\alpha}(s_k)\partial^k_{\widetilde{A_y - \eta_\xi^{-1} q_{\eta_\xi y}\eta_\xi}}\partial^i_{B_\alpha}\tilde{F}(r_\gamma(s_1),\ldots,r_\gamma(s_n)). \end{aligned}$$

Write $\eta_\xi^{-1} q_{\eta_\xi y}\eta_\xi = p_y$. So I have

$$\begin{aligned} &[D_Y D_X - D_X D_Y](\tilde{F}\circ\Psi) \\ &= y^{\beta}(s_j)x^{\alpha}(s_k)\left[\partial^k_{B_\alpha}\partial^i_{B_\beta} - \partial^k_{B_\beta}\partial^i_{B_\alpha}\right]\tilde{F}(r(s_1),\ldots,r(s_n)) \\ &\quad + \left[x^{\alpha}(s_k)\partial^k_{\widetilde{A_y}-\widetilde{p_y}}\partial^i_{B_\alpha} - y^{\alpha}(s_k)\partial^k_{\widetilde{A_x}-\widetilde{p_x}}\partial^i_{B_\alpha}\right]\tilde{F}(r(s_1),\ldots,r(s_n)). \end{aligned} \tag{3.5}$$

If $j \neq k$, then $\partial^k_{B_\alpha}$ and $\partial^j_{B_\beta}$ commute. So only consider $k = j$. Since B_α and B_β are horizontal vector fields, $[B_\alpha, B_\beta]$ is vertical. To show this, compute $\pi_*[B_\alpha, B_\beta] = [\pi_* B_\alpha, \pi_* B_\beta]$. This follows from the Torsion free of Levi-Civita covariant derivative on M, $\nabla^M_X Y - \nabla^M_Y X = [X, Y]$. As horizontal vector fields, $\nabla^M_{\pi_* B_\alpha}\pi_* B_\beta = \nabla^M_{\pi_* B_\beta}\pi_* B_\alpha = 0$. Hence $[\pi_* B_\alpha, \pi_* B_\beta] = 0$. But $\tilde{F}$ only depends on $\pi(u)$, hence this term vanishes.

The first sum in Equation (3.5) vanishes. For the last sum, same reasoning: if $j \neq k$ then the fields commute. So assume $j = k$. Use one of the useful formulas, $[\tilde{V}, B(a)] = B(Va)$ for a skew symmetric matrix V and a vector $a \in \mathbb{R}^d$. Apply this formula, then

$$\left[\partial^j_{\widetilde{A_x}-\widetilde{p_x}}, \partial^j_{B_\alpha}\right] = \partial^j_{(A_x - p_x)e_\alpha}.$$

But on $\tilde{F}$, $\partial^j_{B_\alpha}\partial^j_{\widetilde{p_x}}\tilde{F} = 0$. Hence

$$\partial^j_{\widetilde{p_x}}\partial^j_{B_\alpha}\tilde{F} = \left[\partial^j_{\widetilde{p_x}}, \partial^j_{B_\alpha}\right]\tilde{F} = \partial^j_{B((A_x - p_x)e_\alpha)}\tilde{F}.$$

Therefore, the second sum in Equation (3.5) becomes

$$\begin{aligned} &\left[x^{\alpha}(s_k)\partial^k_{\widetilde{A_y}-\widetilde{p_y}}\partial^i_{B_\alpha} - y^{\alpha}(s_k)\partial^k_{\widetilde{A_x}-\widetilde{p_x}}\partial^i_{B_\alpha}\right]\tilde{F} \\ &\quad = \left[x^{\alpha}(s_j)\partial^j_{B((A_y - p_y)e_\alpha)} - y^{\alpha}(s_j)\partial^j_{B((A_x - p_x)e_\alpha)}\right]\tilde{F} \\ &\quad = \left[\partial^j_{B((A_y - p_y)x)(s_j)} - \partial^j_{B((A_x - p_x)y)(s_j)}\right]\tilde{F}. \end{aligned}$$

