Constant angle surfaces in Minkowski space

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Abstract

A constant angle surface in Minkowski space is a spacelike surface whose unit normal vector field makes a constant hyperbolic angle with a fixed timelike vector. In this work we study and classify these surfaces. In particular, we show that they are flat. Next we prove that a tangent developable surface (resp. cylinder, cone) is a constant angle surface if and only if the generating curve is a helix (resp. a straight line, a circle).

1 Introduction and statement of results

A constant angle surface in Euclidean three-dimensional space $\mathbf{E}^3$ is a surface whose tangent planes make a constant angle with a fixed constant vector field of the ambient space [1, 9]. These surfaces generalize the concept of helix, that is, curves whose tangent lines make a constant angle with a fixed vector of $\mathbf{E}^3$. This kind of surfaces are models to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids [1]. Constant angle surfaces have been studied for arbitrary dimension in Euclidean space $\mathbf{E}^n$ [3, 12] and in different ambient spaces, e.g. $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and Nil_3 [2, 4, 5].

In this work we extend the concept of constant angle surfaces to a Lorentzian ambient space. Let $\mathbf{E}^3_1$ denote the three-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle\, , \,\rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2$$

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where (x_1, x_2, x_3) are the canonical coordinates in $\mathbb{R}^3$. In Minkowski space $\mathbf{E}_1^3$ and due to the variety of causal character of a vector, there is not a natural concept of angle between two arbitrary vectors and only it is possible to define the angle between *timelike* vectors.

Consider a (connected) surface M and a smooth immersion $x : M \to \mathbf{E}_1^3$. We say that x is a *spacelike* immersion if the induced metric on M via x is a Riemannian metric. This is equivalent to saying that any unit normal vector field ξ of M is timelike at each point. In particular, if $x : M \to \mathbf{E}_1^3$ is a spacelike immersion, then the surface M is orientable.

Definition 1.1. *Let $x : M \to \mathbf{E}_1^3$ be a spacelike immersion and let ξ be a unit normal vector field on M. We say that M is a constant angle surface if there is a fixed timelike vector U such that ξ makes a constant hyperbolic angle with U.*

In Theorem 3.4 we give a local description of any constant angle spacelike surface. As a consequence, we prove that they are ruled and flat surfaces (Corollary 3.6). Thus they must be tangent developable surfaces, cylinders and cones. In Section 4 we deal with tangent surfaces showing in Theorem 4.1 that

> *A tangent developable surface is a constant angle surface if and only if the generating curve is a helix.*

Finally we consider in Section 5 cylinders and cones. We show (see Theorems 5.1 and 5.3)

> *The only spacelike cylinders that are constant angle surfaces are planes. A cone is a constant angle surface if and only if the generating curve is a circle contained in a spacelike plane.*

2 Preliminaries

Most of the following definitions can be found in O'Neill's book [11]. Let $\mathbf{E}_1^3$ be the three-dimensional Minkowski space. A vector $v \in \mathbf{E}_1^3$ is said spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, and lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$. The norm (length) of a vector v is given by $|v| = \sqrt{|\langle v, v\rangle|}$.

In Minkowski space $\mathbf{E}_1^3$ one can define the angle between two vectors only if both are timelike. We describe this fact. If $u, v \in \mathbf{E}_1^3$ are two timelike vectors, then $\langle u, v\rangle \neq 0$. We say that u and v lie in the same timelike cone if $\langle u, v\rangle < 0$. This defines an equivalence binary relation with exactly two equivalence classes. If v lies in the same timelike cone than $E_3 := (0, 0, 1)$, we say that v is future-directed. For timelike vectors, we have the Cauchy-Schwarz inequality given by

$$|\langle u, v\rangle| \geq \sqrt{-\langle u, u\rangle}\sqrt{-\langle v, v\rangle}$$

and the equality holds if and only if u and v are two proportional vectors. In the case that both vectors lie in the same timelike cone, there exists a unique number $\theta \geq 0$ such that

$$\langle u, v\rangle = -|u||v| \cosh(\theta).$$

This number θ is called the *hyperbolic angle* between u and v.

Remark 2.1. *We point out that the above reasoning cannot work for other pairs of vectors, even if they are spacelike. For example, the vectors* $u = (\cosh(t), 0, \sinh(t))$ *and* $v = (0, \cosh(t), \sinh(t))$ *are spacelike vectors with* $|u| = |v| = 1$ *for any* t*. However the number* $\langle u, v\rangle = -\sinh(t)^2$ *takes arbitrary values from* 0 *to* $-\infty$*. Thus, there is no* $\theta \in \mathbb{R}$ *such that* $\cos(\theta) = \langle u, v\rangle$*.*

We also need to recall the notion of Lorentzian cross-product $\times : \mathbf{E}^3_1 \times \mathbf{E}^3_1 \to \mathbf{E}^3_1$. If $u, v \in \mathbf{E}^3_1$, the vector $u \times v$ is defined as the unique one that satisfies $\langle u \times v, w\rangle = \det(u, v, w)$, where $\det(u, v, w)$ is the determinant of the matrix whose columns are the vectors u, v and w with respect to the usual coordinates. An easy computation gives

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_2v_1 - u_1v_2).$$

As the cross-product in Euclidean 3-space, the Lorentzian cross-product in Minkowski space has similar algebraic and geometric properties, such as the anti-symmetry or the orthogonality on both factors.

Let $x : M \to \mathbf{E}^3_1$ be an immersion of a surface M into $\mathbf{E}^3_1$. We say that x is spacelike (resp. timelike, lightlike) if the induced metric on M via x is Riemannian (resp. Lorentzian, degenerated). This is equivalent to assert that a (local) normal vector ξ is timelike (resp. spacelike, lightlike). As the concept of angle is given only for timelike vectors, we have to consider those immersions whose unit normal vector is timelike, that is, *spacelike* immersions. Let x be a spacelike immersion. At any point $p \in M$, it is possible to choose a unit normal vector $\xi(p)$ such that $\xi(p)$ is future-directed, i.e. $\langle \xi(p), E_3\rangle < 0$. This shows that if x is a spacelike immersion, the surface M is orientable.

