On a certain generalization of the Carathéodory-Julia-Wolff theorem

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Abstract

Given an analytic self-mapping s of the open unit disk $\mathbb{D}$ and given a Blaschke product b of degree k, we present necessary and sufficient conditions for $s - b$ to have exactly k zeros inside $\mathbb{D}$. As a corollary, we obtain a Carathéodory-Julia-Wolff type theorem for meromorphic functions of the form s/b.

1 Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane and let $\mathbb{T}$ be the unit circle. The class of all functions s analytic on $\mathbb{D}$ and mapping $\mathbb{D}$ into itself will be denoted by $\mathcal{S}$. The values of s and s' at $t_0 \in \mathbb{T}$ will be understood in the sense of nontangential limits

$$s(t_0) := \lim_{z \widehat{\to} t_0} s(z) \quad \text{and} \quad s'(t_0) := \lim_{z \widehat{\to} t_0} s'(z), \tag{1.1}$$

provided the latter limits exist. In (1.1) and in what follows, we write $z \widehat{\to} t_0$ if a point $z \in \mathbb{D}$ tends to a boundary point $t_0 \in \mathbb{T}$ nontangentially, i.e., so that $|z - t_0| < \alpha(1 - |z|)$ for some $\alpha > 1$. We will write $z \to t_0$ if z tends to t_0 unrestrictedly (in $\mathbb{D}$ or in $\mathbb{C}$ which will be clear from the context).

If $s \in \mathcal{S}$ and $\lambda \in \mathbb{T}$, the function $\Re\left(\frac{\lambda + s(z)}{\lambda - s(z)}\right)$ is positive and harmonic in $\mathbb{D}$ and therefore, there exists a non-negative Borel measure $\mu_{s,\lambda}$ (called the *Aleksandrov-*

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Clark measure of s at λ) on $\mathbb{T}$ such that

$$\Re\left(\frac{\lambda+s(z)}{\lambda-s(z)}\right)=\int_{\mathbb{T}}\frac{1-|z|^2}{|z-\zeta|^2}d\mu_{s,\lambda}(\zeta). \tag{1.2}$$

In particular, one can define the measure $\mu_{s,s(t_0)}$ if the limit $s(t_0)$ exists and $|s(t_0)|=1$. On the other hand, if this is the case, then the limit

$$d_s(t_0)=\lim_{z\widehat{\to}t_0}\frac{1-|s(z)|^2}{1-|z|^2} \tag{1.3}$$

also exists (finite or infinite). The following theorem due to G. Julia [7], C. Carathéodory [6] and R. Nevanlinna [9] (see also [10, Chapter 6]) relates the characters from (1.1)–(1.3).

Theorem 1.1. *For $s\in\mathcal{S}$ and $t_0\in\mathbb{T}$, the following are equivalent:*

(1) $d:=\liminf_{z\to t_0}\dfrac{1-|s(z)|^2}{1-|z|^2}<\infty$; (2) $d_s(t_0)<\infty$;

(3) *The limits* (1.1) *exist and satisfy* $|s(t_0)|=1$ *and* $t_0s'(t_0)\overline{s(t_0)}\in\mathbb{R}$.

(4) *The limit $s(t_0)$ exists, $|s(t_0)|=1$, and the corresponding Aleksandrov-Clark measure $\mu_{s,s(t_0)}$ has an atom at t_0.*

Moreover, if these conditions hold, then

$$d=d_s(t_0)=t_0s'(t_0)\overline{s(t_0)}=\frac{1}{\mu_{s,s(t_0)}(\{t_0\})}>0. \tag{1.4}$$

We will denote by $\mathcal{B}_k$ the set of all Blaschke products of degree k. Since every $b\in\mathcal{B}_k$ is analytic on $\mathbb{T}$, it is defined everywhere on $\mathbb{T}$ along with all its derivatives. Furthermore, the existence of the finite limit $d_b(t_0)$ is obvious and the equalities (1.4) are verified directly using the Taylor expansion of b at t_0 and the symmetry relation $b(z)=1/\overline{b(1/\bar{z})}$. The following proposition follows immediately from Theorem 1.1.