So if I plug into Equation (3.5),

$$[D_Y, D_X](\tilde{F}\circ\Psi) = \sum_{j=1}^{n}\left(\partial^j_{B(A_y x - A_x y)(s_j)} - \partial^j_{B(p_y x - p_x y)(s_j)}\right)(\tilde{F}\circ\Psi).$$

Since F is a cylinder function on $H(M)$, I have

$$\begin{aligned}
[D_Y, D_X]F &= [D_Y, D_X](\tilde{F} \circ \Psi) \\
&= \left(\left[\partial^i_{\pi_* B(A_y x)(s_j)} - \partial^i_{\pi_* B(A_x y)(s_j)} \right] - \left[\partial^j_{\pi_* B(p_y x)(s_j)} - \partial^j_{\pi_* B(p_x y)(s_j)} \right] \right) F \\
&= \left(\left[\partial^j_{r A_y x(s_j)} - \partial^j_{r A_x y(s_j)} \right] - \left[\partial^j_{r p_y x(s_j)} - \partial^j_{r p_x y(s_j)} \right] \right) F.
\end{aligned}$$

Thus the following result.

Theorem 3.14. *The Lie bracket of 2 tangent fields, X and Y, is given by*

$$[X, Y] = u\eta_\xi \left(A_x y - A_y x - \left(\eta_\xi^{-1} q_{\eta_\xi x} \eta_\xi y - \eta_\xi^{-1} q_{\eta_\xi y} \eta_\xi x \right) \right).$$

More generally, the Lie bracket of $X = rx$ and $Y = ry$ is given by, under r,

$$\begin{aligned}
[X, Y] &= r \left(\eta_\xi^{-1} \lambda_x \eta_\xi y - \eta_\xi^{-1} \lambda_y \eta_\xi x + A_x y - A_y x - \eta_\xi^{-1} [q_{\eta_\xi x} \eta_\xi y + q_{\eta_\xi y} \eta_\xi x] \right) \\
&= r \left(\omega_x y - \omega_y x \right).
\end{aligned}$$

Definition 3.15. *(Metric on LM.) Define a G^0 metric on TLM by $G^0(X, Y) = \int_0^1 g(X(s), Y(s)) ds$.*

With this definition of the metric, we have the following corollary.

Corollary 3.16. *The connection ∇^{LM} is the Levi-Civita connection.*

3.3 Curvature

Throughout this subsection, fix a $\gamma_0 \in LM$ and a starting point $o = \gamma_0(0)$. Choose an open neighborhood $o \in U \subseteq M$, fix an orthonormal frame u_0 over $U \subseteq M$ and hence define a neighborhood $O \subseteq \bigcup_{x \in U} L_x M$ and a frame $r = u\eta_\xi$ dependent on each $\gamma \in O$, as in Definition 3.4. The goal in this subsection is to compute the curvature at the loop γ_0.

The calculations in this subsection follow those in Section 3 of [Fan01]. Now γ' is a tangent field along the loop γ, so I can write for each $\gamma \in O \subseteq \bigcup_{x \in U} L_x M$, $\gamma' = r(\gamma)\mu'$, for some loop vector field $\mu' \in L\mathbb{R}^d$. Define a map $b' : \gamma \in O \subseteq \bigcup_{x \in U} L_x M \mapsto \mu' \in L\mathbb{R}^d$. To find $D_v b'$ requires going back to the development map, where $v \in TLM$.

