On functions convex in the direction of the real axis with real coefficients

Leopold Koczan Paweł Zaprawa

Abstract

The paper is concerned with the class $X^{(n)}$ consisting of all functions, which are n-fold symmetric, convex in the direction of the real axis and have real coefficients. For this class we determine the Koebe domain, i.e. the set $\bigcap_{f\in X^{(n)}} f(\Delta)$, as well as the covering domain, i.e. the set $\bigcup_{f\in X^{(n)}} f(\Delta)$. The results depend on the parity of $n \in N$. We also obtain the minorant and the majorant for this class. These functions are defined as follows.

If there exists an analytic, univalent function m satisfying the following conditions: $m'(0) > 0$, for every $f \in X^{(n)}$ there is $m \prec f$, and $\bigwedge_{f\in X^{(n)}} [k \prec f \Rightarrow k \prec m]$, then this function is called the minorant of $X^{(n)}$. Similarly, if there exists an analytic, univalent function M such that $M'(0) > 0$, for every $f \in X^{(n)}$ there is $f \prec M$, and $\bigwedge_{f\in X^{(n)}} [f \prec k \Rightarrow M \prec k]$, then this function is called the majorant of $X^{(n)}$.

If these functions exist, then $m(\Delta)$ and $M(\Delta)$ coincide with the Koebe domain and the covering domain for $X^{(n)}$, respectively.

Introduction

In the beginning we recall that an analytic function f is subordinated to an analytic and univalent function F in $\Delta \equiv \{\zeta \in C : |\zeta| < 1\}$ if and only if there exists an analytic function ω such that $\omega(0) = 0$, $\omega(\Delta) \subset \Delta$ and $f(z) = F(\omega(z))$ for $z \in \Delta$. Then we write $f \prec F$.

Received by the editors May 2010.

Communicated by F. Brackx.

2000 *Mathematics Subject Classification :* Primary 30C45; Secondary 30C80.

Key words and phrases : covering domain, Koebe domain, convex in the direction of real axis, real coefficients.

Let S denote the set of all functions f analytic and univalent in Δ and normalized by $f(0) = f'(0) - 1 = 0$. Let $Y \subset S$ be the class of these functions in S which have real coefficient and which are convex in the direction of the imaginary axis. Similarly, let $X \subset S$ consist of the functions with real coefficients in S, which are convex in the direction of the real axis. We call a function f convex in the direction of the straight line l if the intersection of $f(\Delta)$ and each line k parallel to l is either k, or a segment, or a ray or an empty set.

For a given $\mathcal{Y} \subset S$, if there exists an analytic and univalent function m satisfying the following conditions: $m'(0) > 0$,

$$\bigwedge_{f\in\mathcal{Y}} m \prec f \tag{1}$$

and for every analytic function k, $k(0) = 0$, there is

$$\left(\bigwedge_{f\in\mathcal{Y}} k \prec f\right) \Rightarrow k \prec m, \tag{2}$$

then this function is called the minorant of $\mathcal{Y}$. The set $\bigcap_{f\in\mathcal{Y}} f(\Delta)$ is said to be the Koebe domain for $\mathcal{Y}$ and is denoted by $K_{\mathcal{Y}}$. Clearly, if the Koebe domain is a simply connected set, then the minorant exists and $K_{\mathcal{Y}} = m(\Delta)$.

If there exists an analytic and univalent function M such that $M'(0) > 0$,

$$\bigwedge_{f\in\mathcal{Y}} f \prec M \tag{3}$$

and for every analytic function k, $k(0) = 0$, there is

$$\left(\bigwedge_{f\in\mathcal{Y}} f \prec k\right) \Rightarrow M \prec k, \tag{4}$$

then this function is called the majorant of $\mathcal{Y}$. The set $\bigcup_{f\in\mathcal{Y}} f(\Delta)$ is said to be the covering domain for $\mathcal{Y}$ and is denoted by $L_{\mathcal{Y}}$. Notice that if the covering domain is a simply connected set, then the majorant exists. In this case $L_{\mathcal{Y}} = M(\Delta)$.

EXAMPLES.

1. For $\mathcal{Y} = S$ there is $m(z) = \frac{1}{4}z$, $z \in \Delta$, and hence $K_S = \Delta_{1/4}$. In S the majorant does not exist ($L_S = C$).

2. For $\mathcal{Y} = Y$ there is $m(z) = \frac{1}{2}z$, $z \in \Delta$, (McGregor, [5]) and hence $K_Y = \Delta_{1/2}$. The majorant does not exist ($L_Y = C$).

3. $\mathcal{Y} = CVR^{(2)}$, where $CVR^{(2)}$ is the class of univalent, convex and odd functions in Δ with real coefficients. The set $K_{CVR^{(2)}}$ was determined by Krzyż and Reade (see [1]). Then, m maps Δ onto the set $K_{CVR^{(2)}}$ and $m'(0) > 0$. The function $M(z) = \int_0^1 \frac{z}{\sqrt{(1-t^2)(1-t^2z^4)}}dt$ is the majorant of $CVR^{(2)}$ (see [4]).

In [2] the class $Y^{(n)}$ was considered. This is the set of n-fold symmetric functions from Y, i.e.

$$Y^{(n)} \equiv \{f \in Y : f(\varepsilon z) = \varepsilon f(z), z \in \Delta\} \quad , \text{ where } \varepsilon = e^{\frac{2\pi i}{n}}.$$

For functions in $Y^{(n)}$ the property $f(\Delta) = \varepsilon f(\Delta)$ holds. In this case we say that the set $f(\Delta)$ is n-fold symmetric. The symbol aD is understood as $\{az : z \in D\}$. In the above mentioned paper the authors derived the Koebe set and the covering set as well as the minorant and the majorant in $Y^{(n)}$.

Now we are interested in another subclass of S, namely

$$X^{(n)} \equiv \{f \in X : f(\varepsilon z) = \varepsilon f(z), z \in \Delta\} ,$$

where ε is defined as above.