Denote $\mathfrak{X}(M)$ the space of tangent vector fields on M. Let $X, Y \in \mathfrak{X}(M)$. We write by $\tilde{\nabla}$ and ∇ the Levi-Civita connections of $\mathbf{E}^3_1$ and M respectively. Moreover,

$$\nabla_X Y = (\tilde{\nabla}_X Y)^\top$$

where the superscript $^\top$ denotes the tangent part of the vector field $\tilde{\nabla}_X Y$. We define the second fundamental form of x as the tensorial, symmetric map $\sigma : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$ given by

$$\sigma(X, Y) = (\tilde{\nabla}_X Y)^\perp$$

where by $^\perp$ we mean the normal part. The Gauss formula can be written as

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{1}$$

We denote by $A_\xi(X) = A(X)$ the tangent component of $-\tilde{\nabla}_X \xi$, that is, $A_\xi(X) = -(\tilde{\nabla}_X \xi)^\top$. Because $\langle \tilde{\nabla}_X \xi, \xi\rangle = 0$, we have the so-called Weingarten formula

$$\tilde{\nabla}_X \xi = -A_\xi(X). \tag{2}$$

The map $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called the *Weingarten endomorphism* of the immersion x.We have then $\langle AX, Y\rangle = \langle X, AY\rangle$. As a consequence, the Weingarten endomorphism is diagonalizable, that is, if $p \in M$, the map $A_p : T_pM \to T_pM$ defined by $A_p(v) = (AX)_p$ is diagonalizable, where $X \in \mathfrak{X}(M)$ is a vector field that extends v. The eigenvalues of A_p are called the *principal curvatures* and they will be denoted by $\lambda_i(p)$. Moreover, if $X, Y \in \mathfrak{X}(M)$, we have $\langle A(X), Y\rangle = \langle \sigma(X,Y), \xi\rangle$ and

$$\sigma(X,Y) = -\langle \sigma(X,Y), \xi\rangle \xi = -\langle A(X), Y\rangle \xi.$$

$$\widetilde{\nabla}_X Y = \nabla_X Y - \langle A(X), Y\rangle \xi.$$

Let $\{v_1, v_2\}$ be a basis in the tangent plane T_pM and denote

$$\sigma_{ij} = \langle \sigma(v_i, v_j), \xi\rangle = \langle A(v_i), v_j\rangle.$$

If we assume that this basis is orthonormal, we have from (1) and (2)

$$\widetilde{\nabla}_{v_i} V_j = \nabla_{v_i} V_j - \sigma_{ij}\xi. \tag{3}$$

$$\widetilde{\nabla}_{v_i} \xi = \sigma_{i1} v_1 + \sigma_{i2} v_2. \tag{4}$$

where V_i is a tangent vector field that extends v_i.

3 Classification of constant angle surfaces in $\mathbf{E}^3_1$

Let M be a constant angle spacelike surface in $\mathbf{E}^3_1$ whose unit normal vector field ξ is assumed to be future-directed. Without loss of generality, we assume that U is a unitary vector and after an isometry of the ambient space, we can take U as E_3. Denote by θ the hyperbolic angle between ξ and U, that is, $\cosh(\theta) = -\langle \xi, U\rangle$. If $\theta = 0$, then $\xi = U$ on M. This means that x describes the immersion of an affine plane parallel to Ox^1x^2. Throughout this work, we discard the trivial case that $\theta = 0$.

We decompose U as

$$U = U^\top + \cosh(\theta)\xi$$

where $U^\top$ is the projection of U on the tangent plane of M. Let

$$e_1 = \frac{U^\top}{|U^\top|},$$

which defines a unit tangent vector field on M and consider e_2 a unit vector field on M orthogonal to e_1 in such a way that $\{e_1, e_2, \xi\}$ defines a positively oriented unit orthonormal basis for every point of M. We write now the vector U in the following form

$$U = \sinh(\theta) e_1 + \cosh(\theta)\xi. \tag{5}$$

As U is a constant vector field, $\widetilde{\nabla}_{e_2} U = 0$ and (5) gives

$$\sinh(\theta)\widetilde{\nabla}_{e_2} e_1 + \cosh(\theta)\widetilde{\nabla}_{e_2}\xi = 0. \tag{6}$$

Taking the normal component and using (3), we obtain

$$\sinh(\theta)\langle \widetilde{\nabla}_{e_2} e_1, \xi\rangle = -\sinh(\theta)\sigma_{21} = 0.$$

Since $\theta \neq 0$, we conclude $\sigma_{21} = \sigma_{12} = 0$. By combining (4) and (6), it follows that

$$\widetilde{\nabla}_{e_2} e_1 = -\coth(\theta)\sigma_{22}\, e_2.$$

Analogously, we have $\widetilde{\nabla}_{e_1} U = 0$ and (5) yields

$$\sinh(\theta)\ \widetilde{\nabla}_{e_1} e_1 + \cosh(\theta)\ \widetilde{\nabla}_{e_1} \xi = 0.$$

The normal component of the above expression together with (3) gives $\sigma_{11}\sinh(\theta) = 0$, that is, $\sigma_{11} = 0$. We can summarize the above computations with a description of ∇ as follows:

Theorem 3.1. *With the above notations, the Levi-Civita connection ∇ for a constant angle spacelike surface in $\mathbf{E}^3_1$ is given by*

$$\begin{aligned} &\nabla_{e_1} e_1 = 0. \\ &\nabla_{e_1} e_2 = 0,\ \nabla_{e_2} e_1 = -\coth(\theta)\sigma_{22}\, e_2. \\ &\nabla_{e_2} e_2 = \coth(\theta)\sigma_{22}\, e_1. \end{aligned}$$

Moreover, with respect to $\{e_1, e_2\}$, the Weingarten map takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & -\sigma_{22} \end{pmatrix}.$$

At this moment one can choose coordinates u and v such that $\frac{\partial}{\partial u} = e_1$ and $\frac{\partial}{\partial v} = \beta\, e_2$, where β is a certain smooth function on the surface.