The development map I maps $H(\mathbb{R}^d)$ to $H(M)$ and is a diffeomorphism. So, write $I(w) = \gamma \in O \subseteq \bigcup_{x \in U} L_x M$. Note that w may not be a loop. Their relationship is

$$\gamma'(s) = u_\gamma(s) w'(s).$$

The anti-development map is given by

$$I^{-1}(\gamma) = \int_0^\cdot u_\gamma^{-1} \gamma' dr.$$

Differentiate,

$$(I^{-1})'(\gamma) = u_\gamma^{-1} \gamma' = \eta_{\xi(\gamma)} \mu',$$

or by definition of b',

$$(I^{-1})' = \eta_\xi b'.$$

So, $b' = \eta_\xi^{-1}(I^{-1})'$.

Lemma 3.17. *Continue with the notations as above. Assume the connection on M is torsion free. Then*

$$D_v b' = v' + \eta_\xi^{-1}\eta_\xi' v - \Lambda_v b', \tag{3.6}$$

where $\nu = rv$ *with* $v \in L\mathbb{R}^d$.

Proof. Torsion free implies that $\Theta \equiv 0$. From Equation (2.7), let $\psi' = q_h w' + h'$ and write $D_{q,h}$ as the tangent vector associated to ψ, i.e.

$$D_{q,h}F := \frac{d}{dt}\Big|_{t=0} F\left(\int_0^\cdot e^{tq_h(s)} w'(s)ds + th(\cdot)\right)$$

for any smooth function on $H(\mathbb{R}^d)$. Recall that $I = \pi \circ \Phi$. Hence by Theorem 2.18,

$$\begin{aligned}(D_\nu b') \circ I =& D_{q,\eta_\xi v}(b' \circ I) = D_{q,\eta_\xi v}(\eta_\xi^{-1} \circ I \cdot (I^{-1})' \circ I) \\ =& (D_{q,\eta_\xi v}(\eta_\xi^{-1} \circ I)) \cdot (I^{-1})' \circ I + \eta_\xi^{-1} \circ I \cdot D_{q,\eta_\xi v}((I^{-1})' \circ I) \\ =& (D_\nu \eta_\xi^{-1}) \circ I \cdot (I^{-1})' \circ I + \eta_\xi^{-1} \circ I \cdot D_{q,\eta_\xi v}((I^{-1})' \circ I) \\ =& (-A_v \eta_\xi^{-1}(I^{-1})') \circ I + \eta_\xi^{-1} \circ I \cdot D_{q,\eta_\xi v}((I^{-1})' \circ I).\end{aligned}$$

But

$$(I^{-1})' \circ I(w) = u_{I(w)}^{-1} I(w)' = w'.$$

So $(I^{-1})' \circ I : w \to w'$. Therefore, by definition,

$$D_{q,\eta_\xi v}((I^{-1})' \circ I)(w) = D_{q,\eta_\xi v} w' = (\eta_\xi v)' + q_{\eta_\xi v} w'.$$

Hence

$$\begin{aligned}\eta_\xi^{-1} D_{q,\eta_\xi}(I^{-1})' &= v' + \eta_\xi^{-1}\eta_\xi' v + \eta_\xi^{-1} q_{\eta_\xi v} u^{-1}\gamma' \\ &= v' + \eta_\xi^{-1}\eta_\xi' v + \eta_\xi^{-1} q_{\eta_\xi v}\eta_\xi b'.\end{aligned}$$

Therefore,

$$D_\nu b' = v' + \eta_\xi^{-1}\eta_\xi' v - A_v b' + \eta_\xi^{-1} q_{\eta_\xi v}\eta_\xi b'$$

or

$$D_\nu b' = v' + \eta_\xi^{-1}\eta_\xi' v - \Lambda_v b'.$$

Recall I fix a loop $\gamma_0 \in O$. Choose an orthonormal frame field $\{e_a\}_a$ over $U \subseteq M$ such that at the base point $\gamma_0(0) = o \in M$, $\nabla_{e_a}^M e_b = 0$. Hence $\lambda_h \equiv 0$, $h \in L\mathbb{R}^d$ in Definition 3.9, throughout this subsection.

