Convolution of Harmonic Mappings On The Exterior Unit Disk and the Generalized Hypergeometric Functions*

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Abstract

A new class is introduced consisting of harmonic univalent functions on the exterior unit disk defined by convolution. This class generates several known and new subclasses of harmonic univalent functions as special cases. A necessary and sufficient convolution condition is obtained for functions to belong to the class. A corresponding general class of harmonic functions with negative coefficients is also introduced, and coefficient condition that is both necessary and sufficient is obtained for the class. Extreme points are also determined. As applications, starlikeness conditions of the Liu-Srivastava linear operator involving the generalized hypergeometric functions are discussed.

1 Introduction

Complex-valued harmonic univalent functions have recently been studied from the perspective of geometric function theory. These studies were inspired by the seminal works of Clunie and Sheil-Small [6], and also by Sheil-Small [21] on the class S_H consisting of complex-valued harmonic orientation-preserving univalent

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mappings f defined on the open unit disk U, and normalized at the origin by $f(0) = 0$ and $f_z(0) = 1$. Various subclasses of S_H have since been investigated by several authors (see for example [2, 4, 8, 9, 11, 18, 19, 22, 25]).

In [7], Hengartner and Schober investigated the family Σ_H consisting of harmonic orientation-preserving univalent mappings f defined on $\widetilde{U} = \{z : |z| > 1\}$ that map ∞ to ∞. Such a mapping admits a representation of the form

$$f(z) = A log|z| + h(z) + \overline{g(z)},$$

where

$$h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}, \quad \text{and} \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$$

are analytic in $\widetilde{U}$, and $|\alpha| > |\beta|$. In addition, the function defined by $a = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$. By applying an affine transformation $(\overline{\alpha} f - \overline{\beta f} - \overline{\alpha} a_0 + \overline{\beta a_0})/(|\alpha|^2 - |\beta|^2)$, we may restrict our attention to the family $\Sigma_H^{'}$ of harmonic functions of the form

$$f(z) = z + A log|z| + \sum_{n=1}^{\infty} a_n z^{-n} + \overline{\sum_{n=1}^{\infty} b_n z^{-n}}.$$

The subclass with no logarithmic singularity will be denoted by $\Sigma_H^{''} := \{f \in \Sigma_H^{'} : A = 0\}$. Thus functions $f \in \Sigma_H^{''}$ have the representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \tag{1.1}$$

are analytic in $\widetilde{U}$. Several subclasses of the family $\Sigma_H^{''}$ have been studied in [1, 10, 12, 20]. In [10], the class of univalent harmonic functions starlike of a certain order was considered, and sufficient coefficient conditions were obtained. In [20] a class of harmonic functions related to the analytic univalent classes of uniformly convex functions and parabolic starlike functions [17] was investigated.

Now let σ be a real constant satisfying $|\sigma| = 1$, and $\Phi_\sigma = \phi_1 + \sigma\overline{\phi_2}$, where ϕ_1 and ϕ_2 are two analytic functions in $\widetilde{U}$, with

$$\phi_1(z) = z + \sum_{n=0}^{\infty} A_n z^{-n} \quad \text{and} \quad \phi_2(z) = z + \sum_{n=0}^{\infty} B_n z^{-n}. \tag{1.2}$$

In this paper, a new subclass of functions in $\Sigma_H^{''}$ defined by convolution is introduced. This subclass encompasses several classes investigated earlier, particularly those studied in [10, 20]. For that purpose, let us first recall the definition of convolution of two harmonic mappings.