Corollary 3.2. *Given a constant angle spacelike surface M in $\mathbf{E}^3_1$, there exist local coordinates u and v such that the metric on M writes as $\langle\ ,\ \rangle = du^2 + \beta^2 dv^2$, where $\beta = \beta(u, v)$ is a smooth function on M, i.e. the coefficients of the first fundamental form are $E = 1$, $F = 0$ and $G = \beta^2$.*

Now, we will consider that the parametrization $x(u, v)$ given by the above Corollary. We know that $A(x_u) = 0$ and $\sigma_{11} = \sigma_{12} = 0$. From Theorem 3.1 one obtains

$$\begin{aligned} x_{uu} &= 0 \\ x_{uv} &= \frac{\beta_u}{\beta} x_v \\ x_{vv} &= -\beta\beta_u\, x_u + \frac{\beta_v}{\beta}\, x_v + \beta^2\sigma_{22}\xi \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \xi_u &= \widetilde{\nabla}_{x_u}\xi = 0. \\ \xi_v &= \widetilde{\nabla}_{x_v}\xi = \beta\sigma_{22} e_2 = \sigma_{22}\, x_v. \end{aligned}$$

As $\xi_{uv} = \xi_{vu} = 0$, it follows $\widetilde{\nabla}_{x_u}(\sigma_{22}x_v) = 0$. Using the fact that $\sigma_{12} = 0$, $\widetilde{\nabla}_{x_u}x_v = \widetilde{\nabla}_{x_v}x_u$ and Theorem 3.1, we obtain

$$0 = (\sigma_{22})_u x_v + \sigma_{22}\widetilde{\nabla}_{x_u}x_v = (\sigma_{22})_u x_v - \coth(\theta)\sigma_{22}^2 x_v.$$

Therefore

$$(\sigma_{22})_u - \coth(\theta)\sigma_{22}^2 = 0. \tag{7}$$

Also, we use the expression of x_{uv} to conclude that

$$(\sigma_{22})_u + \sigma_{22}\frac{\beta_u}{\beta} = 0$$

that is, $(\beta\sigma_{22})_u = 0$ and then, there exists a smooth function $\varphi = \varphi(v)$ depending only on v such that

$$\beta\sigma_{22} = \varphi(v). \tag{8}$$

Moreover, by combining (7) and (8), we have

$$\frac{\beta_u}{\beta} = -\coth(\theta)\sigma_{22}.$$

Proposition 3.3. *Consider a constant angle spacelike surface $x = x(u,v)$ in $\mathbf{E}_1^3$ where (u,v) are the coordinates given in Corollary 3.2. If $\sigma_{22} = 0$ on M, then x describes an affine plane.*

Proof. We know that $\beta_u = 0$ on M. Thus $x_{uv} = 0$ and hence, x_u is a constant vector. From (5), ξ is a constant vector field along M, and so, x parameterizes a (spacelike) plane. ∎

Here and in the rest of the work, we will assume that $\sigma_{22} \neq 0$. By solving equation (7), we obtain a function $\alpha = \alpha(v)$ such that

$$\sigma_{22}(u,v) = \frac{1}{-\coth(\theta)\ u + \alpha(v)}.$$

Then (8) yields

$$\beta(u,v) = \varphi(v)\Big(-\coth(\theta)\ u + \alpha(v)\Big).$$

Consequently,

$$\begin{aligned} x_{uu} &= 0 && (9)\\ x_{uv} &= \frac{\coth(\theta)}{\coth(\theta)u - \alpha(v)}\, x_v && (10)\\ x_{vv} &= \varphi^2\coth(\theta)(-\coth(\theta)u + \alpha)x_u \\ &+ \Big(\frac{\varphi'}{\varphi} + \frac{\alpha'}{-\coth(\theta)u + \alpha}\Big)x_v + \varphi^2(-\coth(\theta)u + \alpha)\xi. && (11) \end{aligned}$$

From (5) we have

$$\langle x_u, U\rangle = \sinh(\theta), \quad \langle x_v, U\rangle = 0$$

or equivalently

$$\langle x, U\rangle_u = \sinh(\theta), \quad \langle x, U\rangle_v = 0.$$

Then

$$\langle x, U\rangle = \sinh(\theta)u + \mu, \ \mu \in \mathbb{R}.$$

The parametrization of x is now (up to vertical translations)

$$x(u,v) = (x_1(u,v), x_2(u,v), -\sinh(\theta)u).$$

As $E = 1$, there exists a function $\phi : M \to \mathbb{R}$ such that

$$x_u = (\cosh(\theta)\cos\phi(u,v), \cosh(\theta)\sin\phi(u,v), -\sinh(\theta)).$$

Since $x_{uu} = 0$, one obtains $\phi_u = 0$, that is, $\phi = \phi(v)$ and hence

$$\begin{aligned} x_u &= (\cosh(\theta)\cos(\phi(v)), \cosh(\theta)\sin(\phi(v)), -\sinh(\theta)) \\ &= \cosh(\theta)(\cos(\phi(v)), \sin(\phi(v)), 0) - \sinh(\theta)(0,0,1). \end{aligned}$$

Denoting by

$$f(v) = (\cos(\phi(v)), \sin(\phi(v)))$$

we can rewrite x_u as

$$x_u = \cosh(\theta)(f(v), 0) - \sinh(\theta)(0,0,1).$$

We compute x_{uv}:

$$x_{uv} = \cosh(\theta)(f'(v), 0). \tag{12}$$

An integration with respect to u leads to

$$x_v = \cosh(\theta)(uf'(v) + h(v), 0) \tag{13}$$

where $h = h(v)$ is a smooth curve in $\mathbb{R}^2$. From (10) and (13)

$$x_{uv} = \frac{1}{\coth(\theta)u - \alpha(v)} \frac{\cosh^2(\theta)}{\sinh(\theta)}(uf'(v) + h(v), 0).$$