Lemma 3.18. *Suppose for $\gamma \in O$, $\gamma' = (rb')(\gamma)$, $r = u\eta_\xi$ as in Definition 3.4. Write $\nu = rv$ and $\mu = rw$. Given $v(\cdot), w(\cdot) \in L(\mathbb{R}^d)$ independent of $\gamma \in O$,*

$$\begin{aligned}
&D_\nu \left(A_w - \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, b')(\tau) d\tau \cdot \eta_\xi \right) \\
&= \left[A_v, \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, b')(\tau) d\tau \cdot \eta_\xi \right] - \eta_\xi^{-1} \int_0^\cdot [\eta_\xi (\partial_{B(v)} \Omega_r) \eta_\xi^{-1} (w, b')](\tau) d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1} \int_0^\cdot \left[q_{\eta_\xi v}, \Gamma_u(w, b') \right] (\tau) d\tau \cdot \eta_\xi - \eta_\xi^{-1} \int_0^\cdot \Gamma_u(\Lambda_v w, b')(\tau) d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, v')(\tau) d\tau \cdot \eta_\xi - \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, \eta_\xi^{-1} \eta_\xi' v)(\tau) d\tau \cdot \eta_\xi \\
&\quad - A_v A_w + \eta_\xi^{-1} D_\nu D_\mu \eta_\xi.
\end{aligned}$$

Proof.

$$\begin{aligned}
&D_\nu \left(\eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, b')(\tau) d\tau \cdot \eta_\xi \right) \\
&= - A_v \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, b')(\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, b')(\tau) d\tau \cdot \eta_\xi A_v \\
&\quad + \eta_\xi^{-1} \int_0^\cdot (D_\nu \Gamma_u)(w, b')(\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, D_\nu b')(\tau) d\tau \cdot \eta_\xi.
\end{aligned}$$

Now write $\Gamma_u = \eta_\xi \Omega_r \eta_\xi^{-1}$, so $\Omega_r = \eta_\xi^{-1} \Gamma_u \eta_\xi$. Then,

$$D_\nu \Gamma_u = (D_\nu \eta_\xi) \Omega_r \eta_\xi^{-1} + \eta_\xi (D_\nu \Omega_r) \eta_\xi^{-1} + \eta_\xi \Omega_r D_\nu \eta_\xi^{-1}$$

which gives

$$D_\nu \Gamma_u = \eta_\xi A_v \Omega_r \eta_\xi^{-1} + \eta_\xi (D_\nu \Omega_r) \eta_\xi^{-1} - \eta_\xi \Omega_r A_v \eta_\xi^{-1}$$

or

$$D_\nu \Gamma_u = \eta_\xi [A_v, \Omega_r] \eta_\xi^{-1} + \eta_\xi (D_\nu \Omega_r) \eta_\xi^{-1}.$$

Now, I have to compute $D_\nu \Omega_r$. Now $\Omega_r = r^{-1} R(r, r) r$ and r is a frame in $O(M)$. Using the push forward of $\Psi : L_U M \to LO(M)$, $\Psi_* D_\nu$ is given by $D_\nu r$. Thus, by Corollary 3.8,

$$D_\nu \Omega_r = \partial_{B(v)} \Omega_r + \partial_{\widetilde{\Lambda_v}} \Omega_r.$$

But

$$\begin{aligned}
\partial_{\tilde{E}} \Omega_r &= \frac{d}{dt}\Big|_{t=0} (re^{tE})^{-1} R(re^{tE}, re^{tE}) re^{tE} \\
&= -E\Omega_r + r^{-1} R(rE, r) + r^{-1} R(r, rE) + \Omega_r E = -[E, \Omega_r] + \Omega_r(E, \cdot) + \Omega_r(\cdot, E).
\end{aligned}$$