If $f = h + \overline{g}$ is given by (1.1), and Φ_σ by (1.2), then the convolution $\Phi_\sigma * f$ in $\widetilde{U}$ is defined by

$$\begin{aligned} F(z) = (\Phi_\sigma * f)(z) &= (\phi_1 + \sigma\overline{\phi_2}) * (h + \overline{g})(z) \\ &= z + \sum_{n=1}^{\infty} a_n A_n z^{-n} + \sigma\overline{\sum_{n=1}^{\infty} b_n B_n z^{-n}}. \end{aligned}$$

With $F(z) = (\Phi_\sigma * f)(z)$ and $0 \leq \alpha < 1$, the function f is said to belong to the class $\Sigma_H(\Phi_\sigma, \alpha)$ provided $F \in \Sigma_H''$ and

$$\frac{\partial}{\partial\theta} arg(F(re^{i\theta})) > \alpha$$

on $|z| = r$ for each $r > 1$ and $0 \leq \theta < 2\pi$. Specifically, the class $\Sigma_H(\Phi_\sigma, \alpha)$ is given in the following definition:

Definition 1.1. Let σ be a real constant with $|\sigma| = 1$, and $0 \leq \alpha < 1$. Let $\Phi_\sigma(z) = \phi_1(z) + \sigma\overline{\phi_2(z)}$ be a given harmonic function in $\widetilde{U}$, where ϕ_1 and ϕ_2 are of the form (1.2). A harmonic function $f = h + \overline{g}$ where h and g are of the form (1.1), belongs to the class $\Sigma_H(\Phi_\sigma, \alpha)$ if $\Phi_\sigma * f \in \Sigma_H''$ satisfies the inequality

$$\Re\left\{\frac{z(h*\phi_1)'(z) - \sigma\overline{z(g*\phi_2)'(z)}}{(h*\phi_1)(z) + \sigma\overline{(g*\phi_2)(z)}}\right\} > \alpha, \quad (z \in \widetilde{U}). \tag{1.3}$$

Several subclasses of harmonic functions are special cases of the class $\Sigma_H(\Phi_\sigma, \alpha)$. Notable among these subclasses are the subclasses $\Sigma_H^*(\alpha)$ of harmonic starlike functions and $\Sigma_{KH}(\alpha)$ of harmonic convex functions investigated by Jahangiri [10], where

$$\Sigma_H(\Phi_1, \alpha) = \Sigma_H^*(\alpha) \quad \text{and} \quad \Sigma_H(\Phi_{-1}, \alpha) = \Sigma_{KH}(\alpha) \tag{1.4}$$

respectively, with

$$\Phi_1(z) = \frac{z}{1-1/z} + \frac{\overline{z}}{1-1/\overline{z}} = z + \sum_{n=0}^{\infty} z^{-n} + \overline{\left(z + \sum_{n=0}^{\infty} z^{-n}\right)}$$

and

$$\Phi_{-1}(z) = \frac{z-2}{(1-1/z)^2} - \frac{\overline{z}-2}{(1-1/\overline{z})^2} = z - \sum_{n=0}^{\infty} nz^{-n} - \overline{\left(z - \sum_{n=0}^{\infty} nz^{-n}\right)}.$$

Thus the class $\Sigma_H(\Phi_\sigma, \alpha)$ provides a unified treatment of various subclasses of harmonic mappings under appropriate choices of the parameter σ and harmonic function Φ.

In the next section of this paper, a necessary and sufficient convolution condition is obtained for the class $\Sigma_H(\Phi_\sigma, \alpha)$, which as application, yields a sufficient coefficient condition for the class. In Section 3, an appropriate general class of harmonic functions in Σ_H'' with negative coefficients is defined. Necessary and sufficient coefficient conditions are obtained. Growth estimates and extreme points are also determined for the class. In Section 4, starlikeness conditions of the Liu-Srivastava operator involving the generalized hypergeometric functions are investigated. Since many operators can be expressed in terms of the hypergeometric functions, the inclusion results obtained here will be useful for several other operators.

We shall require the following result:

Theorem 1.1. *[12] If f of the form* (1.1) *satisfies the inequality*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 1, \tag{1.5}$$

then f is a harmonic, orientation-preserving univalent function in $\widetilde{U}$.