It is known that if f is in X, then for each $t \in (0,1)$ the function $f(tz)/t$ is also in X. The same is true for functions in $X^{(n)}$. Therefore, the Koebe set for $X^{(n)}$ is, in fact, a domain.

Observe that for even n, all functions from $X^{(n)}$ are odd. Hence

$$f \in X^{(n)} \Leftrightarrow -if(iz) \in Y^{(n)} \quad \text{for} \quad n = 4k-2\,,\ k \in N\,, \tag{5}$$

$$f \in X^{(n)} \Leftrightarrow f \in Y^{(n)} \qquad\qquad \text{for} \quad n = 4k\,,\ k \in N\,. \tag{6}$$

We conclude from (5-6) that one can transfer the results from $Y^{(n)}$ onto $X^{(n)}$.

Every function in $X^{(n)}$ has real coefficients. For this reason the set $f(\Delta)$ is symmetric with respect to the real axis. Another important property of the class $X^{(n)}$ is given in

Lemma 1. *If $f \in X^{(n)}$ then the straight line $k : \zeta = e^{\frac{\pi i}{n}} t\ ,\ t \in R$ is a symmetry axis of the set $f(\Delta)$.*

Proof.
The symmetry with respect to the line $\zeta = e^{\frac{\pi i}{n}} t\ ,\ t \in R$ means that for arbitrary $z, \zeta \in \Delta$, if

$$\overline{ze^{-\frac{\pi i}{n}}} = \zeta e^{-\frac{\pi i}{n}} \tag{7}$$

then

$$\overline{f(z)e^{-\frac{\pi i}{n}}} = f(\zeta)e^{-\frac{\pi i}{n}}\ . \tag{8}$$

Assume that the condition (7) is satisfied. We can write it equivalently in the form

$$\zeta = \bar{z}e^{\frac{2\pi i}{n}} = \bar{z}\varepsilon. \tag{9}$$

From properties of $f \in X^{(n)}$ it follows that

$$\overline{f(z)}\varepsilon = f(\bar{z})\varepsilon = f(\bar{z}\varepsilon)\ .$$

Applying (9) we obtain $\overline{f(z)}\varepsilon = f(\zeta)$. This condition is equivalent to (8). ∎

Corollary 1. *If $f \in X^{(n)}$, then each straight line $\zeta = e^{\frac{\pi i}{n}k} t\ ,\ t \in R,\ k = 0,1,\ldots,2n-1$, is a symmetric axis of $f(\Delta)$.*

The next lemma follows from Lemma 1 and from properties of the class $X^{(n)}$

Lemma 2. *The Koebe domain and the covering domain for $X^{(n)}$ are n-fold symmetric and symmetric with respect to the lines $\zeta = e^{\frac{\pi i}{n}k} t\ ,\ t \in R,\ k = 0,1,\ldots,2n-1$.*

Lemma 3. *The Koebe domain and the covering domain for $X^{(n)}$ are symmetric with respect to the imaginary axis.*

Proof.
If $f \in X^{(n)}$ then $g(z) = -f(-z)$ is also in $X^{(n)}$. Hence the sets $f(\Delta) \cap g(\Delta)$ and $f(\Delta) \cup g(\Delta)$ are symmetric with respect to the imaginary axis. From this

$$\bigcap_{f \in X^{(n)}} f(\Delta) = \bigcap_{f \in X^{(n)}} f(\Delta) \cap (-f(\Delta)) \quad \text{and} \quad \bigcup_{f \in X^{(n)}} f(\Delta) = \bigcup_{f \in X^{(n)}} f(\Delta) \cup (-f(\Delta)) \ .$$

■

From convexity of the functions in $X^{(n)}$ in the direction of the real axis we get

Lemma 4. *The Koebe domain for $X^{(n)}$ is convex in the direction of the real axis.*

For a fixed n we use the notation: $\Lambda_j = \{\zeta \in C : 2(j-1)\pi/n \leq \operatorname{Arg}\zeta \leq 2j\pi/n\}$, $j = 1, 2, ..., n$, and $\Lambda = \{\zeta \in C : 0 \leq \operatorname{Arg}\zeta \leq \pi/n\}$. Furthermore, we will write ∂D to denote the boundary of a set D.

By Lemma 2, we need to determine the boundaries of the Koebe domain and the covering domain for $X^{(n)}$ in the set Λ only.

1 Koebe domain for $X^{(n)}$ and odd n.

Let n be a fixed odd integer, $n \geq 3$. We consider two families of open and n-fold symmetric polygons which are symmetric with respect to the real axis.

The first family consists of polygons such that their successive vertices u, v, w belong to Λ and $\operatorname{Arg} u = 0$, $\operatorname{Arg} v \in (0, \frac{\pi}{n})$, $\operatorname{Arg} w = \frac{\pi}{n}$. The polygons' interior angles corresponding with the vertices u, v, w are of the measure $\pi(1-\frac{1}{n})$, $\pi(1+\frac{1}{n})$ and $\pi(1-\frac{3}{n})$, respectively. It means that the measure of the angle with the vertex lying on the real positive semi-axis is equal to $\pi(1-\frac{1}{n})$. From the above it follows that polygons of the described type have $4n$ sides.

This set of polygons is extended on limiting cases. If $u = v$ (hence $\operatorname{Arg} v = 0$), then we obtain polygons having $2n$ sides of the same length and angles measuring $\pi(1+\frac{1}{n})$ and $\pi(1-\frac{3}{n})$ alternately. If $v = w$ (hence $\operatorname{Arg} v = \frac{\pi}{n}$) then we obtain regular polygons having $2n$ sides and all angles measuring $\pi(1-\frac{1}{n})$.

We denote this family of polygons by $\mathcal{V}_1$. Polygons of this family are shown in Figure 1.

