A note on monotone D-spaces*

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Abstract

A topological space (X, τ) is a D-space if for every function φ: $X \to \tau$ with $x \in \varphi(x)$ for each $x \in X$, $\{\varphi(x) : x \in F\}$ covers X for some closed discrete subset F of X. The Michael line M, one of the most important elementary examples in general topology, is the Euclidean space $\mathbb{R}$ isolating the irrationals. In this note we show that (1) the minimal dense linearly ordered extension of M is hereditarily paracompact, but not monotonically D; (2) the minimal closed linearly ordered extension of M is monotonically D; (3) if the space X is a D-space (resp., a monotone D-space), then so is its Alexandroff duplicate space $\mathscr{A}(X)$ and thus $\mathscr{A}(M)$ is monotonically D.

1 Introduction

The D-property was introduced by E. K. van Douwen in [5] and was studied widely (for instance, [1], [3], [4], [6] or [7]). A neighborhood assignment for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for all $x \in X$. A space X is a D-space if for every neighborhood assignment φ for X, there is a closed discrete subset F of X such that $X = \varphi(F) = \cup\{\varphi(x) : x \in F\}$. It is well-known that a space with a point-countable base is a D-space ([1]) and semi-stratifiable spaces are D-spaces ([3], [4]). Hence σ-spaces, stratifiable spaces, Moore spaces and metrizable spaces are all D-spaces.

In [14], the monotone D-property is introduced and studied. A space X is a monotone D-space if for each neighborhood assignment φ for X, we can pick

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a closed discrete subset $F(\varphi)$ of X with $X = \cup\{\varphi(x) : x \in F(\varphi)\}$ such that if ψ is also a neighborhood assignment for X and $\varphi(x) \subset \psi(x)$ for each $x \in X$, then $F(\psi) \subset F(\varphi)$. Monotone D-spaces are D-spaces, but the converse is not true. The closed unit interval $[0,1]$ is a D-space, but it is not a monotone D-space ([14]). It is well-known that in generalized ordered spaces the D-property is equivalent to paracompactness ([6]). The Michael line M (the real line with the irrationals isolated and the rationals having their usual neighborhoods), one of the most important elementary examples in general topology, is a paracompact generalized ordered space, and so it is a D-space. In [14], it is shown that the Michael line M is also a monotone D-space.

A linearly ordered topological space is a triple $(X, \lambda, \leq)$, where $\leq$ is a linear order on the set X and λ is the open interval topology defined by $\leq$ (that is, λ has a subbase $\{(a, \rightarrow) : a \in X\} \cup \{(\leftarrow, a) : a \in X\}$, where $(a, \rightarrow) = \{x \in X : a < x\}$ and $(\leftarrow, a) = \{x \in X : x < a\}$). For $a, b \in X$, $(a, b) = \{x \in X : a < x < b\}$ is called an open interval. The Euclidean space $\mathbb{R}$ is a linearly ordered topological space. A generalized ordered space is precisely a subspace of a linearly ordered topological space. It happens that for $\mathscr{P} = paracompactness$ (resp., metrizability, Lindelöfness and quasi-developability) a generalized ordered space has $\mathscr{P}$ if and only if its (minimal) closed linearly ordered extension has $\mathscr{P}$. The main results of the note are as follows.

1. *The minimal dense linearly ordered extension of the Michael line is hereditarily paracompact (hence a hereditary D-space), but not a monotone D-space.*

2. *The minimal closed linearly ordered extension of the Michael line is a monotone D-space.*

3. *If X is a D-space (resp., a monotone D-space), so is its Alexandroff duplicate space $\mathscr{A}(X)$. Thus $\mathscr{A}(M)$ is monotonically D for the Michael line M.*

Throughout the note, spaces are topological spaces. We reserve the symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{Z}$ and $\mathbb{Z}^+$ the set of all real numbers, all rational numbers, all irrational numbers, all integers and all positive integers respectively. Let φ and ψ be two neighborhood assignments for a space X, then by φ refining ψ (denoted by $\varphi \prec \psi$) we mean $\varphi(x) \subset \psi(x)$ for each $x \in X$. Undefined terminology and symbols will be found in [10].

2 Main results

For the Michael line M, put

$$\ell(M) = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \{-1, 1\}).$$

Obviously the lexicographic order $\preceq$ on $\ell(M)$ is a linear order on $\ell(M)$. Equip $\ell(M)$ with the open interval topology generated by the linear order $\preceq$ on $\ell(M)$. Then the Michael line M is homeomorphic to the dense subspace $\mathbb{R} \times \{0\}$ of the linearly ordered topological space $\ell(M)$. The space $\ell(M)$ is called a dense linearly ordered extension of M. $\ell(M)$ is also the minimal dense linearly ordered extension of M (see Theorem 2.1 of [13]). Note that the set $\mathbb{R} \times \{0\} \subset \ell(M)$ with the linearly ordered topology generated by the hereditary order from the order on $\ell(M)$ is homeomorphic to the Euclidean space $\mathbb{R}$.