2 Main Results

We now derive a convolution characterization for functions in the class $\Sigma_H(\Phi_\sigma, \alpha)$.

Theorem 2.1. *(Convolution Condition) Let $f = h + \overline{g} \in \Sigma''_H$, and $0 \leq \alpha < 1$. A function f belongs to $\Sigma_H(\Phi_\sigma, \alpha)$ if and only if*

$$(h * \phi_1) * \left[\frac{z + \frac{2\alpha - x - 3}{2 - 2\alpha}}{(1 - 1/z)^2}\right] - \sigma\overline{(g * \phi_2)} * \left[\frac{\frac{(x+\alpha)}{(1-\alpha)}\overline{z} - \frac{3x+1+2\alpha}{2-2\alpha}}{(1 - 1/\overline{z})^2}\right] \neq 0, \quad |x| = 1, |z| > 1.$$

Proof. A necessary and sufficient condition for $f = h + \overline{g}$ to be in the class $\Sigma_H(\Phi_\sigma, \alpha)$, with h and g of the form (1.1), is given by (1.3). The condition (1.3) holds if and only if

$$\frac{1}{(1-\alpha)}\left\{\frac{z(h * \phi_1)'(z) - \sigma\overline{z(g * \phi_2)'(z)}}{(h * \phi_1)(z) + \sigma\overline{(g * \phi_2)(z)}} - \alpha\right\} \neq \frac{x - 1}{x + 1}; \quad |x| = 1, x \neq -1, |z| > 1. \tag{2.1}$$

By a simple algebraic manipulation, (2.1) yields

$$\begin{aligned}
0 &\neq (x+1)[z(h * \phi_1)'(z) - \sigma\overline{z(g * \phi_2)'(z)}] - \alpha(x+1)[(h * \phi_1)(z) + \sigma\overline{(g * \phi_2)(z)}] \\
&\quad - (x-1)(1-\alpha)[(h * \phi_1)(z) + \sigma\overline{(g * \phi_2)(z)}] \\
&= (h * \phi_1) * \left[\frac{(x+1)(z-2)}{(1-1/z)^2} - \frac{(x + 2\alpha - 1)z}{1 - 1/z}\right] \\
&\quad - \sigma\overline{(g * \phi_2)} * \overline{\left[\frac{(\overline{x}+1)(z-2)}{(1-1/z)^2} + \frac{(\overline{x} + 2\alpha - 1)z}{(1 - 1/z)}\right]} \\
&= (h * \phi_1) * \left[\frac{2(1-\alpha)z + (2\alpha - x - 3)}{(1-1/z)^2}\right] \\
&\quad - \sigma\overline{(g * \phi_2)} * \overline{\left[\frac{2(\overline{x}+\alpha)z - (3\overline{x} + 2\alpha + 1)}{(1-1/z)^2}\right]}.
\end{aligned}$$

The latter condition, along with (1.3) for $x = -1$, establishes the result for all $|x| = 1$. ∎

An application of the convolution condition in Theorem 2.1 yields a sufficient coefficient condition for harmonic functions to belong to the class $\Sigma_H(\Phi_\sigma, \alpha)$.

Theorem 2.2. *If $f = h + \overline{g}$ of the form* (1.1) *and $\Phi_\sigma = \phi_1 + \sigma\overline{\phi_2}$ of the form* (1.2) *satisfy the coefficient inequality*

$$\sum_{n=1}^{\infty}(n+\alpha)|a_n||A_n| + \sum_{n=1}^{\infty}(n-\alpha)|b_n||B_n| \leq 1-\alpha,$$

then $f \in \Sigma_H(\Phi_\sigma, \alpha)$.