Comparing with (12) one gets

$$h = -\tanh(\theta)\alpha(v)f'(v)$$

and so,

$$x_v = \cosh(\theta)\big(u - \tanh(\theta)\alpha(v)\big)(f'(v), 0).$$

The value of x_{vv} is now

$$x_{vv} = \cosh(\theta)(u - \tanh(\theta)\alpha(v))(f''(v), 0) - \sinh(\theta)\alpha'(v)(f'(v), 0). \tag{14}$$

Multiplying the two expressions of x_{vv} in (11) and (14) by x_u, we conclude

$$\phi'(v) = \frac{1}{\sinh(\theta)}\varphi(v).$$

We do a change in the variable v to get $\phi' = 1$ for any v, that is, $\phi(v) = v$. It is not difficult to see that this does not change the second derivatives of x in (9), (10) and (11). Then

$$\begin{aligned} x_u &= \cosh(\theta)(\cos(v), \sin(v), 0) - \sinh(\theta)(0,0,1). \\ x_v &= \Big(\cosh(\theta)u - \sinh(\theta)\alpha(v)\Big)(-\sin(v), \cos(v), 0). \end{aligned}$$

The above reasoning can be written in the following

Theorem 3.4. *Let M be a constant angle spacelike surface in Minkowski space $\mathbf{E}_1^3$ which is not totally geodesic. Up to a rigid motion of the ambient space, there exist local coordinates u and v such that M is given by the parametrization*

$$x(u,v) = \Big(u\cosh(\theta)\big(\cos(v), \sin(v)\big) + \psi(v), -u\sinh(\theta)\Big) \tag{15}$$

with

$$\psi(v) = \sinh(\theta)\Big(\int \alpha(v)\sin(v), -\int \alpha(v)\cos(v)\Big) \tag{16}$$

where α is a smooth function on a certain interval I. Here θ is the hyperbolic angle between the unit normal of M and the fixed direction $U = (0,0,1)$.

Proposition 3.5. *A constant angle spacelike surface is flat.*

Proof. At each point $p \in M$, we consider $\{v_1(p), v_2(p)\}$ a basis of eigenvectors of the Weingarten endomorphism A_p. In particular, $\lambda_i(p) = -\sigma_{ii}(p)$. As the function $\langle \xi, U\rangle$ is constant, a differentiation along $v_i(p)$ yields $\langle \widetilde{\nabla}_{v_i(p)}\xi, U\rangle = 0$, $i = 1,2$. Using (4), we obtain

$$\lambda_1(p)\langle v_1(p), U\rangle = \lambda_2(p)\langle v_2(p), U\rangle = 0.$$

Assume that at the point p, $\lambda_1(p)\lambda_2(p) \neq 0$. By using the continuity of the principal curvature functions, we have $\langle v_1(q), U\rangle = \langle v_2(q), U\rangle = 0$ for every point q in a neighborhood $\mathcal{N}_p$ of p. This means that U is a normal vector in $\mathcal{N}_p$ and hence it follows $\theta = 0$: contradiction. Thus $\lambda_1(p)\lambda_2(p) = 0$ for any p, that is, $K = 0$ on M. ■

As in Euclidean space, all flat surfaces are characterized to be locally isometric to planes, cones, cylinders or tangent developable surfaces.

Corollary 3.6. *Any constant angle spacelike surface is isometric to a plane, a cone, a cylinder or a tangent developable surface.*

The fact that a constant angle (spacelike) surface is a ruled surface appears in Theorem 3.4. Exactly, the parametrization (15) writes as

$$x(u,v) = (\psi(v), 0) + u\Big(\cosh(\theta)\big(\cos(v), \sin(v)\big), -\sinh(\theta)\Big),$$

which proves that our surfaces are ruled. Next we present some examples of surfaces obtained in Theorem 3.4.

Example 1. We take different choices of the function α in (16).

1. Let $\alpha(v) = 0$. After a change of variables, $\psi(v) = (0, 0)$ and

$$x(u, v) = u(\cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta)).$$

This surface is a cone with the vertex the origin and whose basis curve is a circle in a horizontal plane. See Figure 1, left.

2. Let $\alpha(v) = 1$. Then $\psi(v) = -\sinh(\theta)(\cos(v), \sin(v))$ and

$$x(u, v) = -\sinh(\theta)(\cos(v), \sin(v), 0) + u(\cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta)).$$

Again, this surface is a cone based in a horizontal circle.

3. Consider $\alpha(v) = 1/\sin(v)$. Then $\psi(v) = \sinh(\theta)(v, -\log(|\sin(v)|))$ and

$$x(u, v) = \sinh(\theta)(v, -\log(|\sin(v)|), 0) + u(\cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta)).$$

See Figure 1, right.

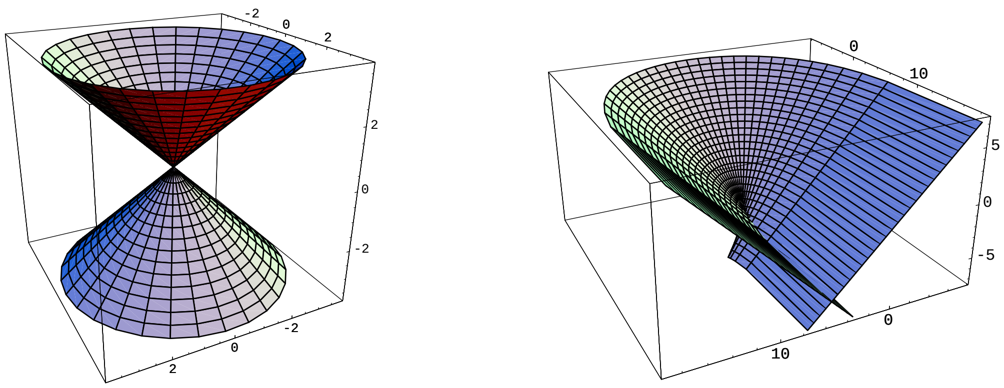


Figure 1: Constant angle surfaces corresponding to several choices of α in Theorem 3.4: $\alpha(v) = 0$ (left) and $\alpha(v) = 1/\sin(v)$ (right).