For $n = 3$ the sets of the family $\mathcal{V}_1$ are unbounded. Every fourth vertex of such a polygon is extended to infinity. For this reason both sides adjacent to every such vertex are parallel. In this way we obtain a star-shaped set with three unbounded strips. The thickness of strips is growing as $\operatorname{Arg} v$ tends to $\frac{\pi}{3}$.

In cases $\operatorname{Arg} v = 0$ and $\operatorname{Arg} v = \frac{\pi}{3}$ these sets become a regular hexagon and a three-pointed unbounded star, respectively (see Figure 2).

Despite the unboundedness of these sets, we still call them polygons (of the generalized type).

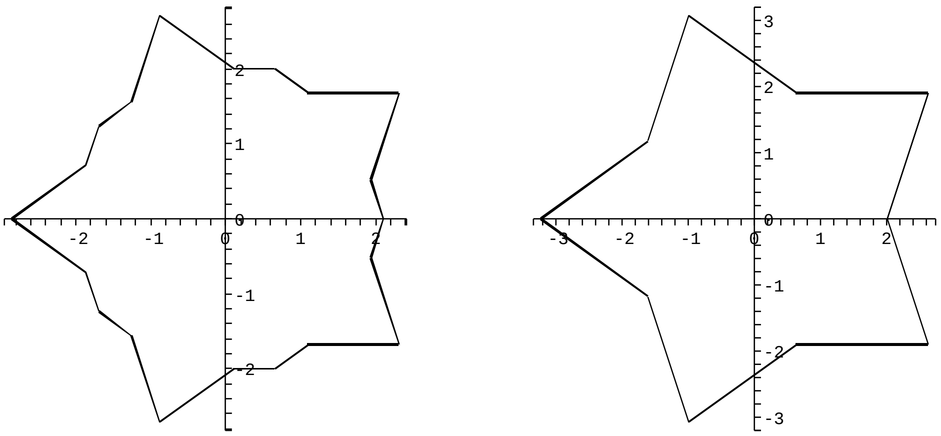


Figure 1: Polygons: **a)** $n = 5$, $\operatorname{Arg} v = \frac{\pi}{12}$ **b)** $n = 5$, $\operatorname{Arg} v = 0$.

The second family of polygons, denoted by $\mathcal{V}_2$, is defined as follows:

$$\mathcal{V}_2 = \{-W : W \in \mathcal{V}_1\} .$$

Let $f \in X^{(n)}$ and let n be an odd integer greater than or equal to 3. Assume that w, $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$, is the omitted value of f. Because of real coefficients, the function f also omits $\overline{w}$. From this and from n-fold symmetry of f, the set

$$\Omega = \{w\varepsilon^j , \overline{w}\varepsilon^j \ : \ j = 0, 1, \ldots , n-1\} \tag{10}$$

is disjoint from $f(\Delta)$.

All the points in Ω have the same modulus. Therefore, they can be arranged in accordance with the increase of the argument as follows:

$$0 \le \arg w \le \arg \overline{w}\varepsilon \le \arg w\varepsilon \le \arg \overline{w}\varepsilon^2 \le \cdots \le \arg w\varepsilon^{n-1} \le \arg \overline{w}\varepsilon^n \le 2\pi \, . \tag{11}$$

Now we take three successive points from Ω (in accordance with the order of (11)) in the following way. By w^* we denote the point which has the greatest imaginary part among the points in Ω and by w_L^* and w_R^* the points directly preceding and succeeding w^*. The choice of w^* is unique because each set $\Lambda e^{\frac{\pi}{n}ki}$, $k = 0, 1, \ldots , 2n-1$, contains only one point of Ω and because the set $\Lambda e^{\frac{\pi}{n}\frac{n-1}{2}i}$ is symmetric with respect to the imaginary axis. It is easy to check that $w^* \in \Lambda_{j_0+1}$, where $j_0 = \operatorname{Ent}(\frac{n}{4})$, and $w_L^* = \overline{w^*}\varepsilon^{\frac{n-1}{2}} = \overline{w^*}e^{\pi(1-\frac{1}{n})i}$, $w_R^* = w_L^*\varepsilon = \overline{w^*}\varepsilon^{\frac{n+1}{2}} = \overline{w^*}e^{\pi(1+\frac{1}{n})i}$.

Additionally, we assume that $w^* \in \partial f(\Delta)$. This means that each point of Ω belongs to $\partial f(\Delta)$. The function f is convex in the direction of the real axis, thus f omits all points lying on the ray $l_R \ : \ \zeta = w^* + t\, , \, t \ge 0$, or on the ray $l_L \ : \ \zeta = w^* - t\, , \, t \ge 0$.

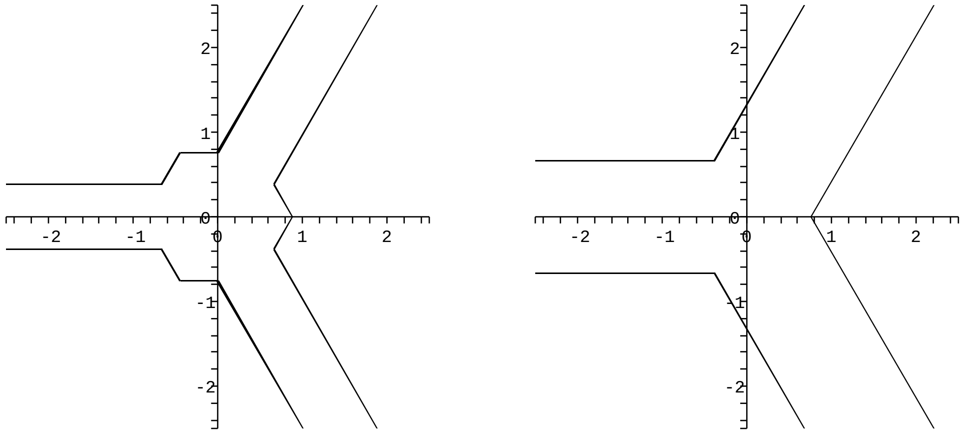


Figure 2: Polygons: **a)** $n = 3$, $\operatorname{Arg} v = \frac{\pi}{6}$ **b)** $n = 3$, $\operatorname{Arg} v = \frac{\pi}{3}$.

