An One Step Smoothing Newton Method for a Class of Stochastic Linear Complementarity Problems*

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Abstract

A new formulation for a class of stochastic linear complementarity problems (SLCPs) with finitely many realizations is proposed, which reformulates the SLCPs as a system of smoothing equations without any constraints by an NCP function. Then, we extend an one step smoothing Newton method to this formulation. Moreover, we show that this algorithm converges globally and local quadratically under mild assumptions.

1 Introduction

The stochastic linear complementarity problem (SLCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x,w) = M(w)x + q(w) \geq 0, \quad x^T F(x,w) = 0, \tag{1}$$

where $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ denotes a vector valued function, $(\Omega, \mathscr{F}, P)$ is a probability space with $\Omega \subseteq \mathbb{R}^m$, $M(w) \in \mathbb{R}^{n\times n}$ and $q(w) \in \mathbb{R}^n$ for $w \in \Omega$ are random

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matrices and vectors. Because of the existence of a random elements w, however, we cannot generally expect that there exists a vector x^* satisfying (1). That is (1) may not have a solution in general. Therefore, to present an appropriate deterministic formulation of SLCP is an important issue. There have been proposed three types of formulations of SLCP, the expected value (EV) formulation [1, 2], the expected residual minimization (ERM) formulation [3, 4, 5, 6], and the SMPEC formulation [7, 8].

This paper considers the following class of stochastic linear complementarity problems in which Ω only has finitely many elements. Let $\Omega = \{w_1, w_2, \cdots, w_m\}$. Find an $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ F(x, w_i) = M(w_i)x + q(w_i) \geq 0,\ x^T F(x, w_i) = 0,\ i = 1, 2, \cdots, m,\ m > 1. \quad (2)$$

We suppose $p_i = P\{w_i \in \Omega\} > 0$, $i = 1, 2, \cdots, m$. In [6], problem (2) was formulated equivalent to (3)-(4),

$$x \geq 0, \quad \overline{M}x + \overline{q} \geq 0, \quad x^T(\overline{M}x + \overline{q}) = 0, \tag{3}$$

$$M(w_i)x + q(w_i) \geq 0, \quad i = 1, 2, \cdots, m, \tag{4}$$

where $\overline{M} = \sum_{i=1}^{m} p_i M(w_i)$ and $\overline{q} = \sum_{i=1}^{m} p_i q(w_i)$. Let $\overline{F}(x)$ be the expectation function of the random function $F(x, w)$, then $\overline{F}(x) = E[F(x, w)] = \overline{M}x + \overline{q}$. Paper [6] reformulated the problem (2) as a system of nonsmooth equations with nonnegative constraints.

In this paper, we aim at modifying the constrained minimization reformulation for (2) in [6]. By using the smoothing symmetric perturbed Fischer function (for short, denoted as the SSPF-function) [9], we propose a new formulation for SLCP (2) with smoothing parameter, which reformulates (3)-(4) as a system of smoothing equations without any constraints. We then extend the one step smoothing Newton method in [10] to this problem. Under the assumptions that $\overline{M}$ is a P_0-matrix and the solution set of the linear complementarity problem (3) is nonempty and bounded, the smoothing Newton algorithm is convergent globally and local quadratically.

Throughout this paper, we use the following notation. All vectors (vector functions) are column vectors (vector functions). $\mathbb{R}_{++}$ denotes the positive orthant in $\mathbb{R}$. For any vector $u \in \mathbb{R}^n$, we denote by $\text{diag}\{u_i, i = 1, 2, \cdots, n\}$ the diagonal matrix whose ith diagonal element is u_i. The symbol $\|\cdot\|$ stands for the 2-norm. For any locally Lipschitzian function $F : \mathbb{R}^n \to \mathbb{R}^n$, by

$$\partial_B F(x) = \{\lim_{x^k \in D_F, x^k \to x} F'(x^k)\},\ \partial F(x) = \text{co}(\partial_B F(x))$$

denote the B-subdifferential and the Clark subdifferential of F at x respectively, in which $D_F \subset \mathbb{R}^n$ denotes the set of points at which F is differentiable.

The following definitions will be used in this paper.

Definition 1. *The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a P_0-function if there is an index i such that*

$$x_i \neq y_i \text{ and } (x_i - y_i)\big[F_i(x) - F_i(y)\big] \geq 0, \quad for\ all\ x, y \in \mathbb{R}^n, x \neq y.$$

Definition 2. *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a P_0-matrix if all its principal minors are nonnegative.*

Definition 3. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitzian function. F is said to be semismooth at x if F is directionally differentiable at x and for any $V \in \partial F(x+h)$ and $\|h\| \to 0$,*

$$\|F(x+h) - F(x) - Vh\| = o(\|h\|).$$

F is said to be strongly semismooth at x if F is semismooth at x and

$$\|F(x+h) - F(x) - Vh\| = O(\|h\|^2).$$

2 The New Formulation and Algorithm

In this section, we reformulate the problem (2) as an unconstrained optimization problem and then extend the one step smoothing Newton method in [10] to this problem.