Lemma 1.2. *Let $s\in\mathcal{S}$, $b\in\mathcal{B}_k$, $t_0\in\mathbb{T}$ and let us assume that the boundary limit $s(t_0)$ exists and equals $b(t_0)$. Then the following are equivalent:*

1. *The limit $s'(t_0)$ exist and satisfies* $t_0\overline{b(t_0)}\left(b'(t_0)-s'(t_0)\right)\geq 0$.
2. *The limit $d_s(t_0)$ exists and satisfies* $d_s(t_0)\leq d_b(t_0)$.
3. *The Aleksandrov-Clark measures $\mu_{s,b(t_0)}$ and $\mu_{b,b(t_0)}$ have atoms at t_0 which satisfy* $\mu_{s,b(t_0)}(\{t_0\})\geq\mu_{b,b(t_0)}(\{t_0\})$.

Let us consider the function f of the form $f=s-b$ where $s\in\mathcal{S}$, $b\in\mathcal{B}_k$ and let us denote by $N_{\mathbb{D}}(f)$ the number of zeros of f (counted with multiplicities) in $\mathbb{D}$. It follows from the Schwarz-Pick lemma that if $s\not\equiv b$, then $N_{\mathbb{D}}(s-b)\leq k$. The following theorem is the main result of this note.

Theorem 1.3. *Let $s \in \mathcal{S}$ and $b \in \mathcal{B}_k$ and let us assume that $s \not\equiv b$. Then $N_{\mathbb{D}}(s-b) < k$ if and only if there exists a point $t_0 \in \mathbb{T}$ such that the boundary limits $s(t_0)$ and $d_s(t_0)$ exist and satisfy*

$$s(t_0) = b(t_0) \quad \text{and} \quad d_s(t_0) \le d_b(t_0). \tag{1.5}$$

Moreover, if $N_{\mathbb{D}}(s-b) = n < k$, then there are at most $k-n$ points $t_0 \in \mathbb{T}$ subject to (1.5).

Observe that by Lemma 1.2, the second condition in (1.5) can be equivalently replaced by inequality $t_0\overline{b(t_0)}\left(b'(t_0) - s'(t_0)\right) \ge 0$ or by inequality $\mu_{s,b(t_0)}(\{t_0\}) \ge \mu_{b,b(t_0)}(\{t_0\})$.

Theorem 1.3 clarifies how distinct s and b must be on $\mathbb{T}$ in order to guarantee $N_{\mathbb{D}}(s-b) = k$. Using the boundary interpolation results from [5] it can be shown that for each $b \in \mathcal{B}_k$ and any sequence $\{t_i\}_{i\ge 1} \subset \mathbb{T}$, there exists $s \in \mathcal{S}$ such that

$$s(z) - b(z) = O(z - t_i) \quad \text{as} \;\; z\widehat{\rightarrow}t_i \;\; \text{for} \;\; i = 1, 2, \ldots \tag{1.6}$$

and still $N_{\mathbb{D}}(s-b) = k$. Theorem 1.3 shows that in this case we have necessarily $d_s(t_i) > d_b(t_i)$ for every $i \ge 1$.

To conclude the introduction we remark that in case $b(z) \equiv z$, Theorem 1.3 amounts to the Carathéodory-Julia-Wolff theorem: *If $s \in \mathcal{S}$ ($s \ne id$) has no fixed points in $\mathbb{D}$, then there exists a unique point $t_0 \in \mathbb{T}$ such that $s(t_0) = t_0$ and $d_s(t_0) = s'(t_0) \le 1$.* In Section 3 we will extend this theorem to the class of meromorphic functions of the form s/b where $s \in \mathcal{S}$ and b is a finite Blaschke product.