I. Suppose that $f(\Delta) \cap l_R = \emptyset$. From the symmetry of $f \in X^{(n)}$ with respect to the straight line $\zeta = t\varepsilon^{j_0}$, $t \geq 0$, the ray $k_R \, : \, \zeta = (\overline{w^*} + t)\varepsilon^{2j_0}$, $t \geq 0$, is also disjoint from $f(\Delta)$. From the n-fold symmetry of f, each ray of the form $l_R\varepsilon^j$ and $k_R\varepsilon^j$, $j = 0, 1, \ldots, n-1$, is disjoint from $f(\Delta)$.

Moreover, since $w_L^* \notin f(\Delta)$, one of two rays starting from w_L^* and parallel to the real axis is also disjoint from $f(\Delta)$. This ray appears to be $p_R \, : \, \zeta = w_L^* + t$, $t \geq 0$.
Indeed, if the ray $\zeta = w_L^* - t$, $t \geq 0$, were disjoint from $f(\Delta)$, then, from the symmetry with respect to the straight line $\zeta = te^{\frac{\pi}{2}(1-\frac{1}{n})i}$, $t \geq 0$ (by Corollary 1), the ray $\zeta = w^* - te^{\pi(1-\frac{1}{n})i}$, $t \geq 0$, would be disjoint from $f(\Delta)$. From this w^* and w_L^* would not belong to $\partial f(\Delta)$, a contradiction.
From the properties of $X^{(n)}$ it follows that each straight line $p_R\varepsilon^j$, $j = 0, 1, \ldots,$ $n-1$, and its reflection in the real axis have no common points with $f(\Delta)$.

We conclude from the above argument that $f(\Delta)$ is contained in a polygon with one vertex in w^*. One can verify that this polygon belongs to the family $\mathcal{V}_1$ when $n = 4k + 1$, $k \in N$, and to the family $\mathcal{V}_2$ when $n = 4k - 1$, $k \in N$.

II. If $f(\Delta) \cap l_L = \emptyset$ then each ray $l_L\varepsilon^j$, $j = 0, 1, \ldots, n-1$, and its reflection in the real axis have no common points with $f(\Delta)$. Similarly as in **I.**, it can be proved that $f(\Delta)$ is disjoint from $q_L \, : \, \zeta = w_R^* - t$, $t \geq 0$. From the properties of $X^{(n)}$ it follows that each ray $q_L\varepsilon^j$, $j = 0, 1, \ldots, n-1$, and its reflection in the real axis have no common points with $f(\Delta)$.

From above, $f(\Delta)$ is contained in a polygon with one vertex in w^*. This polygon is a member of $\mathcal{V}_2$ when $n = 4k + 1$, $k \in N$, and is a member of $\mathcal{V}_1$ when $n = 4k - 1$, $k \in N$.

By the Schwarz-Christoffel formulae there exists exactly one analytic function which maps Δ univalently onto a fixed polygon of the family $\mathcal{V}_1$ and has positive derivative in 0. This function is

$$\Delta \ni z \mapsto A \int_0^z \sqrt[n]{\frac{(\zeta^n - e^{in\varphi})(\zeta^n - e^{-in\varphi})}{(\zeta^n + 1)^3(\zeta^n - 1)}} d\zeta \ \ , \text{ for a suitable } \varphi \in \left[0, \frac{\pi}{n}\right] . \tag{12}$$

From now on we choose the principal branch of the n-th root. It can be easily checked that the above formula is still valid for $\varphi = 0$ and $\varphi = \frac{\pi}{n}$.

Putting suitable A into (12) we get the function with classical normalization

$$\Delta \ni z \mapsto \int_0^z \sqrt[n]{\frac{(1 - \zeta^n e^{-in\varphi})(1 - \zeta^n e^{in\varphi})}{(1 + \zeta^n)^3(1 - \zeta^n)}} d\zeta \ . \tag{13}$$

We denote this function by $F_{1,\varphi}$ and the polygon $F_{1,\varphi}(\Delta)$ by $A_{1,\varphi}$. With this notation $\mathcal{V}_1 = \{\lambda A_{1,\varphi} \ : \ \lambda > 0 \, , \ \varphi \in [0, \frac{\pi}{n}]\}$.

Moreover, let

$$v_1(\varphi) \equiv F_{1,\varphi}(e^{i\varphi}). \tag{14}$$

For a fixed φ, the point $v_1(\varphi)$ coincides with the vertex of the polygon $A_{1,\varphi}$ such that its argument is from the range $[0, \frac{\pi}{n}]$. Hence, v_1 is given by the formula

$$v_1 \ : \ \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1 - t^n)(1 - t^n e^{2in\varphi})}{(1 + t^n e^{in\varphi})^3(1 - t^n e^{in\varphi})}} dt \ , \tag{15}$$

and it is an injective function on $[0, \frac{\pi}{n}]$.

In a similar way, there is exactly one analytic function which maps Δ univalently onto a fixed polygon of the family $\mathcal{V}_2$ and has positive derivative in 0. By the definitions of $\mathcal{V}_1$ and $\mathcal{V}_2$, a function f maps Δ onto a polygon of the family $\mathcal{V}_1$ if and only if a function g, satisfying $g(z) = -f(-z)$, maps Δ onto a polygon of the family $\mathcal{V}_2$. Therefore, $F_{2,\varphi} \ : \ z \mapsto -F_{1,\varphi}(-z)$ is typically normalized and $F_{2,\varphi}(\Delta) \in \mathcal{V}_2$.