Thus,

$$(D_\nu \Gamma_u) = \eta_\xi [A_v, \Omega_r] \eta_\xi^{-1} - \eta_\xi [\Lambda_v, \Omega_r] \eta_\xi^{-1} + \Gamma_u(\Lambda_v \cdot, \cdot) + \Gamma_u(\cdot, \Lambda_v \cdot) + \eta_\xi (\partial_{B(v)} \Omega_r) \eta_\xi^{-1}$$

Therefore, using Equation (3.6),

$$
\begin{aligned}
&D_\nu \left(\eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,b')(\tau) d\tau \cdot \eta_\xi \right) \\
&= - \left[A_v, \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,b')(\tau) d\tau \cdot \eta_\xi \right] + \eta_\xi^{-1} \int_0^\cdot [\eta_\xi (\partial_{B(v)} \Omega_r) \eta_\xi^{-1}(w,b')](\tau) d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1} \int_0^\cdot \left[\eta_\xi [\Lambda_v, \Omega_r] \eta_\xi^{-1}(w,b') \right] (\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(\Lambda_v w, b')(\tau) d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,v')(\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, \eta_\xi^{-1} \eta_\xi' v)(\tau) d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1} \int_0^\cdot \left[\eta_\xi [A_v, \Omega_r] \eta_\xi^{-1}(w,b') \right] (\tau) d\tau \cdot \eta_\xi .
\end{aligned}
$$

Also,

$$
\begin{aligned}
D_\nu A_w = D_\nu(\eta_\xi^{-1} D_\mu \eta_\xi) &= (D_\nu \eta_\xi^{-1}) D_\mu \eta_\xi + \eta_\xi^{-1} (D_\nu D_\mu) \eta_\xi \\
&= -\eta_\xi^{-1} (D_\nu \eta_\xi) \eta_\xi^{-1} D_\mu \eta_\xi + \eta_\xi^{-1} D_\nu D_\mu \eta_\xi \\
&= -A_v A_w + \eta_\xi^{-1} D_\nu D_\mu \eta_\xi .
\end{aligned}
$$

So

$$
\begin{aligned}
&D_\nu \left(\eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,b')(\tau) d\tau \cdot \eta_\xi - A_w \right) \\
&= - \left[A_v, \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,b')(\tau) d\tau \cdot \eta_\xi \right] + \eta_\xi^{-1} \int_0^\cdot [\eta_\xi (\partial_{B(v)} \Omega_r) \eta_\xi^{-1}(w,b')](\tau) d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1} \int_0^\cdot \left[\eta_\xi [\Lambda_v, \Omega_r] \eta_\xi^{-1}(w,b') \right] (\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(\Lambda_v w, b')(\tau) d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,v')(\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, \eta_\xi^{-1} \eta_\xi' v)(\tau) d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1} \int_0^\cdot \left[\eta_\xi [A_v, \Omega_r] \eta_\xi^{-1}(w,b') \right] (\tau) d\tau \cdot \eta_\xi \\
&\quad + A_v A_w - \eta_\xi^{-1} D_\nu D_\mu \eta_\xi .
\end{aligned}
$$

But note that

$$
[\Lambda_v, \Omega_r] = [A_v, \Omega_r] - [\eta_\xi^{-1} q_{\eta_\xi v} \eta_\xi, \eta_\xi^{-1} \Gamma_u \eta_\xi] = [A_v, \Omega_r] - \eta_\xi^{-1} [q_{\eta_\xi v}, \Gamma_u] \eta_\xi .
$$

This then simplify the above expression to

$$
\begin{aligned}
&D_\nu \left(\eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,b')(\tau) d\tau \cdot \eta_\xi - A_w \right) \\
&= - \left[A_v, \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,b')(\tau) d\tau \cdot \eta_\xi \right] + \eta_\xi^{-1} \int_0^\cdot [\eta_\xi (\partial_{B(v)} \Omega_r) \eta_\xi^{-1}(w,b')](\tau) d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1} \int_0^\cdot [q_{\eta_\xi v}, \Gamma_u(w,b')](\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(\Lambda_v w, b')(\tau) d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w,v')(\tau) d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(w, \eta_\xi^{-1} \eta_\xi' v)(\tau) d\tau \cdot \eta_\xi + A_v A_w \\
&\quad - \eta_\xi^{-1} D_\nu D_\mu \eta_\xi .
\end{aligned}
$$