Proof. The given condition shows that the coefficients of $\Phi_\sigma * f$ satisfy

$$\sum_{n=1}^{\infty} n(|a_n||A_n| + |b_n||B_n|) \leq 1.$$

It follows from (1.5) in Theorem 1.1 that $\Phi_\sigma * f \in \Sigma_H''$. For h and g given by (1.1), Theorem 2.1 gives

$$\left|(h*\phi_1) * \left[\frac{z+\frac{2\alpha-x-3}{2-2\alpha}}{(1-1/z)^2}\right] - \sigma\overline{(g*\phi_2)} * \left[\frac{\frac{(x+\alpha)}{(1-\alpha)}\overline{z} - \frac{3x+1+2\alpha}{2-2\alpha}}{(1-1/\overline{z})^2}\right]\right|$$

$$= \left|z + \sum_{n=1}^{\infty}\left[n+2+(n+1)\frac{2\alpha-x-3}{2-2\alpha}\right]a_nA_nz^{-n}\right.$$
$$\left. -\sigma\sum_{n=1}^{\infty}\left[(n+2)\frac{x+\alpha}{1-\alpha} - (n+1)\frac{3x+2\alpha+1}{2-2\alpha}\right]\overline{b_nB_nz^{-n}}\right|$$
$$> |z|\left[1 - \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}|a_n||A_n| - |\sigma|\sum_{n=1}^{\infty}\frac{n-\alpha}{1-\alpha}|b_n||B_n|\right].$$

The last expression is non-negative by hypothesis, and hence by Theorem 2.1, it follows that $f \in \Sigma_H(\Phi_\sigma, \alpha)$. ∎

Remark 2.1. The coefficient bound in Theorem 2.2 can also be found in [10]. However the approach is different in this paper.

Using the relations (1.4), along with Theorem 2.2 yield the following two corollaries:

Corollary 2.1. *[10] Let $f = h + \overline{g}$ be of the form* (1.1), *and $0 \leq \alpha < 1$. If*

$$\sum_{n=1}^{\infty}\left[(n+\alpha)|a_n| + (n-\alpha)|b_n|\,\right] \leq 1-\alpha,$$

then $f \in \Sigma_H^*(\alpha)$.

Corollary 2.2. *[10] Let $f = h + \overline{g}$ be of the form* (1.1), *and $0 \leq \alpha < 1$. If*

$$\sum_{n=1}^{\infty} n\left[(n+\alpha)|a_n| + (n-\alpha)|b_n|\,\right] \leq 1-\alpha,$$

then $f \in \Sigma_{KH}(\alpha)$.

3 Harmonic mappings with negative coefficients

In this section, we shall devote attention to an appropriate subclass of harmonic functions with negative coefficients. Let us denote by $T\Sigma_H''$ the class consisting of functions $f = h + \overline{g} \in \Sigma_H''$, where

$$h(z) = z + \sigma \sum_{n=1}^{\infty} a_n z^{-n}, \quad \text{and} \quad g(z) = -\sum_{n=1}^{\infty} b_n z^{-n}, \quad a_n \geq 0, b_n \geq 0. \tag{3.1}$$

Let $\Phi_\sigma = \phi_1 + \sigma\overline{\phi_2}$, where

$$\phi_1(z) = z + \sigma \sum_{n=0}^{\infty} A_n z^{-n}, \quad \phi_2(z) = z + \sigma \sum_{n=0}^{\infty} B_n z^{-n}, \quad (A_n \geq 0, B_n \geq 0), \tag{3.2}$$

are given analytic functions in $\widetilde{U}$, and the real constant σ satisfies $|\sigma| = 1$.

We shall use the notation

$$T\Sigma_H(\Phi_\sigma, \alpha) := \Sigma_H(\Phi_\sigma, \alpha) \cap T\Sigma_H'',$$

and for the harmonic starlike situation, we let

$$T\Sigma_H^*(\alpha) := \Sigma_H^*(\alpha) \cap T\Sigma_H''.$$

A necessary and sufficient coefficient condition is obtained for the class $T\Sigma_H(\Phi_\sigma, \alpha)$.