Let

$$v_2(\varphi) \equiv F_{2,\varphi}(e^{i\varphi}) \ . \tag{16}$$

Hence, we can write

$$v_2 \ : \ \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1 - t^n)(1 - t^n e^{2in\varphi})}{(1 + t^n e^{in\varphi})(1 - t^n e^{in\varphi})^3}} dt \ . \tag{17}$$

Let us define

$$F_1(z) = z \int_0^1 \sqrt[n]{\frac{(1 - t^n)(1 - t^n z^{2n})}{(1 + t^n z^n)^3(1 - t^n z^n)}} dt \ ,$$

and

$$F_2(z) = z \int_0^1 \sqrt[n]{\frac{(1 - t^n)(1 - t^n z^{2n})}{(1 + t^n z^n)(1 - t^n z^n)^3}} dt \ ,$$

Theorem 1. *Let $n \geq 3$ be odd.*

1. *The minorant of the class $\{f \in X^{(n)} \ : \ f(\Delta) \in \mathcal{V}_1\}$ is*
   a) F_1 for $n = 4k-1$, $k \in N$,
   b) F_2 for $n = 4k+1$, $k \in N$,
2. *The minorant of the class $\{f \in X^{(n)} \ : \ f(\Delta) \in \mathcal{V}_2\}$ is*
   a) F_2 for $n = 4k-1$, $k \in N$,
   b) F_1 for $n = 4k+1$, $k \in N$.

Proof.
For $n = 4k-1$, $k \in N$, and for a fixed $\varphi \in [0, \frac{\pi}{n}]$ we have

$$F_1(e^{i\varphi}) = e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1-t^n)(1-t^n e^{2in\varphi})}{(1+t^n e^{in\varphi})(1-t^n e^{in\varphi})^3}} dt \, .$$

Hence, values $F_1(e^{i\varphi})$ and $v_1(\varphi)$ are equal. Moreover, F_1 is n-fold symmetric and one-to-one on the boundary of Δ. This means that F_1 is univalent in whole Δ and from this reason F_1 is the minorant of $\{f \in X^{(n)} \ : \ f(\Delta) \in \mathcal{V}_1\}$.
Analogously, one can prove the theorem in other cases. ■

Theorem 2. *Let $n \geq 3$ be odd. Then $K_{X^{(n)}} = F_1(\Delta) \cap F_2(\Delta)$.*

Proof.
Let $n \geq 3$ be an odd fixed number. Let us denote by K the Koebe domain for $X^{(n)}$. From Theorem 1 we know that

$$K \subset F_1(\Delta) \cap F_2(\Delta) \, . \tag{18}$$

Suppose that $w = \varrho e^{i\varphi} \in \Lambda$ is a boundary point of K. Then the point w^*, which has the greatest imaginary part among the points of Ω, belongs to

$$K \cap \left\{ \zeta \in C \ : \ \frac{n-1}{2n}\pi \leq \arg \zeta \leq \frac{n+1}{2n}\pi \right\}$$

From Lemma 3, $-\overline{w^*}$ also belongs to this set. Without a loss of generality we can assume that

$$\operatorname{Re} -\overline{w^*} \leq 0 \leq \operatorname{Re} w^* \, .$$

We shall discuss three possibilities.

If the open segment with endpoints w^* and $-\overline{w^*}$ is contained in K, then $w^* \neq -\overline{w^*}$ and there exists a function $f \in X^{(n)}$ such that $w^* \in \partial f(\Delta)$. Hence

$$\{w^* + t \ : \ t \geq 0\} \cap f(\Delta) = \emptyset \quad \text{and} \quad \{-\overline{w^*} - t \ : \ t \geq 0\} \cap g(\Delta) = \emptyset \, ,$$

where $g(z) \equiv -f(-z)$. This implies

$$f \prec F_{1,\varphi} \quad \text{and} \quad g \prec F_{2,\varphi} \, ,$$

but the normalization of f leads to $f \equiv F_{1,\varphi}$ and $g \equiv F_{2,\varphi}$. Therefore,

$$\partial K \subset \partial F_1(\Delta) \cup \partial F_2(\Delta) \,.$$

This and (18) results in $K = F_1(\Delta) \cap F_2(\Delta)$.

In the second case, if the open segment with endpoints w^* and $-\overline{w^*}$ is disjoint from K, then the whole straight line passing through these points is also disjoint from K. There exist functions $f, h \in X^{(n)}$ such that $w^* \in \partial f(\Delta)$, $-\overline{w^*} \in \partial h(\Delta)$ and

$$\{w^* + t \;:\; t \geq 0\} \cap f(\Delta) = \emptyset \quad \text{and} \quad \{-\overline{w^*} + t \;:\; t \geq 0\} \cap h(\Delta) = \emptyset \,.$$

Now we conclude that

$$f \prec F_{1,\varphi} \quad \text{and} \quad h \prec F_{1,\varphi} \,.$$

Then $f \equiv F_{1,\varphi} \equiv h$, and consequently $w^* = -\overline{w^*}$, a contradiction.

Finally, if $w^* = -\overline{w^*}$, i.e. $\operatorname{Arg} w^* = \frac{\pi}{2}$, then $w^* \in \partial F_{1,\varphi}(\Delta)$ and $w^* \in \partial F_{2,\varphi}(\Delta)$. ■

The functions F_1 and F_2 are n-fold symmetric and connected by relation $F_1(-z) = -F_2(z)$, $z \in \Delta$. Observe that for all $z \in \Delta$

$$F_1(e^{i\frac{\pi}{n}}) = e^{i\frac{\pi}{n}} F_2(z) \,.$$

From the argument similar to this used in the proof of Lemma 1, the curves $\{F_1(e^{i\theta})\ ,\ \theta \in [0, \frac{\pi}{n}]\}$ and $\{F_2(e^{i\theta})\ ,\ \theta \in [0, \frac{\pi}{n}]\}$ are symmetric with respect to the ray $\zeta = e^{\frac{\pi i}{2n}} t\ ,\ t \geq 0$. This and Lemma 2 result in