4 Tangent developable constant angle surfaces

In this section we study tangent developable surfaces that are constant angle surfaces (see [10] for the Euclidean ambient space). Given a regular curve $\gamma : I \to \mathbf{E}_1^3$, we define the tangent surface M generated by γ as the surface parameterized by

$$x(s, t) = \gamma(s) + t\gamma'(s), \quad (s, t) \in I \times \mathbb{R}.$$

The tangent plane at a point (s, t) of M is spanned by $\{x_s, x_t\}$, where

$$x_s = \gamma'(s) + t\gamma''(s), \qquad x_t = \gamma'(s).$$

The surface is regular at those points where $t(\gamma'(s) \times \gamma''(s)) \neq 0$. Without loss of generality, we will assume that $t > 0$.

On the other hand, since M is a spacelike surface and $\gamma(s) \in M$, the curve γ must be spacelike. We parameterize γ such that s is the arc-length parameter, that is, $\langle \gamma'(s), \gamma'(s) \rangle = 1$ for every s. As a consequence, $\gamma''(s)$ is orthogonal to $\gamma'(s)$. We point out that although γ is a spacelike curve, the acceleration vector $\gamma''(s)$ can be of any causal character, that is, spacelike, timelike or lightlike. However, the surface M is spacelike, which implies that γ is not an arbitrary curve. Indeed, by computing the first fundamental form $\{E, G, F\}$ of M with respect to basis $\{x_s, x_t\}$, we obtain

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}(s,t) = \begin{pmatrix} 1 + t^2\langle \gamma''(s), \gamma''(s) \rangle & 1 \\ 1 & 1 \end{pmatrix}.$$

M is spacelike if and only if $EG - F^2 > 0$. This is equivalent to $\langle \gamma''(s), \gamma''(s) \rangle > 0$, that is, $\gamma''(s)$ is spacelike for any s.

The tangent vector $\mathbf{T}(s)$ and the normal vector $\mathbf{N}(s)$ are defined by $\mathbf{T}(s) = \gamma'(s)$, $\mathbf{N}(s) = \gamma''(s)/\kappa(s)$, respectively, where $\kappa(s) = |\gamma''(s)| > 0$ is the curvature of γ at s. The Frenet Serret frame of γ at each point s associates an orthonormal basis $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$, where $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ is called the binormal vector ([6, 8]). We remark that $\mathbf{B}(s)$ is a unit timelike vector. The corresponding Frenet equations are

$$\begin{cases} \mathbf{T}' = & & \kappa\mathbf{N} & \\ \mathbf{N}' = & -\kappa\mathbf{T} & & +\tau\mathbf{B} \\ \mathbf{B}' = & & \tau\mathbf{N}. & \end{cases}$$

The function $\tau(s) = -\langle \mathbf{N}'(s), \mathbf{B}(s) \rangle$ is called the torsion of γ at s. For tangent surfaces x, the unit normal vector field ξ to M is $\xi = (x_s \times x_t)/\sqrt{EG - F^2} = -\mathbf{B}(s)$.

In order to give the next result, recall the concept of a helix in Minkowski space. A spacelike (or timelike) curve $\gamma = \gamma(s)$ parameterized by the arc-length is called *a helix* if there exists a vector $v \in \mathbf{E}_1^3$ such that the function $\langle \gamma'(s), v \rangle$ is constant. This is equivalent to saying that the function τ/κ is constant.

Theorem 4.1. *Let M be a tangent developable spacelike surface generated by γ. Then M is a constant angle surface if and only if γ is a helix with $\tau^2 < \kappa^2$. Moreover the direction U with which M makes a constant hyperbolic angle θ can be taken such that*

$$U = \frac{1}{\sqrt{\kappa^2 - \tau^2}}\Big(- \tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s)\Big) \tag{17}$$

and the angle θ is determined by the relation

$$\cosh(\theta) = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}. \tag{18}$$

Proof. 1. Assume that M makes a constant angle with a fixed direction U, with $\langle U, U \rangle = -1$. Then $\langle \mathbf{B}(s), U \rangle$ is a constant function c with $c < 0$. By differentiation with respect to s, and using the Frenet equation, we have

$\tau\langle \mathbf{N}(s), U\rangle = 0$ for any s. If $\langle \mathbf{N}(s_0), U\rangle\rangle \neq 0$ at some point s_0, then $\tau = 0$ in a neighborhood of s_0. This means that the binormal $\mathbf{B}(s)$ is a constant vector V, γ is a planar curve and $\xi = -V$ is constant on M. Thus, ξ makes constant angle not only with the vector U (which is fix from the beginning), but with any timelike vector. Hence U could be replaced by other vector, for example by V. Equations (17) and (18) are trivial. Finally, γ is a helix with $\tau^2 < \kappa^2$ and the surface is a (spacelike) affine plane.

If $\langle \mathbf{N}(s), U\rangle = 0$ on I, and because $\langle U, U\rangle = -1 = \langle \mathbf{T}(s), U\rangle^2 - c^2$, the function $\langle \mathbf{T}(s), U\rangle$ is a constant function. Therefore γ is a helix in $\mathbf{E}_1^3$ again. A differentiation of $\langle \mathbf{N}(s), U\rangle = 0$ gives $\langle \mathbf{T}(s), U\rangle = c\tau/\kappa$. Thus $-1 = c^2\tau^2/\kappa^2 - c^2$, which shows that $\tau^2 < \kappa^2$. Moreover, $c = -\kappa/\sqrt{\kappa^2 - \tau^2}$. As $U = \langle \mathbf{T}(s), U\rangle \mathbf{T}(s) - c\mathbf{B}(s)$, we get the expression (17). Finally (18) is trivial.