Theorem 3.1. *Let f be of the form (3.1), and $0 \leq \alpha < 1$. The function f belongs to $T\Sigma_H(\Phi_\sigma, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} a_n A_n + \sum_{n=1}^{\infty} \frac{n-\alpha}{1-\alpha} b_n B_n \leq 1. \tag{3.3}$$

Proof. If f belongs to $T\Sigma_H(\Phi_\sigma, \alpha)$, then (1.3) is equivalent to

$$\Re\left\{ \frac{(1-\alpha)z - \sigma^2 \sum_{n=1}^{\infty}(n+\alpha)a_n A_n z^{-n} - \sigma^2 \sum_{n=1}^{\infty}(n-\alpha) b_n B_n \overline{z}^{\,-n}}{z + \sigma^2 \sum_{n=1}^{\infty} a_n A_n z^{-n} - \sigma^2 \sum_{n=1}^{\infty} b_n B_n \overline{z}^{\,-n}} \right\} > 0$$

for $z \in \widetilde{U}$. Letting $z \to 1^+$ through real values yields condition (3.3). The fact that condition (3.3) is sufficient is obtained from Theorem 2.2. ■

From (1.4), Theorem 3.1 yields the following result:

Corollary 3.1. *[10] Let f be of the form (3.1), and $0 \leq \alpha < 1$. Then $f \in T\Sigma_H^*(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left[(n+\alpha)a_n + (n-\alpha)b_n\right] \leq 1 - \alpha.$$

Also $f \in T\Sigma_{KH}(\alpha)$ if and only if

$$\sum_{n=1}^{\infty} n\left[(n+\alpha)a_n + (n-\alpha)b_n\right] \leq 1 - \alpha.$$

Theorem 3.2. *Let Φ_σ be of the form* (3.2) *with $A_n \geq A_1 > 0$, $B_n \geq B_1 > 0$, and $1 \leq B_1 \leq A_1$. If $f \in T\Sigma_H(\Phi_\sigma, \alpha)$, then for $|z| = r > 1$,*

$$r - \frac{1}{B_1} r^{-1} \leq |f(z)| \leq r + \frac{1}{B_1} r^{-1}.$$

Proof. First note that by assumptions,

$$(1-\alpha)B_1 \left[\sum_{n=1}^{\infty} (|\sigma| a_n + b_n)\right] \leq \sum_{n=1}^{\infty} [(n+\alpha)a_n A_n + (n-\alpha) b_n B_n] \leq 1 - \alpha.$$

Thus,

$$\begin{aligned} |f(z)| &= \left| z + \sigma \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z}^{-n} \right| \\ &\leq r + r^{-1} \left[\sum_{n=1}^{\infty} (|\sigma| a_n + b_n)\right] \\ &\leq r + \frac{1}{B_1} r^{-1}. \end{aligned}$$

The lower bound is obtained in a similar manner. ∎

The lower bound is sharp with equality for $f(z) = z - \frac{1}{\sigma^2 B_1} \overline{z}^{-1}$. The estimates given in the corollary below improve the bounds obtained by Jahangiri [10].

Corollary 3.2. *If $f \in T\Sigma_H^*(\alpha)$ or $f \in T\Sigma_{KH}(\alpha)$, then*

$$r - r^{-1} \leq |f(z)| \leq r + r^{-1}, \quad |z| = r > 1.$$

The class $T\Sigma_H(\Phi_\sigma, \alpha)$ is clearly convex. We now determine its extreme points.