Corollary 2. *The set $K_{X^{(n)}}$ for odd $n \geq 3$ is $2n$-fold symmetric.*

Since $K_{X^{(n)}} \cap \Lambda e^{\frac{n-1}{2n}\pi i}$, or equivalently,

$$K_{X^{(n)}} \cap \left\{ \zeta \in C \;:\; \frac{n-1}{2n}\pi \leq \arg \zeta \leq \frac{n+1}{2n}\pi \right\}$$

is convex in the direction of the real axis, each point of the boundary of $K_{X^{(n)}}, n = 4k-1$, which has argument from $\frac{n-1}{2n}\pi$ to $\frac{\pi}{2}$, is a vertex of some polygon of the family $\mathcal{V}_1$ and each point which has argument from $\frac{\pi}{2}$ to $\frac{n+1}{2n}\pi$ is a vertex of some polygon of the family $\mathcal{V}_2$. The same is true in the case $n = 4k+1$ but with exchanged families $\mathcal{V}_1$ and $\mathcal{V}_2$. Combining this and Theorem 2 we obtain

Theorem 3. *Let $n \geq 3$ be odd. The boundary of the Koebe domain for $X^{(n)}$ in the set Λ coincides with*

$$\{F_1(e^{i\theta})\ ,\ \theta \in [0, \tfrac{\pi}{2n}]\} \cap \{F_2(e^{i\theta})\ ,\ \theta \in [\tfrac{\pi}{2n}, \tfrac{\pi}{n}]\} \,.$$

Considering $2n$-fold symmetry of this boundary it is sufficient to describe this curve in any sector of the measure $\frac{\pi}{n}$. The boundary of the Koebe domain for $X^{(n)}$ can be written simply as follows:

Corollary 3. *Let $n \geq 3$ be odd. The boundary of the Koebe domain for $X^{(n)}$ is of the form*

$$\bigcup_{j=0,\dots,2n-1} e^{j\frac{\pi}{n}} \cdot \{F_1(e^{i\theta})\ ,\ \theta \in [-\tfrac{\pi}{2n}, \tfrac{\pi}{2n}]\} \,.$$

At the end of this paragraph it is interesting to look at one special case of the polygons discussed above. For $n = 3$ and $\varphi = 0$ the function $F_{2,0}$ takes form

$$F_{2,0}(z) = \int_0^z \frac{\sqrt[3]{1+\zeta^3}}{1-\zeta^3}\, d\zeta\, . \tag{19}$$

Since $F_{2,0}(\Delta) = -F_{1,0}(\Delta)$, the set $F_{2,0}(\Delta)$ is a three-pointed unbounded star (in Figure 2b the set $F_{1,0}(\Delta)$ is shown). All three bounded vertices of this polygon lie on the circle of the radius

$$a = |F_{2,0}(-1)| = \int_0^1 \frac{\sqrt[3]{1-t^3}}{1+t^3}\, dt = \frac{\sqrt[3]{2}}{6} B(\tfrac{1}{3}, \tfrac{2}{3}) = \frac{\sqrt[3]{2}}{6}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3})\, .$$

The symbols B and Γ stand for the Beta and the Gamma functions.

Therefore,

$$a = \frac{\sqrt[3]{2}}{3\sqrt{3}}\pi = 0.761\ldots\,,$$

which yields that the width of each strip of this star equals

$$d = \frac{\sqrt[3]{2}}{3}\pi = 1.319\ldots\,.$$

The function (19) will also appear in paragraph 4.

2 Koebe domain for $X^{(n)}$ and even n.

Let n be a fixed even integer, $n \geq 2$. From (5-6) and Theorem 4 established in [2] we obtain

Theorem 4. *Let $n \geq 2$ be even. The minorant of the class $X^{(n)}$ is of the form*

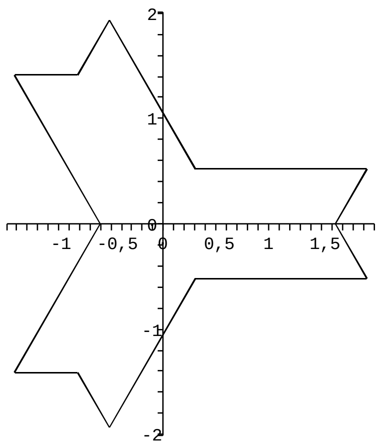
1. $G_1(z) = z \displaystyle\int_0^1 \sqrt[n]{\frac{(1-t^n)^2(1-t^n z^{2n})^2}{(1+t^n z^n)^4 (1-t^n z^n)^2}}dt$ *for* $n = 4k-2\,,\ k \in N$,
2. $G_2(z) = z \displaystyle\int_0^1 \sqrt[n]{\frac{(1-t^n)^2(1-t^n z^{2n})^2}{(1-t^n z^n)^4 (1+t^n z^n)^2}}dt$ *for* $n = 4k\,,\ k \in N$.

From this theorem we get the corollaries

Corollary 4. *Let n be a fixed even integer, $n = 4k-2\,,\ k \in N$.*

1. *$G_1(\Delta)$ is the Koebe domain for $X^{(n)}$,*
2. *The boundary of the Koebe domain for $X^{(n)}$ in Λ_1 is $v_2^1([0, \frac{\pi}{n}])$, where v_2^1 is given by*

$$v_2^1 \,:\, \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1-t^n)^2(1-t^n e^{2in\varphi})^2}{(1+t^n e^{in\varphi})^4(1-t^n e^{in\varphi})^2}}dt.$$



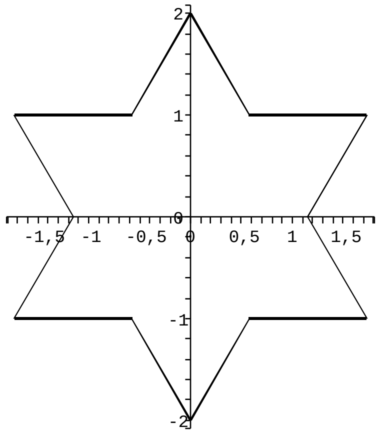


Figure 3: Polygons: **a)** $n = 3$, $\operatorname{Arg} v = \frac{\pi}{12}$ **b)** $n = 3$, $\operatorname{Arg} v = \frac{\pi}{6}$.