2 Proof of Theorem 1.3

To prove Theorem 1.3 we will use the following auxiliary construction. Let us assume that $N_{\mathbb{D}}(s-b) = n \le k = \deg b$ and let $z_1, \ldots, z_\ell$ be the zeros of the function $s - b$ of respective multiplicities $n_1, \ldots, n_\ell$ so that $n_1 + \ldots + n_\ell = n$. Then s and b have the same n_i first Taylor coefficients at z_i for $i = 1, \ldots, \ell$. Let us denote these Taylor coefficients by c_{ij}:

$$\frac{s^{(j)}(z_i)}{j!} = \frac{b^{(j)}(z_i)}{j!} = c_{ij} \quad \text{for} \quad j = 0, \ldots, n_i - 1;\; i = 1, \ldots, \ell. \tag{2.1}$$

Let $T = \operatorname{diag}\{T_1, \ldots, T_\ell\}$ be the diagonal block matrix with the diagonal block T_i equal the upper triangular $n_i \times n_i$ Jordan block with the number $\overline{z}_i \in \mathbb{D}$ on the main diagonal, let E be the row vector

$$E = \begin{bmatrix} E_1 & \ldots & E_\ell \end{bmatrix}, \quad \text{where} \quad E_i = \begin{bmatrix} 1 & 0 & \ldots 0 \end{bmatrix} \in \mathbb{C}^{1\times n_i}$$

and let $C \in \mathbb{C}^n$ be defined from the numbers c_{ij} as follows:

$$C = \begin{bmatrix} C_1 & \ldots & C_\ell \end{bmatrix}, \quad \text{where} \quad C_i = \begin{bmatrix} \overline{c}_{i,0} & \ldots \overline{c}_{i,n_i-1} \end{bmatrix} \in \mathbb{C}^{1\times n_i}.$$

We next let $P \in \mathbb{C}^{n\times n}$ to denote the Schwarz-Pick matrix

$$P = \left[\left[\frac{1}{m!r!} \frac{\partial^{m+r}}{\partial z^m \partial \bar{\zeta}^r} \frac{1 - b(z)\overline{b(\zeta)}}{1 - z\bar{\zeta}} \Bigg|_{\substack{z = z_i \\ \zeta = z_j}} \right]_{m=0,\ldots,n_i-1}^{r=0,\ldots,n_j-1} \right]_{i,j=1}^{\ell} \tag{2.2}$$

which is known to be positive definite whenever $n := n_1 + \ldots + n_\ell \le k := \deg b$. This matrix can be alternatively defined as the unique solution of the Stein equation

$$P - T^*PT = E^*E - C^*C \tag{2.3}$$

where T, E and C are defined as above. The verification of (2.3) for P of the form (2.2) is straightforward and the uniqueness follows from the fact that all the eigenvalues of T are in $\mathbb{D}$. We next define the 2×2 matrix function

$$\Theta(z) = I_2 - (1 - z\bar{\mu})K(z,\mu)J, \quad \text{where} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.4}$$

μ is an arbitrary point in $\mathbb{T}$ and

$$K(z,\mu) = \begin{bmatrix} E \\ C \end{bmatrix} (I_n - zT)^{-1} P^{-1} (I_n - \bar{\mu}T^*)^{-1} \begin{bmatrix} E^* & C^* \end{bmatrix}.$$

An easy computation based solely on the Stein identity (2.3) shows that

$$J - \Theta(z)J\Theta(z)^* = (1 - |z|^2)K(z,z) \tag{2.5}$$

which implies in particular that Θ is J-inner in $\mathbb{D}$:

$$\Theta(z)J\Theta(z)^* \le J \ \text{ if } z \in \mathbb{D}, \quad \Theta(t)J\Theta(t)^* = J \ \text{ if } t \in \mathbb{T}. \tag{2.6}$$

Another calculation based on (2.3) gives

$$\det \Theta(z) = \prod_{i=1}^{\ell} \left(\frac{(z - z_i)(\bar{\mu} - \bar{z}_i)}{(1 - z\bar{z}_i)(1 - \bar{\mu}z_i)} \right)^{n_i}. \tag{2.7}$$

The role of the function Θ in interpolation theory is justified by the following well-known result. In its formulation, we use the symbol $\mathcal{B}H^\infty$ to denote the closed unit ball of the Hardy space H^∞ of the unit disk.