Theorem 3.3. *Let*

$$h_0(z) := z, \quad h_n(z) := z + \frac{\sigma(1-\alpha)}{(n+\alpha)A_n} z^{-n},$$

and

$$g_0(z) := z, \quad g_n(z) := z - \frac{1-\alpha}{(n-\alpha)B_n} \overline{z}^{\,-n}, \quad (n = 1, 2, \cdots).$$

A function $f \in T\Sigma_H(\Phi_\sigma, \alpha)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} (\lambda_n h_n(z) + \gamma_n g_n(z)),$$

where $\lambda_n \geq 0$, $\gamma_n \geq 0$, and $\sum_{n=0}^{\infty} (\lambda_n + \gamma_n) = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(\lambda_n h_n(z)+\gamma_n g_n(z)) \\ &= z+\sigma\sum_{n=1}^{\infty}\lambda_n\frac{1-\alpha}{(n+\alpha)A_n}z^{-n}-\sum_{n=1}^{\infty}\gamma_n\frac{1-\alpha}{(n-\alpha)B_n}\overline{z}^{\,-n}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}\lambda_n\frac{1-\alpha}{(n+\alpha)A_n}A_n+\sum_{n=1}^{\infty}\frac{n-\alpha}{1-\alpha}\gamma_n\frac{1-\alpha}{(n-\alpha)B_n}B_n \\ &=\sum_{n=1}^{\infty}(\lambda_n+\gamma_n)=1-\lambda_0-\gamma_0\le 1, \end{aligned}$$

it follows from Theorem 3.1 that $f\in T\Sigma_H(\Phi_\sigma,\alpha)$.

Conversely, if $f\in T\Sigma_H(\Phi_\sigma,\alpha)$, then

$$a_n\le\frac{1-\alpha}{(n+\alpha)A_n},\quad \text{and}\quad b_n\le\frac{1-\alpha}{(n-\alpha)B_n}.$$

For $n\ge 1$, set

$$\lambda_n=\frac{n+\alpha}{1-\alpha}a_nA_n,\ \gamma_n=\frac{n-\alpha}{1-\alpha}b_nB_n,\ \ 0\le\lambda_0\le 1,$$

and

$$\gamma_0=1-\lambda_0-\sum_{n=1}^{\infty}(\lambda_n+\gamma_n).$$

Then it is easily seen that $\sum_{n=0}^{\infty}(\lambda_n h_n(z)+\gamma_n g_n(z))=f(z)$. ■

4 The Liu-Srivastava Linear Operator

As applications in this final section, we take the operator Φ_σ discussed in the earlier sections to be the Liu-Srivastava operator involving the generalized hypergeometric functions. For that purpose, first let us denote by $\widetilde{\Sigma}$ the class of all analytic functions f in $\widetilde{U}$ of the form

$$f(z)=z+\sum_{k=0}^{\infty}a_kz^{-k}.$$

For $\alpha_j\in\mathbb{C}\quad(j=1,2,\cdots,l)$ and $\beta_k\in\mathbb{C}\setminus\{0,-1,-2,\cdots\}$ $(k=1,2,\cdots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z)$ in $\widetilde{U}$ is defined by the infinite series

$${}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z):=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_l)_k}{(\beta_1)_k\ldots(\beta_m)_k}\frac{z^{-k}}{k!}$$

$(l \leq m+1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \cdots\})$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

It is known [23, p.43] that the ${}_lF_m$ series is absolutely convergent in $\mathbb{C}$ if $l \leq m$, and in $\widetilde{U}$ if $l = m+1$. Furthermore, if

$$\Re\left(\sum_{j=1}^{m} \beta_j - \sum_{j=1}^{l} \alpha_j\right) > 0,$$

then the ${}_lF_m$ series is absolutely convergent for $|z| = 1$. Corresponding to the function $z\,{}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$, the Liu-Srivastava operator [5, 15, 16]

$$H^{(l,m)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) : \widetilde{\Sigma} \to \widetilde{\Sigma}$$

is defined by the Hadamard product

$$\begin{aligned} H^{(l,m)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) f(z) &:= z\,{}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z) * f(z) \\ &= z + \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n+1} \ldots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \cdots (\beta_m)_{n+1}} \frac{a_n z^{-n}}{(n+1)!}. \end{aligned}$$