In the rest of this paper, we assume that $\overline{M}$ is a P_0- matrix. Obviously, equation (3) is a standard linear complementarity problem. Therefore, we can reformulate it as a system of smooth equation by an NCP function. Here, we consider the smoothing symmetric perturbed Fischer function (SSPF-function) [9] $\phi : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\phi(\mu, a, b) = (1+\mu)(a+b) - \sqrt{(a+\mu b)^2 + (\mu a + b)^2 + \mu^2}. \tag{5}$$

For $\mu = 0$, $\phi(0, a, b)$ is the Fischer-Burmeister function with the following property

$$\phi(0, a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0.$$

Define the function $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by

$$\Phi(\mu, x) = \begin{pmatrix} \phi(\mu, x_1, \overline{F_1}(x)) \\ \vdots \\ \phi(\mu, x_n, \overline{F_n}(x)) \end{pmatrix}. \tag{6}$$

Then, (3) is equivalent to the equation $\Phi(0, x) = 0$.

For any $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, define the function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}^n$, respectively, as

$$f(x) = (e^{|x_1| - x_1} - 1, e^{|x_2| - x_2} - 1, \cdots, e^{|x_n| - x_n} - 1)^T$$

and

$$g(\mu,x) = (e^{\sqrt{x_1^2+\mu^2}-x_1}-1, e^{\sqrt{x_2^2+\mu^2}-x_2}-1, \cdots, e^{\sqrt{x_n^2+\mu^2}-x_n}-1)^T. \tag{7}$$

Thus, $f(x) \geq 0$, for all $x \in \mathbb{R}^n$.

Set $y = [y_{\cdot 1}, y_{\cdot 2}, \cdots, y_{\cdot m}]^T \in \mathbb{R}^{mn}$, where $y_{\cdot i} \in \mathbb{R}^n$, $i = 1, 2, \cdots, m$. Let $z = (\mu, x, y) \in \mathbb{R}^{n(m+1)+1}$ and

$$H(z) = \begin{bmatrix} \mu \\ \Phi(\mu,x) \\ M(w_1)x + q(w_1) - g(\mu, y_{\cdot 1}) \\ M(w_2)x + q(w_2) - g(\mu, y_{\cdot 2}) \\ \vdots \\ M(w_m)x + q(w_m) - g(\mu, y_{\cdot m}) \end{bmatrix}. \tag{8}$$

Hence, (3)-(4) is equivalent to finding a root of the following equation

$$H(z) = 0. \tag{9}$$

Lemma 1. *Let $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $H : \mathbb{R}^{n(m+1)+1} \to \mathbb{R}^{n(m+1)+1}$ be defined by (6) and (8), respectively. Then*

(i) *Φ is continuously differentiable at any $(\mu, x) \in \mathbb{R}^{n+1}$ with $\mu \neq 0$. For $\mu = 0$, Φ is semismooth on $\mathbb{R}^{n+1}$.*

(ii) *H is continuously differentiable and its Jacobian H' is nonsingular at any $z = (\mu, x, y) \in \mathbb{R}_{++} \times \mathbb{R}^n \times \mathbb{R}^{mn}$.*

(iii) *H is locally Lipschitzian and strongly semismooth on $\mathbb{R}^{n(m+1)+1}$.*

Proof. (i) By Lemma 2.4 (a) in [10], (i) holds.

(ii) It follows from (i) and the definition (7) that H is continuously differentiable on $\mathbb{R}_{++} \times \mathbb{R}^n \times \mathbb{R}^{mn}$. For any $z = (\mu, x, y) \in \mathbb{R}_{++} \times \mathbb{R}^n \times \mathbb{R}^{mn}$, we have that

$$H'(z) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \Phi'_\mu(\mu,x) & \Phi'_x(\mu,x) & 0 & 0 & \cdots & 0 \\ u_1(z) & M(w_1) & \triangle_1 & 0 & \cdots & 0 \\ u_2(z) & M(w_2) & 0 & \triangle_2 & \cdots & 0 \\ & & \vdots & & & \\ u_m(z) & M(w_m) & 0 & 0 & \cdots & \triangle_m \end{bmatrix}, \tag{10}$$

where $u_i(z) = g'_\mu(\mu, y_{\cdot i})$ and $\triangle_i = g'_y(\mu, y_{\cdot i})$, $i = 1, 2, \cdots, m$.