Theorem 2.1. *Let* $\Theta = \begin{bmatrix} \theta_{11} & \theta_{21} \\ \theta_{12} & \theta_{22} \end{bmatrix}$ *be defined as in* (2.4). *Then the linear fractional formula*

$$g = \mathbf{T}_\Theta[\sigma] := \frac{\theta_{11}\sigma + \theta_{12}}{\theta_{21}\sigma + \theta_{22}}, \quad \sigma \in \mathcal{B}H^\infty, \tag{2.8}$$

establishes a one-to-one correspondence between $\mathcal{B}H^\infty$ *and the set of all functions* $g \in \mathcal{B}H^\infty$ *such that*

$$g^{(j)}(z_i) = j!c_{ij} \quad \text{for} \quad j = 0, \ldots, n_i - 1;\ i = 1, \ldots, \ell. \tag{2.9}$$

Furthermore, if $\sigma \in \mathcal{B}_q$*, then* $\mathbf{T}_\Theta[\sigma] \in \mathcal{B}_{n+q}$.

The set $\mathcal{B}H^\infty$ (sometimes called the *Schur class*) consists of all analytic functions mapping $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$ so that the inclusion $\mathcal{S} \subset \mathcal{B}H^\infty$ is clear. On the other hand, if a function $f \in \mathcal{B}H^\infty$ does not belong to $\mathcal{S}$, it follows from the maximum modulus principle that f is a unimodular constant function (that is, $f \in \mathcal{B}_0$). Thus, $\mathcal{B}H^\infty = \mathcal{S} \cup \mathcal{B}_0$. We supplement Theorem 2.1 by several simple observations. We first observe that for g and σ related as in (2.8),

$$\begin{bmatrix} 1 & -g \end{bmatrix} \Theta = u_g \begin{bmatrix} 1 & -\sigma \end{bmatrix}, \quad \text{where} \quad u_g := \theta_{11} - \theta_{21}g. \tag{2.10}$$

It follows from (2.10) that if $u_g(\zeta) = 0$, then $\Theta(\zeta)$ is not invertible so that $\det \Theta(\zeta) = 0$. Thus we conclude from (2.7) that $u_g(z) \ne 0$ for every $z \notin \{z_1, \ldots, z_\ell\}$.

Lemma 2.2. *Let g and σ be related as in* (2.8) *and let $t_0 \in \mathbb{T}$. Then*

1. *The limit $g(t_0)$ exists if and only if $\sigma(t_0)$ exists.*
2. *$|g(t_0)| = 1$ if and only if $|\sigma(t_0)| = 1$.*
3. *In the latter case, the limits $d_g(t_0)$ and $d_\sigma(t_0)$ are related by*

$$d_g(t_0) = \begin{bmatrix} 1 & -g(t_0) \end{bmatrix} K(t_0, t_0) \begin{bmatrix} 1 \\ -\overline{g(t_0)} \end{bmatrix} + |u_g(t_0)|^2 d_\sigma(t_0). \tag{2.11}$$

Proof: The first statement follows directly from (2.8). The second statement follows from (2.10) since $\Theta(t_0)$ is J-unitary (see the second formula in (2.6)). To complete the proof we multiply both parts of (2.5) by the row-vector $\begin{bmatrix} 1 & -g(z) \end{bmatrix}$ on the left, by its adjoint on the right, divide the resulting equality by $1 - |z|^2$ and take into account formula (2.4) for J to get

$$\frac{1 - |g(z)|^2}{1 - |z|^2} = \begin{bmatrix} 1 & -g(z) \end{bmatrix} K(z, z) \begin{bmatrix} 1 \\ -\overline{g(z)} \end{bmatrix} + |u_g(z)|^2 \frac{1 - |\sigma(z)|^2}{1 - |z|^2}. \tag{2.12}$$