2. Conversely, let $\gamma = \gamma(s)$ be a helix and let $x = x(s,t)$ be the corresponding tangent surface. We know that τ/κ is a constant function. If $\tau = 0$, γ is a planar curve. Then the tangent surface generated by γ is a plane, which is a constant angle surface. If $\tau \neq 0$, let us define

$$U(s) = -\frac{\tau}{\kappa}\mathbf{T}(s) + \mathbf{B}(s).$$

Using the Frenet equations, we have $dU/ds = 0$, that is, U is a constant vector. Moreover, $\langle \xi, U\rangle = -\langle \mathbf{B}(s), U\rangle = 1$. Thus M is a constant angle surface. The hyperbolic angle θ is given by

$$\cosh(\theta) = \frac{\langle \xi, U\rangle}{\sqrt{-\langle U, U\rangle}} = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}. \qquad \blacksquare$$

We present two examples of constant angle surfaces that are tangent surfaces. After an isometry of the ambient space, we assume that $U = E_3$. From (18) if $\tau/\kappa = a$, with $|a| < 1$, then $\cosh(\theta) = 1/\sqrt{1-a^2}$. Moreover $\langle \mathbf{T}(s), U\rangle = -\sinh(\theta)$ and $\langle \gamma(s), E_3\rangle = -\sinh(\theta)s + b$, with $b \in \mathbb{R}$. After an appropriate change of variables, we take $b = 0$ and we write

$$\gamma(s) = (\gamma_1(s), \gamma_2(s), \sinh(\theta)s).$$

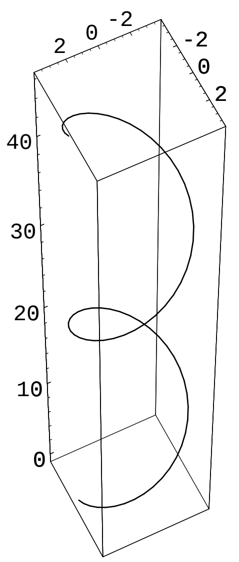
Because s is the arc-length parameter, there exists a smooth function $\lambda(s)$ such that $\gamma'(s) = (\cosh(\theta)\cos(\lambda(s)), \cosh(\theta)\sin(\lambda(s)), \sinh(\theta))$. An easy computation leads to

$$\begin{aligned}
\mathbf{N}(s) &= (-\sin(\lambda(s)), \cos(\lambda(s)), 0)\\
\mathbf{B}(s) &= (-\sinh(\theta)\cos(\lambda(s)), -\sinh(\theta)\sin(\lambda(s)), -\cosh(\theta)).
\end{aligned}$$

The curvature is $\kappa(s) = \cosh(\theta)\lambda'(s)$ and the torsion is $\tau(s) = -\sinh(\theta)\lambda'(s)$.

Example 2. We take $\lambda(s) = s$. An integration yields

$$\gamma(s) = (\cosh(\theta)\sin(s), -\cosh(\theta)\cos(s), \sinh(\theta)s).$$



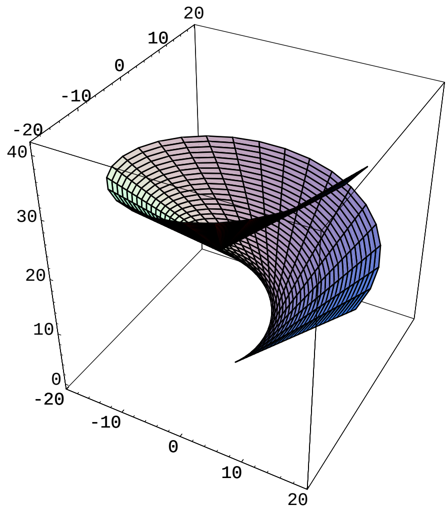


Figure 2: A constant angle tangent developable surface with $\kappa(s) = \cosh(\theta)$ and $\tau(s) = -\sinh(\theta)$. Here $\theta = 2$ and $U = (0, 0, 1)$.

Here $\kappa(s) = \cosh(\theta)$ and $\tau(s) = -\sinh(\theta)$ and γ is a helix where both the curvature and torsion functions are constant. A picture of the curve γ and the corresponding tangent surface appears in Figure 2.

Example 3. We take $\lambda(s) = s^2$. Recall that the Fresnel functions are defined as

$$\mathrm{FrS}(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \qquad \mathrm{FrC}(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt.$$

Then

$$\gamma(s) = \left(\sqrt{\tfrac{\pi}{2}} \cosh(\theta) \mathrm{FrC}\left(\sqrt{\tfrac{2}{\pi}}\, s\right), \sqrt{\tfrac{\pi}{2}} \cosh(\theta) \mathrm{FrS}\left(\sqrt{\tfrac{2}{\pi}}\, s\right), \sinh(\theta) s \right)$$

is a helix where $\kappa(s) = 2\cosh(\theta)s$ and $\tau(s) = -2\sinh(\theta)s$. Figure 3 shows the curve γ and the generated tangent surface.

Remark. We can extend the concept of constant angle surfaces for tangent developable *timelike* surfaces. Let M be a tangent surface generated by a curve γ such that M is timelike. Then γ is a spacelike curve (with γ'' timelike) or γ is a timelike curve (with γ'' spacelike). Assume that γ is parameterized by the arc-length s. Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the Frenet frame of γ, that is, $\mathbf{T}(s) = \gamma'(s)$, $\mathbf{N}(s) = \gamma''(s)/\kappa(s)$, with $\kappa(s) = |\gamma''(s)|$ and $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$. The Frenet equations are

$$\begin{cases} \mathbf{T}' = & & \kappa\mathbf{N} & \\ \mathbf{N}' = & \kappa\mathbf{T} & & +\tau\mathbf{B} \\ \mathbf{B}' = & & \epsilon\tau\mathbf{N} & \end{cases}$$