Corollary 5. *Let n be a fixed even integer, $n = 4k$, $k \in N$.*

1. *$G_2(\Delta)$ is the Koebe domain for $X^{(n)}$,*
2. *The boundary of the Koebe domain for $X^{(n)}$ in Λ_1 is $v_2^2([0, \frac{\pi}{n}])$, where v_2^2 is given by*

$$v_2^2 \ : \ \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1-t^n)^2(1-t^n e^{2in\varphi})^2}{(1-t^n e^{in\varphi})^4(1+t^n e^{in\varphi})^2}} dt.$$

3 Covering domain for $X^{(n)}$ and odd n.

Let n be a fixed odd integer, $n \geq 3$. We consider a family of open and n-fold symmetric polygons such that their successive vertices u, v, w belong to Λ and $\operatorname{Arg} u = 0$, $\operatorname{Arg} v \in (0, \frac{\pi}{n})$, $\operatorname{Arg} w = \frac{\pi}{n}$. The polygons' interior angles are of the measure $\pi(1 + \frac{1}{n})$ and $\pi(1 - \frac{2}{n})$ alternately. The measure of the angle with the vertex lying on the real positive semi-axis is equal to $\pi(1 + \frac{1}{n})$. From the above it follows that polygons of the described type have $4n$ sides.

For $n \neq 3$, this family of polygons is extended on limiting cases. If $u = v$ (hence $\operatorname{Arg} v = 0$), then we obtain polygons having $2n$ sides of the same length and angles measuring $\pi(1 - \frac{3}{n})$ and $\pi(1 + \frac{1}{n})$ alternately. If $v = w$ (hence $\operatorname{Arg} v = \frac{\pi}{n}$), then we obtain polygons having $2n$ sides of the same length and angles measuring $\pi(1 + \frac{1}{n})$ and $\pi(1 - \frac{3}{n})$ alternately.

In case $n = 3$ the limiting polygons become three-pointed stars described in paragraph 2.

We denote this family of polygons by $\mathcal{U}$. The polygons of this family are shown in Figures 3 and 4.

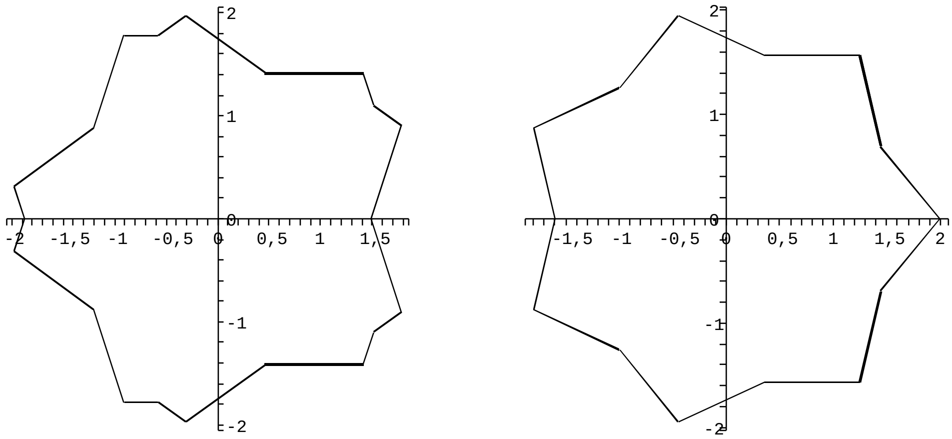


Figure 4: Polygons: **a)** $n = 5$, $\operatorname{Arg} v = \frac{3\pi}{20}$ **b)** $n = 7$, $\operatorname{Arg} v = 0$.

Let $f \in X^{(n)}$ and let n be odd integer greater than or equal to 3. Assume that $w \in \partial f(\Delta)$ and $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$ for $n \neq 3$ or $\operatorname{Arg} w \in (0, \frac{\pi}{n})$ for $n = 3$. Because of real coefficients, $\overline{w}$ also belongs to $\partial f(\Delta)$. From this and from n-fold symmetry of f, the set Ω given by (10) is contained in $\partial f(\Delta)$.

Like in the case of the Koebe domain, we choose three successive, in accordance with the order of (11), points from Ω: the w^* point which has the greatest imaginary part among the points in Ω and the w_L^*, w_R^* points directly preceding and succeeding w^*. One can check that $w_L^* \in \Lambda_{k_0}$ and $w_R^* \in \Lambda_{k_0+1}$, where $k_0 = \operatorname{Ent}(\frac{n+2}{4})$.

We claim that the segment $s_L = \{\zeta = w_L^* - t, t \geq 0\} \cap \Lambda_{k_0}$ is contained in $\operatorname{cl}(f(\Delta))$.

Assume that it is not the case. Hence, there exists $w_0 \in s_L$ such that $w_0 \notin f(\Delta)$. It follows that each ray $\zeta = (w_0 - t)\varepsilon^j, t \geq 0, j = 0, 1, \ldots, n-1$, and its reflection in the real axis are disjoint from $f(\Delta)$. Therefore, $f(\Delta)$ is contained in the polygon which has sides included in these rays. It means that $w_L^* \notin \partial f(\Delta)$, a contradiction.

Similarly, we can prove that $s_R = \{\zeta = w_R^* + t, t \geq 0\} \cap \Lambda_{k_0+1}$ is contained in $\operatorname{cl}(f(\Delta))$.

By Corollary 1, the segments $s_L\varepsilon^j$ and $s_R\varepsilon^j$, $j = 0, 1, \ldots, n-1$, and their reflection in the real axis are contained in the closure of $f(\Delta)$. Consequently, $f(\Delta)$ is contained in some polygon of the family $\mathcal{U}$.