Upon passing to the limit as $z \widehat{\rightarrow} t_0$ in the latter equality we get (2.11). Since the first term on the right hand side of (2.12) tends to a finite limit and since $u(t_0) \neq 0$, the limits $d_g(t_0)$ and $d_\sigma(t_0)$ in (2.11) are finite or infinite simultaneously. ∎

Lemma 2.3. *Let $s \in \mathcal{S}$ and $b \in \mathcal{B}_k$ meet conditions* (2.1). *Then*

$$s = \mathbf{T}_\Theta[\widetilde{s}] \quad \text{and} \quad b = \mathbf{T}_\Theta[\widetilde{b}] \quad \text{for some } \widetilde{s} \in \mathcal{B}H^\infty \text{ and } \widetilde{b} \in \mathcal{B}_{k-n}. \tag{2.13}$$

Furthermore, the limits $s(t_0)$ and $d_s(t_0)$ exist and satisfy (1.5) *if and only if the limits $\widetilde{s}(t_0)$ and $d_{\widetilde{s}}(t_0)$ exist and satisfy*

$$\widetilde{s}(t_0) = \widetilde{b}(t_0) \quad \text{and} \quad d_{\widetilde{s}}(t_0) \leq d_{\widetilde{b}}(t_0). \tag{2.14}$$

Proof: The first statement follows from Theorem 2.1. The existence part of the second statement follows from Lemma 2.2. The equivalence of the first equalities in (1.5) and (2.14) follows since Θ is analytic and invertible at t_0. Now let us assume that all the limits in (1.5) and (2.14) exist and that $s(t_0) = b(t_0)$. By part (3) in Lemma 2.2,

$$d_s(t_0) = \begin{bmatrix} 1 & -s(t_0) \end{bmatrix} K(t_0, t_0) \begin{bmatrix} 1 \\ -\overline{s(t_0)} \end{bmatrix} + |u_s(t_0)|^2 d_{\widetilde{s}}(t_0), \tag{2.15}$$

$$d_b(t_0) = \begin{bmatrix} 1 & -b(t_0) \end{bmatrix} K(t_0, t_0) \begin{bmatrix} 1 \\ -\overline{b(t_0)} \end{bmatrix} + |u_b(t_0)|^2 d_{\widetilde{b}}(t_0), \tag{2.16}$$

where according to (2.10), $u_s = \theta_{11} - \theta_{21} s$ and $u_b = \theta_{11} - \theta_{21} b$. Due to the assumption $s(t_0) = b(t_0)$, the first terms on the right in (2.15) and (2.16) are equal and also $u_s(t_0) = u_b(t_0)$. Subtracting (2.16) from (2.15) we get

$$d_s(t_0) - d_b(t_0) = |u_b(t_0)|^2 \left(d_{\widetilde{s}}(t_0) - d_{\widetilde{b}}(t_0) \right)$$

and since $u(t_0) \neq 0$, the equivalence of inequalities in (1.5) and (2.14) follows. ■

Proof of Theorem 1.3: To prove the sufficiency part we will argue via contradiction. Let us assume that $N_{\mathbb{D}}(s-b) = k$ and that (1.5) holds for some $t_0 \in \mathbb{T}$. By Lemma 2.3, s and b are of the form (2.13) where $\widetilde{b} \in \mathcal{B}_{k-k} = \mathcal{B}_0$. Thus $\widetilde{b} \equiv \gamma \in \mathbb{T}$ and therefore, $d_{\widetilde{b}}(t_0) = 0$. By Lemma 2.3, $\widetilde{s}(t_0) = \gamma$ and $0 \leq d_{\widetilde{s}}(t_0) \leq d_{\widetilde{b}}(t_0) = 0$. Since $|s(t_0)| = 1$ and $d_{\widetilde{s}}(t_0) = 0$, we conclude by the Julia lemma [7] that $\widetilde{s} \equiv \gamma$ which implies that $s \equiv b$. This contradicts the assumption of the theorem and completes the proof of the sufficiency part.