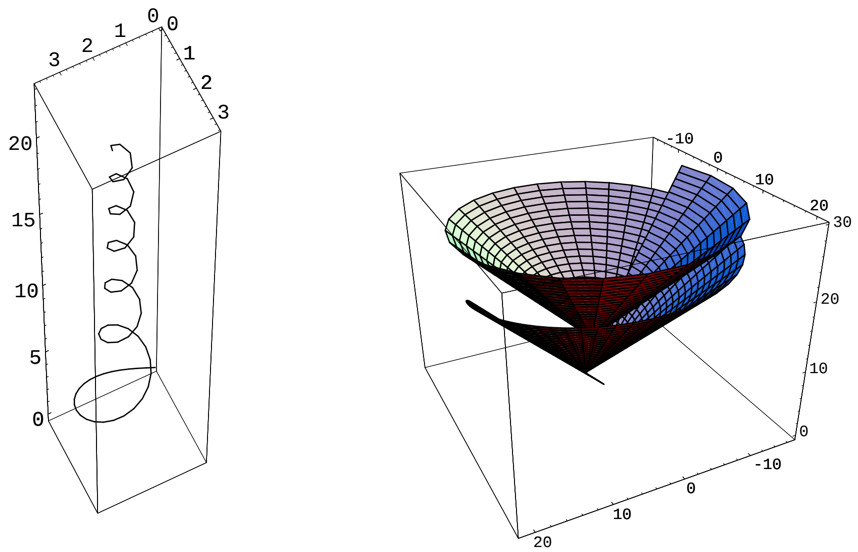


Figure 3: A constant angle tangent developable surface with $\kappa(s) = 2s\cosh(\theta)$ and $\tau(s) = -2s\sinh(\theta)$. Here $\theta = 2$ and $U = (0,0,1)$.

where $\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s)\rangle$ and $\langle \mathbf{T}(s), \mathbf{T}(s)\rangle = \epsilon = -\langle \mathbf{N}(s), \mathbf{N}(s)\rangle$, $\epsilon \in \{1,-1\}$. Anyway, $\mathbf{B}$ is always spacelike. We assume that there exists a fixed vector $U \in \mathbf{E}^3_1$ such that the function $\langle \xi, U\rangle$ is constant. Then it is not difficult to show that this condition is equivalent to saying that γ is a planar curve ($\tau = 0$, and M is an affine plane), or $\langle \mathbf{N}(s), U\rangle = 0$ for any s. In this case, the first Frenet equation yields $\langle \mathbf{T}'(s), U\rangle = 0$ and thus, $\langle \mathbf{T}(s), U\rangle$ is a constant function. This means that γ is a helix of $\mathbf{E}^3_1$. This generalizes Theorem 4.1 for tangent timelike surfaces.

We point out that our parametrization of M, $x(s,t) = \gamma(s) + t\gamma'(s)$ where γ is a helix given by

$$\gamma(s) = \left(\cosh(\theta)\int \cos(\lambda(s)), \cosh(\theta)\int \sin(\lambda(s)), \sinh(\theta)s\right)$$

does not satisfy the conditions of Corollary 3.2 since $F \neq 0$. In order to obtain the parametrization given in Theorem 3.4, we do a change of parameters given by

$$u = -(s+t)\ ,\ v = \pi + \lambda(s).$$

Now we obtain $x_s = -x_u + \lambda' x_v$ and $x_t = -x_u$.

But $x_t = (\cosh(\theta)\cos(\lambda(s)), \cosh(\theta)\sin(\lambda(s)), \sinh(\theta))$ or, in terms of u and v

$$x_u = (\cosh(\theta)\cos(v), \cosh(\theta)\sin(v), -\sinh(\theta)).$$

Similarly $x_s = x_t + t\lambda'(s)\ (-\cosh(\theta)\sin(\lambda(s)), \cosh(\theta)\cos(\lambda(s)), 0)$. It follows

$$x_v = \left(u + \lambda^{-1}(v-\pi)\right)\cosh(\theta)\ (-\sin(v), \cos(v), 0).$$

Consequently, the function α involved in the general formula can be expressed as

$$\alpha(v) = -\coth(\theta)\ \lambda^{-1}(v-\pi).$$

5 Constant angle cylinders and cones

In this section we consider cylinders and cones that are constant angle (spacelike) surfaces. A ruled surface is called a *cylinder* if it can be parameterized by $x(s,t) = \gamma(s) + tv$, where γ is a regular curve and v is a fixed vector. The regularity of the cylinder is given by the fact that $\gamma'(s) \times v \neq 0$. A *cone* is a ruled surface that can be parameterized by $x(s,t) = t\gamma(s)$, where γ is a regular curve. The vertex of the cone is the origin and the surface is regular wherever $t\big(\gamma(s) \times \gamma'(s)\big) \neq 0$.

Theorem 5.1. *The only constant angle (spacelike) cylinders are planes.*

Proof. Let M be a spacelike cylinder generated by a curve γ and a fixed direction v. As the surface is spacelike, v is a spacelike vector, for which we will assume $|v| = 1$. We can suppose that γ is contained in a plane Π such that v is orthogonal to Π. In particular, Π is a timelike plane. The unit normal vector is $\xi(s,t) = \xi(s) = \gamma'(s) \times v$.