■

Recall the Lie bracket of D_μ and D_ν is given by, in terms of r,

$$[rv, rw] = r\left(A_v w - A_w v - \eta_\xi^{-1}[q_{\eta_\xi v}\eta_\xi w + q_{\eta_\xi w}\eta_\xi v]\right) = r(\Lambda_v w - \Lambda_w v).$$

Lemma 3.19.

$$\Lambda_v \wedge \Lambda_w = [A_v, A_w] - [A_v, \eta_\xi^{-1}q_{\eta_\xi w}\eta_\xi] - [\eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi, A_w] + [q_{\eta_\xi v}\eta_\xi, q_{\eta_\xi w}\eta_\xi].$$

Proof. Now $\Lambda_v = A_v - \eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi$, so

$$\begin{aligned}
&[A_v - \eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi, A_w - \eta_\xi^{-1}q_{\eta_\xi w}\eta_\xi] \\
&\quad = [A_v, A_w] - [A_v, \eta_\xi^{-1}q_{\eta_\xi w}\eta_\xi] - [\eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi, A_w] + [\eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi, \eta_\xi^{-1}q_{\eta_\xi w}\eta_\xi] \\
&\quad = [A_v, A_w] - [A_v, \eta_\xi^{-1}q_{\eta_\xi w}\eta_\xi] - [\eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi, A_w] + \eta_\xi^{-1}[q_{\eta_\xi v}, q_{\eta_\xi w}]\eta_\xi.
\end{aligned}$$ ∎

Theorem 3.20. *Let $\gamma_0 \in O \subseteq \bigcup_{x\in U} L_x M$ and define $r = u\eta_\xi$ as in Definition 3.4. Let $v, w, z \in L\mathbb{R}^d$. The curvature R^{LM} of ∇^{LM} at γ_0 is given by*

$$R^{LM}(rv, rw)rz = R^M(rv(\cdot), rw(\cdot))rz(\cdot),$$

whereby R^M is the curvature of the underlying manifold M.

Proof.

$$\begin{aligned}
&D_\nu\Lambda_w - D_\mu\Lambda_v \\
&= [A_v, \eta_\xi^{-1}q_{\eta_\xi w}\eta_\xi] - [A_w, \eta_\xi^{-1}q_{\eta_\xi v}\eta_\xi] \\
&\quad - \int_0^\cdot [\eta_\xi(\partial_{B(v)}\Omega_r)\eta_\xi^{-1}(w, b')](\tau)d\tau + \eta_\xi^{-1}\int_0^\cdot [\eta_\xi(\partial_{B(w)}\Omega_r)\eta_\xi^{-1}(v, b')](\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot [q_{\eta_\xi v}, \Gamma_u(w, b')](\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\int_0^\cdot [q_{\eta_\xi w}, \Gamma_u(v, b')](\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(\Lambda_v w - \Lambda_w v, b')(\tau)d\tau \cdot \eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, v')(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, w')(\tau)d\tau \cdot \eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, \eta_\xi^{-1}\eta_\xi' v)(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, \eta_\xi^{-1}\eta_\xi' w)(\tau)d\tau \cdot \eta_\xi - [A_v, A_w] + \eta_\xi^{-1}[D_\nu, D_\mu]\eta_\xi.
\end{aligned}$$