The necessity part will be first proved for the case $N_{\mathbb{D}}(s-b) = 0$, that is, under the assumption that $s(z) \neq b(z)$ for every $z \in \mathbb{D}$. Define

$$f_r(z) = \frac{r-1}{r}s(z) - b(z) \quad \text{for} \quad r \geq 1.$$

By Rouche theorem, $N_{\mathbb{D}}(f_r) = k$ for every r. Let us denote by ζ_r one (any one) of the zeros of f_r. If the set $\{\zeta_r\}$ had an accumulation point $\zeta \in \mathbb{D}$, then we would have $s(\zeta) = b(\zeta)$ and $f(\zeta) = 0$ which contradicts the assumption $N_{\mathbb{D}}(s-b) = 0$. Thus, $\{\zeta_r\}$ has an accumulation point $t_0 \in \mathbb{T}$. Take a sequence $\{\zeta_{r_i}\}$ converging to t_0. Thus,

$$\frac{r_i - 1}{r_i}s(\zeta_{r_i}) = b(\zeta_{r_i}) \tag{2.17}$$

and therefore,

$$\frac{1-|s(\zeta_{r_i})|^2}{1-|\zeta_{r_i}|^2} = \frac{1-\frac{r_i^2}{(r_i-1)^2}|b(\zeta_{r_i})|^2}{1-|\zeta_{r_i}|^2} \leq \frac{1-|b(\zeta_{r_i})|^2}{1-|\zeta_{r_i}|^2}. \tag{2.18}$$

Since b is a finite Blaschke product, the limit of the rightmost ratio in (2.18) exists and equals $d_b(t_0)$. Now we conclude from (2.18) that

$$d := \liminf_{z \to t_0} \frac{1-|s(z)|^2}{1-|z|^2} \leq d_b(t_0) < \infty. \tag{2.19}$$

Then by Theorem 1.1, the nontangential limits $s(t_0)$ and $d_s(t_0)$ exist and satisfy $s(t_0) = b(t_0)$ (due to (2.17)) and $d_s(t_0) = d \leq d_b(t_0)$ (by (2.19)).

For the general case, let us assume that $N_{\mathbb{D}}(s-b) = n < k$ and let $z_1, \ldots, z_\ell \in \mathbb{D}$ be the zeros of the function $s - b$ of respective multiplicities $n_1, \ldots, n_\ell$ so that $n_1 + \ldots + n_\ell = n$. By Lemma 2.3, s and b are of the form (2.13) where $\widetilde{s} \in \mathcal{B}H^\infty$ and $\widetilde{b} \in \mathcal{B}_{k-n}$. Since $s(\zeta) \neq b(\zeta)$ and $\det \Theta(\zeta) \neq 0$ for every $\zeta \in \mathbb{D} \setminus \{z_1, \ldots, z_\ell\}$, it is readily seen that $\widetilde{s}(\zeta) \neq \widetilde{b}(\zeta)$ for every such point ζ. On the other hand, it is well known (see e.g., [3]) that the value $\sigma(z_i)$ of the parameter σ in (2.8) at the interpolation node z_i completely determines the (n_i+1)-th Taylor coefficient $g^{(n_i)}(z_i)/n_i!$ of $g = \mathbf{T}_\Theta(\sigma)$. Since we assumed that $s - b$ has zero of multiplicity n_i at z_i, i.e., that $s^{(n_i)}(z_i) \neq s^{(n_i)}(z_i)$, it then follows that $\widetilde{s}(z_i) \neq \widetilde{b}(z_i)$ for $i = 1, \ldots, k$. Thus $N_{\mathbb{D}}(\widetilde{s} - \widetilde{b}) = 0$ and by the first part of the proof, there exists a point $t_0 \in \mathbb{T}$ such that the limits $\widetilde{s}(t_0)$ and $d_{\widetilde{s}}(t_0)$ exist and satisfy relations (2.14). But then it follows from Lemma 2.3 that the limits $s(t_0)$ and $d_s(t_0)$ exist and satisfy relations (1.5).