By contradiction, we assume that γ is not a straight line, that is, $\kappa(s) \neq 0$ at some interval. We consider $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the Frenet frame of γ. As γ is a planar curve, $\mathbf{B}(s) = \pm v$ and so, $\xi(s) = \pm\mathbf{N}(s) := \gamma''(s)/\kappa(s)$. Let U be the unit (timelike) vector such that the function $\langle \xi(s), U\rangle$ is constant, that is, $\langle \mathbf{N}(s), U\rangle$ is constant. By differentiation with respect to s, using the Frenet equations and since γ is a planar curve, we obtain $\langle \mathbf{T}(s), U\rangle = 0$ for any s. A new differentiation gives $\kappa(s)\langle \mathbf{N}(s), U\rangle = 0$ for any s. As $\kappa(s) \neq 0$, we have $\langle \mathbf{N}(s), U\rangle = 0$, for any s. However, $\mathbf{N}(s)$ and U are both timelike vectors and thus, the product $\langle \mathbf{N}(s), U\rangle$ can never vanish: contradiction. Consequently, $\kappa(s) = 0$ for any s, that is, γ is a straight line and then M is a (spacelike) plane. ∎

Remark 5.2. *We point out that this result is more restrictive than the corresponding in Euclidean space* $\mathbf{E}^3$. *In* $\mathbf{E}^3$, *any cylinder is a constant angle surface: it suffices to take U as the vector that defines the rulings of the cylinder. The difference in Lorentzian ambient is that our surfaces are spacelike and the vector U is timelike, which imposes extra conditions.*

For the next result concerning cones, we recall that a (spacelike) circle in Minkowski space is a planar curve with constant curvature [7, 8]. We also point out that the plane Π containing the circle can be of any causal character. Indeed, after a rigid motion of $\mathbf{E}^3_1$, a spacelike circle can be viewed as follows: a Euclidean circle in a horizontal plane (if Π is spacelike), a hyperbola in a vertical plane (if Π is timelike) and a parabola in a $\pi/4$-inclined plane (if Π is lightlike).

Theorem 5.3. *Let M be a (spacelike) cone. Then M is a constant angle surface if and only if the generating curve is a circle in a spacelike plane or it is a straight line (and M is a plane).*

Proof. Let M be a cone, for which one can assume that its vertex is the origin of $\mathbb{R}^3$. Let $x(s,t) = t\gamma(s)$ be a parametrization of M, where $t \neq 0$ and $\gamma(s) \neq 0$, $s \in I$. As $x_s = t\gamma'(s)$ is spacelike, $\langle \gamma'(s), \gamma'(s)\rangle > 0$. On the other hand, x_t must be spacelike, this means that $\langle \gamma(s), \gamma(s)\rangle > 0$. We can change $\gamma(s)$ by a proportional vector and suppose that γ lies in the unit Minkowski sphere of $\mathbf{E}^3_1$, that is, in the de Sitter space $\mathbb{S}^2_1 = \{x \in \mathbf{E}^3_1; x_1^2 + x_2^2 - x_3^2 = 1\}$. Thus, $|\gamma(s)| = 1$ for any $s \in I$. Without loss of generality, we suppose that $\gamma = \gamma(s)$ is parameterized by the arc-length. Then $\gamma(s)$ and $\gamma''(s)$ are orthogonal to $\gamma'(s)$. The unit normal vector field ξ on M is collinear to $x_s \times x_t$. Denoting by $\mathbf{T}(s) = \gamma'(s)$, we have $\xi = \mathbf{T}(s) \times \gamma(s)$. In particular,

$$\gamma''(s) = -\gamma(s) - \langle \gamma''(s), \xi(s) \rangle \xi(s). \tag{19}$$

Assume that M is a constant angle surface and let U be the unit timelike vector such that $\langle \xi(s), U \rangle$ is constant. By differentiation with respect to s, we have

$$\langle \gamma''(s) \times \gamma(s), U \rangle = 0 \tag{20}$$

for any s. Substituting in (20) the value of $\gamma''(s)$ obtained in (19), we get

$$\langle \gamma''(s), \gamma'(s) \times \gamma(s) \rangle \langle \gamma'(s), U \rangle = 0.$$

We discuss the two possibilities:

1. If $\langle \gamma''(s), \gamma'(s) \times \gamma(s) \rangle \neq 0$ at some point, then $\langle \gamma'(s), U \rangle = 0$ for any s. This means that $\gamma(s)$ lies in a plane orthogonal to U and so, this plane must be spacelike. Thus the acceleration $\gamma''(s)$ is a spacelike vector. Then we can take the Frenet frame of γ, namely $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, where $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is a timelike vector. Moreover, $\mathbf{B}(s) = \pm U$. If $\kappa(s) = 0$ for any s, then γ is a straight line and the surface is a plane. On the contrary, since $\langle \mathbf{T}(s), \gamma(s) \rangle = 0$, by taking the derivative, one obtains $\kappa(s) \langle \mathbf{N}(s), \gamma(s) \rangle + 1 = 0$. On the other hand, because γ is a planar curve ($\tau = 0$), the derivative of the function $\langle \mathbf{N}(s), \gamma(s) \rangle$ vanishes. This means that $\langle \mathbf{N}(s), \gamma(s) \rangle$ is constant and so, $\kappa(s)$ is constant.

2. Assume $\langle \gamma''(s), \gamma'(s) \times \gamma(s) \rangle = 0$ for any s. As $\gamma(s)$ and $\gamma'(s)$ are orthogonal spacelike vectors, then $\gamma''(s)$ is a spacelike vector. Again, we consider the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ where $\mathbf{B}$ is a timelike vector. The above equation writes now as $\kappa(s) \langle \mathbf{B}(s), \gamma(s) \rangle = 0$. If $\kappa(s) = 0$ for any s, then γ is a straight line again. Suppose now $\langle \mathbf{B}(s), \gamma(s) \rangle = 0$. Similar to the previous case, because $\gamma(s) \in \mathbb{S}^2_1$, it follows $\langle \mathbf{T}(s), \gamma(s) \rangle = 0$ and $\kappa(s) \langle \mathbf{N}(s), \gamma(s) \rangle + 1 = 0$. In particular, $\langle \mathbf{N}(s), \gamma(s) \rangle \neq 0$ and then, the derivative of $\langle \mathbf{B}(s), \gamma(s) \rangle$ implies $\tau = 0$, that is, γ is a planar curve. Finally, the derivative of $\langle \mathbf{N}(s), \gamma(s) \rangle$ is zero, namely $\langle \mathbf{N}(s), \gamma(s) \rangle$ is constant, and then, $\kappa(s)$ is constant too. ∎

As an example of constant angle cones, Figure 1 (left) shows a cone based on a circle contained in a (horizontal) spacelike plane.

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