For convenience, we write

$$z\,{}_lF_m[\alpha; \beta; z] := z\,{}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z),$$

$$H^{l,m}[\alpha; \beta] f(z) := H^{(l,m)}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) f(z).$$

Special cases of the Liu-Srivastava linear operator include the Carlson-Shaffer linear operator $\mathfrak{L}(a, c) := H^{(2,1)}(1, a; c)$ (studied among others by Liu and Srivastava [14], Liu [13], and Yang [27]), the operator $D^{n+1} := \mathfrak{L}(n+1, 1)$, which is analogous to the Ruscheweyh derivative operator (investigated by Yang [26]), and the operator

$$J_c := \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt = \mathcal{L}(c, c+1) \quad (c > 0)$$

(studied by Uralegaddi and Somanatha [24]).

Corresponding to $f = h + \overline{g}$ given by (1.1), we define an operator $\mathcal{L}$ on f given by

$$\mathcal{L}[f] = \Phi_\sigma * f = (\phi_1 + \sigma\overline{\phi_2}) * (h + \overline{g}), \tag{4.1}$$

where

$$\phi_1(z) = z\,{}_lF_m[\lambda; \beta; z] = z + \sum_{n=0}^{\infty} A_n z^{-n}, \qquad \phi_2(z) = z\,{}_pF_q[c; d; z] = z + \sum_{n=0}^{\infty} B_n z^{-n}, \tag{4.2}$$

and

$$A_n = \frac{(\lambda_1)_{n+1} \cdots (\lambda_l)_{n+1}}{(\beta_1)_{n+1} \cdots (\beta_m)_{n+1}} \frac{1}{(n+1)!}, \qquad B_n = \frac{(c_1)_{n+1} \cdots (c_p)_{n+1}}{(d_1)_{n+1} \cdots (d_q)_{n+1}} \frac{1}{(n+1)!}. \tag{4.3}$$

Of course here we are assuming that none of the denominator parameters can be zero or a negative integer. A similar operator to $\mathcal{L}$ defined by (4.1) was recently studied in the unit disk by Ahuja *et al.* [3].

Theorem 4.1. *Let $f = h + \overline{g} \in \Sigma''_H$ be of the form* (1.1), *where the coefficients a_n and b_n satisfy*

$$|a_n| \leq \frac{1-\alpha}{n+\alpha}, \quad \text{and} \quad |b_n| \leq \frac{1-\alpha}{n-\alpha}, \ (n \geq 1). \tag{4.4}$$

Let ϕ_1 and ϕ_2 of the form (4.2) *satisfy*

$$\sum_{j=1}^{m} \beta_j > \sum_{j=1}^{l} |\lambda_j|, \qquad \sum_{j=1}^{q} d_j > \sum_{j=1}^{p} |c_j|,$$

where $\beta_j > 0$ $(j = 1, \ldots, m)$ and $d_j > 0$ $(j = 1, \ldots, q)$. If

$${}_lF_m[|\lambda|;\beta;1] - \frac{\prod_{j=1}^{l} |\lambda_j|}{\prod_{j=1}^{m} \beta_j} +{}_p F_q[|c|;d;1] - \frac{\prod_{j=1}^{p} |c_j|}{\prod_{j=1}^{q} d_j} \leq 3 \tag{4.5}$$

holds, then $\mathcal{L}[f] \in \Sigma^*_H(\alpha)$.