To prove the last statement of the theorem (again via contradiction), we assume that $N_{\mathbb{D}}(s-b) = n < k$ and that there exist $r := k-n+1$ points $t_1, \dots, t_r \in \mathbb{T}$ such that

$$s(t_i) = b(t_i) \quad \text{and} \quad d_s(t_i) \le d_b(t_i) \quad \text{for} \quad i = 1, \dots, r.$$

Then the functions $\widetilde{s} \in \mathcal{B}H^\infty$ and $\widetilde{b} \in \mathcal{B}_{k-n}$ from representations (2.13) meet conditions

$$\widetilde{s}(t_i) = \widetilde{b}(t_i) \quad \text{and} \quad d_{\widetilde{s}}(t_i) \le d_{\widetilde{b}}(t_i) \quad \text{for} \quad i = 1, \dots, r, \tag{2.20}$$

by Lemma 2.3. The $r \times r$ boundary Schwarz-Pick matrix

$$P = [p_{ij}]_{i,j=1}^r \quad \text{with entries} \quad p_{ij} = \begin{cases} d_{\widetilde{b}}(t_i) & \text{if} \quad i = j, \\ \dfrac{1 - \widetilde{b}(t_i)\overline{\widetilde{b}(t_j)}}{1 - t_i \bar{t}_j} & \text{if} \quad i \neq j, \end{cases}$$

constructed from b is positive semidefinite. By Lemma 2.1 in [4],

$$\text{rank} P = \min\{r, \deg \widetilde{b}\}. \tag{2.21}$$

Let us think for a moment that b is given and we are looking for a function $\widetilde{s} \in \mathcal{B}H^\infty$ satisfying interpolation conditions (2.20). Then we have a well-known boundary Nevanlinna-Pick problem [9] which has a unique solution if and only if the matrix P introduced just above is positive semidefinite and singular; see e.g., [2, 3, 5]. This is exactly what we have since by (2.21), $\text{rank}P = \deg b = n - k < r$. Thus, the only function $\widetilde{s} \in \mathcal{B}H^\infty$ satisfying conditions (2.20) is the function $\widetilde{b}$ itself. Therefore, conditions (2.20) imply that $\widetilde{s} \equiv \widetilde{b}$ and therefore, that $s = \mathbf{T}_\Theta[\widetilde{s}] \equiv \mathbf{T}_\Theta[\widetilde{b}] = b$ which gives the desired contradiction. ■

3 The Carathéodory-Julia-Wolff theorem for generalized Schur functions

In this concluding section we demonstrate that a version of Theorem 1.3 can be formulated in terms of fixed points of meromorphic functions g of the form $g = s/\vartheta$ where $s \in \mathcal{B}H^\infty$ and a finite Blaschke product ϑ do not have common zeros in $\mathbb{D}$. These functions (commonly known as generalized Schur functions) appeared in [1, 11] in certain interpolation context and have been studied later in [8]. We denote by $\mathcal{S}_k$ the class of generalized Schur functions g with the denominator $\vartheta \in \mathcal{B}_k$ in the above representation. Let us say that a point $z_0 \in \mathbb{D}$ is a fixed point of g of multiplicity (fixed point index) m if the function $z \to g(z) - z$ has zero of multiplicity m at z_0.

Theorem 3.1. *Let $g \in \mathcal{S}_k$. If g has less than $k+1$ fixed points in $\mathbb{D}$ counted with multiplicities, then there exists a boundary fixed point $t_0 \in \mathbb{T}$ such that the angular derivative $g'(t_0)$ exists and satisfies $g'(t_0) \le 1$.*

Proof: The statement trivially holds true if g is a unimodular constant (i.e., $g \in \mathcal{B}_0$). Also it is easily verified if g is of the form $g = \gamma/\vartheta$ for $\gamma \in \mathcal{B}_0$ and $\vartheta \in \mathcal{B}_k$ $(k > 0)$. Indeed, every g of this form has no fixed points in $\mathbb{C} \setminus \mathbb{T}$ and it has at least one fixed point $t_0 \in \mathbb{T}$. Then $\vartheta(t_0) = \gamma\bar{t}_0$ and by (1.4),

$$d_\vartheta(t_0) = t_0\vartheta'(t_0)\overline{\vartheta(t_0)} = \overline{\gamma}t_0^2\vartheta'(t_0) > 0. \tag{3.1}$$