For any $x \in \mathbb{R}^n$, $\mu \neq 0$, a straightforward calculation yields

$$g'_x(\mu,x) = \mathrm{diag}\Big\{ \Big(\frac{x_i}{\sqrt{x_i^2+\mu^2}} - 1\Big) e^{\sqrt{x_i^2+\mu^2}-x_i}, \ i = 1, 2, \cdots, n \Big\}.$$

It is not difficult to see that $\big(\frac{x_i}{\sqrt{x_i^2+\mu^2}}-1\big)e^{\sqrt{x_i^2+\mu^2}-x_i}\neq 0$, then we have

$$|\triangle_i|\neq 0,\ \ i=1,2,\cdots,m. \tag{11}$$

Since $\overline{M}$ is a P_0-matrix, $\overline{F}(x)$ must be a P_0- function. In view of Lemma 2.4 (b) in [10], the matrix $\Phi_x'(\mu,x)$ is nonsingular. This together with (11) imply $H'(x)$ is nonsingular at any $z=(\mu,x,y)\in \mathbb{R}_{++}\times \mathbb{R}^n\times \mathbb{R}^{mn}$.

(iii) Since $\overline{F}(x)=\overline{M}x+\overline{q}$, there must exist a constant $L>0$ such that

$$\|\overline{F}'(x^1)-\overline{F}'(x^2)\|=\|\overline{M}-\overline{M}\|\leq L\|x^1-x^2\|,\ \ \forall\, x^1,x^2\in \mathbb{R}^n.$$

By Lemma 2.4 (c) in [10], H is strongly semismooth on $\mathbb{R}^{n(m+1)+1}$. ∎

Next, we employ the algorithm in [10] to solve the equation (9).

Define

$$\rho(z)=\gamma\|H(z)\|\cdot\min\{1,\|H(z)\|\},\ \ 0<\gamma<1.$$

Algorithm

Step 0 Choose $0<\mu_0<1$ and $\delta,\sigma\in(0,1)$. Let $\overline{u}=(\mu_0,0)\in \mathbb{R}_{++}\times \mathbb{R}^{n(m+1)}$ and $(x^0,y^0)\in \mathbb{R}^n\times \mathbb{R}^{mn}$, Let $z^0=(\mu_0,x^0,y^0)$. Choose $\gamma\in(0,1)$ such that $\gamma\|H(z^0)\|<1$. Set $k=0$.

Step 1 If $||H(z^k)||=0$, stop. Otherwise, let $\rho_k=\rho(z^k)$.

Step 2 Compute $\triangle z^k=(\triangle\mu_k,\triangle x^k,\triangle y^k)\in \mathbb{R}^{n(m+1)+1}$ by

$$H(z^k)+H'(z^k)\triangle z^k=\rho_k\overline{u}. \tag{12}$$

Step 3 Let m_k be the smallest nonnegative integer such that

$$\|H(z^k+\delta^{m_k}\triangle z^k)\|\leq[1-\sigma(1-\gamma)\delta^{m_k}]\|H(z^k)\|. \tag{13}$$

Let $\alpha_k=\delta^{m_k}$.

Step 4 Set $z^{k+1}=z^k+\alpha_k\triangle z^k$ and $k=k+1$. Go to Step 1.

Like the analysis in [10], in order to prove the algorithm is well-defined, we should define the set

$$\overline{\Omega}=\{z\in \mathbb{R}^{n(m+1)+1}|\mu\geq\rho(z)\mu_0\}.$$

Lemma 2. *The algorithm is well-defined and generates an infinite sequence $\{z^k=(\mu_k,x^k,y^k)\}$ with $\mu_k\in \mathbb{R}_{++}$ and $z^k\in\overline{\Omega}$ for all $k\geq 0$.*

Proof. The result can be obtained immediately from Theorem 2.5 in [10].

3 Convergence Analysis

The convergence properties of this method are summarized in the following theorem.