So

$$\begin{aligned}
&D_\nu\Lambda_w - D_\mu\Lambda_v + \Lambda_v \wedge \Lambda_w \\
&= -\int_0^\cdot [\eta_\xi(\partial_{B(v)}\Omega_r)\eta_\xi^{-1}(w, b')](\tau)d\tau + \eta_\xi^{-1}\int_0^\cdot [\eta_\xi(\partial_{B(w)}\Omega_r)\eta_\xi^{-1}(v, b')](\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot [q_{\eta_\xi v}, \Gamma_u](w, b')(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\int_0^\cdot [q_{\eta_\xi w}, \Gamma_u](v, b')(\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(\Lambda_v w - \Lambda_w v, b')(\tau)d\tau \cdot \eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, v')(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, w')(\tau)d\tau \cdot \eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, \eta_\xi^{-1}\eta_\xi' v)(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, \eta_\xi^{-1}\eta_\xi' w)(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}[D_\nu, D_\mu]\eta_\xi + \eta_\xi^{-1}[q_{\eta_\xi v}, q_{\eta_\xi w}]\eta_\xi.
\end{aligned}$$

But

$$d[q_{\eta_\xi v}, q_{\eta_\xi w}]/ds = [\Gamma_u(v, b'), q_{\eta_\xi w}] + [q_{\eta_\xi v}, \Gamma_u(w, b')].$$

Hence this simplifies to

$$\begin{aligned}
&D_\nu \Lambda_w - D_\mu \Lambda_v + \Lambda_v \wedge \Lambda_w \\
&= -\int_0^\cdot [\eta_\xi(\partial_{B(v)}\Omega_r)\eta_\xi^{-1}(w, b')](\tau)d\tau + \eta_\xi^{-1}\int_0^\cdot [\eta_\xi(\partial_{B(w)}\Omega_r)\eta_\xi^{-1}(v, b')](\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(\Lambda_v w - \Lambda_w v, b')(\tau)d\tau \cdot \eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, v')(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, w')(\tau)d\tau \cdot \eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, \eta_\xi^{-1}\eta_\xi' v)(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, \eta_\xi^{-1}\eta_\xi' w)(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}[D_\nu, D_\mu]\eta_\xi
\end{aligned}$$

Now

$$\begin{aligned}
\Lambda_{r^{-1}[\nu,\mu]} &= A_{r^{-1}[\nu,\mu]} - \eta_\xi^{-1} q_{r^{-1}[\nu,\mu]}\eta_\xi \\
&= \eta_\xi^{-1}[D_\nu, D_\mu]\eta_\xi - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(\Lambda_v w - \Lambda_w v, b')(\tau)d\tau \cdot \eta_\xi.
\end{aligned}$$

So

$$\begin{aligned}
&D_\nu \Lambda_w - D_\mu \Lambda_v - \Lambda_{r^{-1}[\nu,\mu]} + \Lambda_v \wedge \Lambda_w \\
&= -\eta_\xi^{-1}\int_0^\cdot [\eta_\xi(\partial_{B(v)}\Omega_r)\eta_\xi^{-1}(w, b')](\tau)d\tau + \eta_\xi^{-1}\int_0^\cdot [\eta_\xi(\partial_{B(w)}\Omega_r)\eta_\xi^{-1}(v, b')](\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, v')(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, w')(\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, \eta_\xi^{-1}\eta_\xi' v)(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, \eta_\xi^{-1}\eta_\xi' w)(\tau)d\tau \cdot \eta_\xi.
\end{aligned}$$

By the second Bianchi identity,

$$\partial_{B(v)}\Omega_r(w, b') - \partial_{B(w)}\Omega_r(v, b') = \partial_{B(b')}\Omega_r(w, v)$$