On the other hand, $g'(t_0) = -\frac{\gamma\vartheta'(t_0)}{\vartheta(t_0)^2} = -\frac{\gamma\vartheta'(t_0)}{\gamma^2\bar{t}_0^2} = -\overline{\gamma}t_0^2\vartheta'(t_0)$ which together with (3.1) implies $g'(t_0) < 0$, that is, even more than wanted.

Since $\mathcal{B}H^\infty = \mathcal{S} \cup \mathcal{B}_0$, it remains to consider the case where g is of the form $g = s/\vartheta$ for some $s \in \mathcal{S}$ and $\vartheta \in \mathcal{B}_k$ having no common zeros in $\mathbb{D}$. Let $b := z\vartheta \in \mathcal{B}_{k+1}$. Then every zero of the function $s - b$ is a fixed point for g and vice versa. Then we have from the assumption of the theorem that $N_{\mathbb{D}}(s - b) < k + 1$; so we conclude from Theorem 1.3 that there is a point $t_0 \in \mathbb{T}$ such that the limits (1.1) exist and satisfy

$$s(t_0) = b(t_0) = t_0\vartheta(t_0) \quad \text{and} \quad t_0\overline{b(t_0)}\left(b'(t_0) - s'(t_0)\right) \geq 0. \tag{3.2}$$

Therefore the boundary limits $g(t_0)$ and $g'(t_0)$ exist. It follows from the first equality in (3.2) that $g(t_0) = t_0$ so that t_0 is a fixed boundary point for g. We now use equalities $b = z\vartheta$ and $s = g\vartheta$ to write the second relation in (3.2) in terms of g and ϑ as

$$\begin{aligned} 0 &\leq t_0\overline{b(t_0)}\left(b'(t_0) - s'(t_0)\right) \\ &= t_0\overline{t_0\vartheta(t_0)}\left(t_0\vartheta'(t_0) + \vartheta(t_0) - g'(t_0)\vartheta(t_0) - g(t_0)\vartheta'(t_0)\right) = 1 - g'(t_0) \end{aligned}$$

where the last equality follows since $g(t_0) = t_0$ and $|t_0| = |\vartheta(t_0)| = 1$. Thus, $g'(t_0) \leq 1$ as desired. ■

Note that in the classical case ($k = 0$), the boundary derivative $g'(t_0)$ is necessarily nonnegative at any boundary fixed point and thus, the bound $g'(t_0) \leq 1$ for $g \in \mathcal{B}H^\infty$ means that $|g'(t_0)| \leq 1$. On the other hand, if $g \in \mathcal{B}H^\infty$ has a (unique) fixed point z_0 in $\mathbb{D}$, then $|g'(z_0)| \leq 1$ by the Schwarz-Pick lemma. It therefore follows that every function $g \in \mathcal{B}H^\infty$ has a unique fixed point $z_0 \in \overline{\mathbb{D}}$ (the *Denjoy-Wolff point* of g) such that $|g'(z_0)| \leq 1$. From complex dynamics point of view, it might be of interest to characterize meromorphic (or at least rational) functions $g \in \mathcal{S}_\kappa$ having a Denjoy-Wolff point (maybe not unique). The following example shows that in general, such a point may not exist. Indeed, the function

$$g(z) = \frac{z}{\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}} = \frac{z(2-z)}{2z-1}$$

belongs to $\mathcal{S}_1$ and has two fixed points $z_0 = 0$ and $t_0 = 1$. Furthermore, $g'(z) = \frac{-2z^2+2z-2}{(2z-1)^2}$ and thus $g'(0) = g'(1) = -2$ (which of course is consistent with Theorem 3.1).

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