Assumption 1. *The solution set of (3) is nonempty and bounded.*

Theorem 1. *Suppose that Assumption 1 is satisfied and $\overline{M}$ is a P_0-matrix. Let $\{z^k = (\mu_k, x^k, y^k)\}$ be the iteration sequence generated by the algorithm. Then*

(i) *$\{z^k\}$ is bounded.*

(ii) *The sequences $\{\|H(z^k)\|\}$ and $\{\mu_k\}$ tend to zero and hence it has at least one accumulation point $z^* = (\mu_*, x^*, y^*)$ with $H(z^*) = 0$. Therefore, x^* is a solution of (3)-(4).*

(iii) *If all $V \in \partial H(z^*)$ are nonsingular, then the whole sequence $\{z^k\}$ converges to z^* and the rate of convergence is quadratic.*

Proof. (i) Define the level set

$$L(c) = \{z \in \mathbb{R}^{n(m+1)+1} | \|H(z)\| \leq \|H(z^0)\| = c\}.$$

Let $\overline{L}(c) = \{x \in \mathbb{R}^n | \|\Phi(\mu, x)\| \leq c\}$. Then $\overline{L}(c) \subset L(c)$. From line search (13), we know $\{z^k\} \subset L(c)$. It follows from Assumption 1 and Theorem 3.6 (ii) in [10] that $\{\mu_k, x^k\}$ is bounded. So, we only to prove $\{y^k\}$ is bounded. Suppose $\|y^k\| \to \infty$, then

$$\|M(w_i)x^k + q(w_i) - g(\mu_k, y^k_{\cdot i})\| \to \infty, \ \ i = 1, 2, \cdots, m.$$

This contradicts the fact that $z^k \in L(c)$.

(ii) From Lemma 2, we known

$$\mu_{k+1} = \mu_k + \alpha_k \triangle \mu_k = (1 - \alpha_k)\mu_k + \alpha_k \rho_k \mu_0 \leq \mu_k,$$

which implies that $\{\mu_k\}$ is monotonically decreasing and bounded. Thus, $\{\mu_k\}$ is convergent. On the other hand, by (i) and (13), $\{\|H(z^k)\|\}$ is also monotonically decreasing and bounded and hence is convergent. Let $z^* = (\mu_*, x^*, y^*)$ be an accumulation point of $\{z^k\}$. Without loss of generality, we assume that $\{z^k\}$ converges to z^*. Then

$$\lim_{k\to\infty} \|H(z^k)\| = \|H(z^*)\|, \ \lim_{k\to\infty} \mu_k = \mu_*, \ \lim_{k\to\infty} \rho_k = \rho_* = \gamma\|H(z^*)\| \min\{1, \|H(z^*)\|\}.$$

If $\{\|H(z^k)\|\}$ does not converge to zero. Then

$$\|H(z^*)\| > 0, \ \rho_* > 0, \ 0 < \rho_* \mu_0 \leq \mu_* \leq \mu_0. \tag{14}$$

So $\lim\limits_{k\to\infty} \alpha_k = 0$ from (13). Thus, the stepsize $\widetilde{\alpha} = \dfrac{\alpha_k}{\delta}$ does not satisfy the line search criterion in Step 3 for any sufficiently large k, i.e., the following inequality holds

$$\|H(z^k + \widetilde{\alpha} \triangle z^k)\| > \left[1 - \sigma(1-\gamma)\widetilde{\alpha}\right] \|H(z^k)\|$$

for any sufficiently large k, which implies that

$$\frac{\|H(z^k+\widetilde{\alpha}\triangle z^k)\|-\|H(z^k)\|}{\widetilde{\alpha}} > -\sigma(1-\gamma)\|H(z^k)\|.$$

From $\mu_* \neq 0$, we know that $H(\cdot)$ is continuously differentiable at z^*. Letting $k\to\infty$, then above inequality gives

$$\frac{1}{\|H(z^*)\|}(H(z^*))^T H'(z^*)\triangle z^* \geq -\sigma(1-\gamma)\|H(z^*)\|. \tag{15}$$

Additionally, by taking the limit on (12), we get

$$H'(z^*)\triangle z^* = -H(z^*)+\rho_*\overline{u}. \tag{16}$$

Combining (15) with (16), we have

$$\rho_*\|H(z^*)\|\mu_0 \geq [1-\sigma(1-\gamma)]\|H(z^*)\|^2.$$

Noting $0<\mu_0<1$ and the definition of ρ_k, then

$$\gamma\|H(z^*)\|^2 \geq [1-\sigma(1-\gamma)]\|H(z^*)\|^2. \tag{17}$$

(17) indicates $(1-\sigma)(1-\gamma)\leq 0$, which contradicts the fact that $\sigma,\gamma\in(0,1)$. Thus, $H(z^*)=0$ and $\mu_*=0$. According to the definition of H, we obtain

$$M(w_i)x^*+q(w_i)\geq 0,\ i=1,2,\cdots,m \ \text{ and } \ \Phi(0,x^*)=0.$$

Therefore, x^* is a solution of (3)-(4).

(iii) These results can be proved by following from Theorem 3.7 in [10]. ■

Remark The conditions of global and local convergence in this paper are weaker than those required by [6].

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