and by the symmetry of curvature tensor, at $\gamma_0 \in O$,

$$\begin{aligned}
&[D_\nu \Lambda_w - D_\mu \Lambda_v - \Lambda_{[\eta_\xi v, \eta_\xi w]} + \Lambda_v \wedge \Lambda_w + D_\nu \lambda_w - D_\mu \lambda_v](\gamma_0) \\
&= \Big[\eta_\xi^{-1}\int_0^\cdot [\eta_\xi(\partial_{B(b')}\Omega_r)\eta_\xi^{-1}(v, w)](\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v', w)(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, w')(\tau)d\tau \cdot \eta_\xi \\
&\quad - \eta_\xi^{-1}\int_0^\cdot \Gamma_u(w, \eta_\xi^{-1}\eta_\xi' v)(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\int_0^\cdot \Gamma_u(v, \eta_\xi^{-1}\eta_\xi' w)(\tau)d\tau \cdot \eta_\xi \\
&\quad + \eta_\xi^{-1}\Omega_{r(0)}(v(0), w(0))\eta_\xi\Big](\gamma_0).
\end{aligned}$$

Now I have to compute r'. Note that by definition, $r = u\eta_\xi$ and $\pi(u\eta_\xi) = \gamma$. Thus I seek to find a horizontal vector field B such that $\pi_* B = \gamma'$. But $\gamma' = u\eta_\xi b' = rb'$.

Therefore $\pi_* B = rb'$ which implies $\vartheta(r') = b'$. The vertical vector field is clearly $-\widetilde{\xi}$. Thus,

$$r' = (b')^\alpha B_\alpha - \widetilde{\xi}.$$

Hence,

$$\begin{aligned}\Omega_r' &= (b')^\alpha B_\alpha \Omega_r - \partial_{\widetilde{\xi}} \Omega_r \\ &= (b')^\alpha B_\alpha \Omega_r + [\xi, \Omega_r] - \Omega_r(\xi\cdot, \cdot) - \Omega_r(\cdot, \xi\cdot).\end{aligned}$$

Therefore,

$$\begin{aligned}&d(\eta_\xi \Omega_r(v,w)\eta_\xi^{-1})/ds \\ &= -\,[\xi, \eta_\xi \Omega_r(v,w)\eta_\xi^{-1}] + \eta_\xi (b')^\alpha B_\alpha \Omega_r(v,w)\eta_\xi^{-1} + \eta_\xi[\xi, \Omega_r(v,w)]\eta_\xi^{-1} \\ &\quad + \eta_\xi \Omega_r(\eta_\xi^{-1}\eta_\xi' v, w)\eta_\xi^{-1} + \eta_\xi \Omega_r(v, \eta_\xi^{-1}\eta_\xi' w)\eta_\xi^{-1} + \eta_\xi \Omega_r(v', w)\eta_\xi^{-1} + \eta_\xi \Omega_r(v, w')\eta_\xi^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}&\Omega_r(v,w) \\ &= \eta_\xi^{-1} \int_0^\cdot [\eta_\xi(\partial_{B(b')}\Omega_r)(v,w)](\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(v', w)(\tau)d\tau \cdot \eta_\xi \\ &\quad + \eta_\xi^{-1} \int_0^\cdot \Gamma_u(v, w')(\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1} \int_0^\cdot [\eta_\xi \Omega_r(v, \eta_\xi^{-1}\eta_\xi' w)\eta_\xi^{-1}](\tau)d\tau \\ &\quad + \eta_\xi^{-1} \int_0^\cdot [\eta_\xi \Omega_r(\eta_\xi^{-1}\eta_\xi' v, w)\eta_\xi^{-1}](\tau)d\tau \cdot \eta_\xi + \eta_\xi^{-1}\Omega_{r(0)}(v(0), w(0))\eta_\xi,\end{aligned}$$

or at $\gamma_0 \in O \subseteq \bigcup_{x\in U} L_x M$,

$$[D_v \Lambda_w - D_\mu \Lambda_v - \Lambda_{[\eta_\xi v, \eta_\xi w]} + \Lambda_v \wedge \Lambda_w + D_v \lambda_w - D_\mu \lambda_v](\gamma_0) = \Omega_r(v,w)(\gamma_0).$$

■

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