The only analytic function which maps Δ univalently onto a fixed polygon of the family $\mathcal{U}$ and has positive derivative in 0 is of the form

$$\Delta \ni z \mapsto B \int_0^z \sqrt[n]{\frac{(\zeta^n + 1)(\zeta^n - 1)}{(\zeta^n - e^{in\varphi})^2 (\zeta^n - e^{-in\varphi})^2}} d\zeta \ , \text{ for a suitable } \varphi \in \left[0, \frac{\pi}{n}\right]. \quad (20)$$

We take the principal branch of the n-th root. The above formula is still valid for $\varphi = 0$ and $\varphi = \frac{\pi}{n}$.

Putting suitable B into (20) we get the function with typical normalization

$$\Delta \ni z \mapsto \int_0^z \sqrt[n]{\frac{(1+\zeta^n)(1-\zeta^n)}{(1-\zeta^n e^{-in\varphi})^2(1-\zeta^n e^{in\varphi})^2}}d\zeta \,. \tag{21}$$

We denote this function by G_φ and the polygon $G_\varphi(\Delta)$ by B_φ. With this notation $\mathcal{U} = \{\lambda B_\varphi \,:\, \lambda > 0\,,\, \varphi \in [0, \frac{\pi}{n}]\}$.

Moreover, let

$$u_1(\varphi) \equiv G_\varphi(e^{i\varphi}) \,. \tag{22}$$

The point $u_1(\varphi)$ coincides with the vertex of the polygon B_φ such that the argument of this vertex is from the range $[0, \frac{\pi}{n}]$. Hence u_1 is given by the formula

$$u_1 \,:\, \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto \int_0^1 \sqrt[n]{\frac{(1+t^n e^{in\varphi})(1-t^n e^{in\varphi})}{(1-t^n)^2(1-t^n e^{2in\varphi})^2}}dt \,. \tag{23}$$

and it is an injective function on $[0, \frac{\pi}{n}]$.

The following theorem can be proved in the same way as Theorems 1-2.

Theorem 5. *Let n be a fixed odd integer, $n \geq 3$. The function*

$$G(z) = z \int_0^1 \sqrt[n]{\frac{(1+t^n z^n)(1-t^n z^n)}{(1-t^n)^2(1-t^n z^{2n})^2}}dt$$

is the majorant for the class $X^{(n)}$.

Theorem 6. *For odd n, $n \geq 3$, there is $L_{X^{(n)}} = G(\Delta)$.*

One can easily check that $|G(z)| < |G(1)|$ for $z \in \Delta$. Hence,

Corollary 6.

$$\sup\left\{|f(z)| \,:\, f \in X^{(n)}, z \in \Delta\right\} = \begin{cases} \frac{B\left(\frac{1}{n}, \frac{n-3}{2n}\right)}{n\sqrt[n]{4}} & \textit{for} \quad n \geq 5 \\ \infty & \textit{for} \quad n = 3. \end{cases}$$

4 Covering domain for $X^{(n)}$ and even n.

From (5-6) and from Corollary 13 in [2] we get

Theorem 7. *Let n be a fixed even integer, $n \geq 4$. The majorant of the class $X^{(n)}$ is of the form*

1. $H_1(z) = z \int_0^1 \sqrt[n]{\frac{(1-t^n z^n)^2}{(1-t^n)^2(1-t^n z^{2n})^2}}dt \qquad \textit{for} \quad n = 4k-2\,,\, k \in N,$
2. $H_2(z) = z \int_0^1 \sqrt[n]{\frac{(1+t^n z^n)^2}{(1-t^n)^2(1-t^n z^{2n})^2}}dt \qquad \textit{for} \quad n = 4k\,,\, k \in N.$

This results in

Corollary 7. *Let n be a fixed even integer, $n = 4k - 2$, $k = 2, 3, \ldots$.*

1. *$H_1(\Delta)$ is the covering domain for $X^{(n)}$,*
2. *The boundary of the covering domain for $X^{(n)}$ in Λ_1 coincides with $u_2^1([0, \frac{\pi}{n}])$, where u_2^1 is given by the formula*

$$u_2^1 \ : \ \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1 - t^n e^{in\varphi})^2}{(1 - t^n)^2 (1 - t^n e^{2in\varphi})^2}} dt.$$

Corollary 8. *Let n be a fixed even integer, $n = 4k$, $k \in N$.*

1. *$H_2(\Delta)$ is the covering domain for $X^{(n)}$,*
2. *The boundary of the covering domain for $X^{(n)}$ in Λ_1 coincides with $u_2^2([0, \frac{\pi}{n}])$, where u_2^2 is given by the formula*

$$u_2^2 \ : \ \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1 + t^n e^{in\varphi})^2}{(1 - t^n)^2 (1 - t^n e^{2in\varphi})^2}} dt.$$

Theorem 8. *The covering domain for $X^{(2)}$ is whole C.*

The latter is a simple consequence of

$$C = h_0(\Delta) \cup h_1(\Delta) \subset \bigcup_{f \in X^{(2)}} f(\Delta) \, ,$$

where $h_0(z) = \frac{z}{1+z^2}$ and $h_1(z) = \frac{1}{2} \log \frac{1+z}{1-z}$. Both functions h_0 and h_1 belong to $X^{(2)}$.

Directly from Corollary 15 in [2] we get

Corollary 9. *For even n we have*

$$\sup\{|f(z)| \ : \ f \in X^{(n)}, z \in \Delta\} = \begin{cases} \frac{B\left(\frac{1}{n}, \frac{n-4}{2n}\right)}{n \sqrt[n]{4}} & \text{for} \quad n \geq 6 \\ \infty & \text{for} \quad n = 2 \quad \text{or} \quad n = 4. \end{cases}$$

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Department of Applied Mathematics,
Lublin University of Technology,
Nadbystrzycka 38D, 20-618 Lublin, Poland
l.koczan@pollub.pl, p.zaprawa@pollub.pl