Proof. In view of Theorem 2.2, it suffices to show that $S \leq 1-\alpha$, where

$$S := \sum_{n=1}^{\infty} (n+\alpha)|a_n||A_n| + \sum_{n=1}^{\infty} (n-\alpha)|b_n||B_n|, \tag{4.6}$$

where A_n and B_n are given by (4.3). Thus

$$\begin{aligned} S &\leq (1-\alpha) \sum_{n=1}^{\infty} [|A_n| + |B_n|] \\ &\leq (1-\alpha) \left\{ \sum_{n=1}^{\infty} \frac{(|\lambda_1|)_{n+1} \cdots (|\lambda_l|)_{n+1}}{(\beta_1)_{n+1} \cdots (\beta_m)_{n+1}} \frac{1}{(n+1)!} \right. \\ &\qquad\qquad \left. + \sum_{n=1}^{\infty} \frac{(|c_1|)_{n+1} \cdots (|c_p|)_{n+1}}{(d_1)_{n+1} \cdots (d_q)_{n+1}} \frac{1}{(n+1)!} \right\} \\ &= (1-\alpha) \left\{ {}_lF_m[|\lambda|;\beta;1] - 1 - \frac{\prod_{j=1}^{l} |\lambda_j|}{\prod_{j=1}^{m} \beta_j} +{}_p F_q[|c|;d;1] - 1 - \frac{\prod_{j=1}^{p} |c_j|}{\prod_{j=1}^{q} d_j} \right\} \\ &\leq 1-\alpha, \end{aligned}$$

provided (4.5) holds. ∎

Note that the hypergeometric condition (4.5) is independent of α.

Example 4.1. Let $l = 2 = p$, $m = 1 = q$, $\beta > 1 + |\lambda|$, and $d > 1 + |c|$ in Theorem 4.1. The Gauss summation formula [23, p.30] gives

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad Re(c-a-b) > 0. $$

Using the property that $\Gamma(z+1) = z\Gamma(z)$ and the Gauss summation formula, the condition (4.5) reduces to

$$ \frac{\beta-1}{\beta-|\lambda|-1} - \frac{|\lambda|}{\beta} + \frac{d-1}{d-|c|-1} - \frac{|c|}{d} \leq 3. $$

Let $\mathcal{M}(\alpha)$ denote the class consisting of functions $f = h + g$ of the form (1.1) satisfying

$$ \sum_{n=1}^{\infty} [(n+\alpha)|a_n| + (n-\alpha)|b_n|] \leq 1 - \alpha. $$

It follows from Corollary 2.1 that $\mathcal{M}(\alpha) \subset \Sigma_H^*(\alpha)$, and under conditions (4.5), the proof of Theorem 4.1 shows that $L[\mathcal{M}(\alpha)] \subset \mathcal{M}(\alpha)$ also holds true. In particular, with $\mathcal{M}(\alpha) = T\Sigma_H^*(\alpha)$, the following corollary is obtained:

Corollary 4.1. *Let $\mathcal{L}[f]$ be given by* (4.1) *with $\sigma = 1$. Further let ϕ_1 and ϕ_2 of the form* (4.2) *satisfy*

$$ \sum_{j=1}^{m} \beta_j > \sum_{j=1}^{l} \lambda_j, \qquad \sum_{j=1}^{q} d_j > \sum_{j=1}^{p} c_j, $$

where $\lambda_j \geq 0$, $\beta_j > 0$, and $c_j \geq 0$, $d_j > 0$. Then $\mathcal{L}[T\Sigma_H^*(\alpha)] \subset T\Sigma_H^*(\alpha)$ if

$$ {}_lF_m[\lambda;\beta;1] - \frac{\prod_{j=1}^{l} \lambda_j}{\prod_{j=1}^{m} \beta_j} + {}_pF_q[c;d;1] - \frac{\prod_{j=1}^{p} c_j}{\prod_{j=1}^{q} d_j} \leq 3. \tag{4.7} $$

Proof. It follows from Corollary 3.1 that the coefficients a_n and b_n satisfy the conditions (4.4) of Theorem 4.1. If the condition (4.7) holds true, it follows that $S \leq 1 - \alpha$, where S is given by (4.6). Corollary 3.1 now gives $\mathcal{L}[f] \in T\Sigma_H^*(\alpha)$. ∎

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