It is well-known that the minimal dense linearly ordered extension $\ell(X)$ of a paracompact space X may not be paracompact, however for the minimal dense linearly ordered extension $\ell(M)$ of the Michael line M, we have the following Theorem.

Theorem 1. *The space $\ell(M)$ is hereditarily paracompact, and hence a hereditary D-space.*

Proof. Let Y be a subspace of $\ell(M)$. Now we will show that Y is paracompact. Suppose not. Then Y has a closed subspace F homeomorphic to a stationary subset T of some uncountable regular cardinal. Let $f : F \to T$ be a homeomorphic mapping. Since $\mathbb{P} \times \{0\}$ is a discrete open subset of $\ell(M)$, $F \setminus (\mathbb{P} \times \{0\})$ is a closed subspace of Y and $f(F \setminus (\mathbb{P} \times \{0\}))$ is still a stationary subset. So we suppose $F \cap (\mathbb{P} \times \{0\}) = \emptyset$. Let $M_1 = \ell(M) \setminus (\mathbb{P} \times \{-1, 0\})$ and $M_2 = \ell(M) \setminus (\mathbb{P} \times \{0, 1\})$. Put $Y_1 = Y \cap M_1$ and $Y_2 = Y \cap M_2$. Let $(\mathbb{R}, \tau_1)$ be generated by the base $\mathscr{B}_1 = \lambda \cup \{[a, b) : a \in \mathbb{P}, b \in \mathbb{R}, a < b\}$ and $(\mathbb{R}, \tau_2)$ be generated by the base $\mathscr{B}_2 = \lambda \cup \{(a, b] : b \in \mathbb{P}, a \in \mathbb{R}, a < b\}$, where λ is the usual topology on $\mathbb{R}$. Then for $i \in \{1, 2\}$, M_i as a subspace of $\ell(M)$ is homeomorphic to $(\mathbb{R}, \tau_i)$ and thus its subspace Y_i can be considered as the subspace of $(\mathbb{R}, \tau_i)$. Since $(\mathbb{R}, \lambda)$ is second countable it is hereditarily separable. Let C_i' be the countable dense subset of Y_i considered as a separable subspace of $(\mathbb{R}, \lambda)$ and $C_i = C_i' \cup \{y \in Y_i : \textit{y has a predecessor or a successor}\}$. Then for $i \in \{1, 2\}$, the countable C_i is dense in Y_i and thus Y_i as the subspace of $(\mathbb{R}, \tau_i)$ is separable. Noticing that $F = F \cap Y = (F \cap Y_1) \cup (F \cap Y_2)$, we see that $f(F \cap Y_1)$ or $f(F \cap Y_2)$ is stationary. That is, a closed subset of Y_1 or Y_2 is homeomorphic to a stationary subset. Hence Y_1 or Y_2 is not paracompact. This contradicts the separability of Y_1 and Y_2 (separable generalized ordered spaces are paracompact). In [6] it is shown that in generalized ordered spaces the D-property is equivalent to paracompactness, and thus $\ell(M)$ is a hereditary D-space. ∎

For the Michael line M, put

$$M^* = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \mathbb{Z}).$$

Let $\preceq$ be the lexicographic order on M^*. Equip M^* with the open interval topology generated by the linear order $\preceq$ on M^*. Then the Michael line M is homeomorphic to the closed subspace $\mathbb{R} \times \{0\}$ of the linearly ordered topological space M^*. The space M^* is called a closed linearly ordered extension of M ([12]). By Theorem 9 of [16], the space M^* is the minimal closed linearly ordered extension of M.

Theorem 2. *The space M^* is a monotone D-space.*

Proof. For a neighborhood assignment φ' for M^*, define a neighborhood assignment φ^* for M^* such that $\varphi^* \prec \varphi'$ as follows. Let $x^* = \langle x, k \rangle \in M^*$, if $x \in \mathbb{P}$, define $\varphi^*(x^*) = \{x^*\}$; if $x \in \mathbb{Q}$, then $k = 0$. Let I_x be the maximal open convex subset of M^* such that $\langle x, 0 \rangle \in I_x \subset \varphi'(x^*) = \varphi'(\langle x, 0 \rangle)$. If $I_x = M^*$, define $\varphi^*(x^*) = M^*$. Now suppose that I_x is one of the following, where $q_x < x < r_x$, $s_x < x < t_x$ and $\{q_x, r_x\} \subset \mathbb{R}$ while $\{s_x, t_x\} \subset \mathbb{P}$:

(1) $(\leftarrow, \langle r_x, m\rangle)$; (2) $\{\langle y,i\rangle \in M^* : y \leq t_x\}$; (3) $\{\langle y,i\rangle \in M^* : y < t_x\}$;
(4) $(\langle q_x, j\rangle, \rightarrow)$; (5) $\{\langle y,i\rangle \in M^* : y \geq s_x\}$; (6) $\{\langle y,i\rangle \in M^* : y > s_x\}$;
(7) $(\langle q_x, k\rangle, \langle r_x, l\rangle)$; (8) $\{\langle y,i\rangle \in M^* : s_x \leq y \leq t_x\}$;
(9) $\{\langle y,i\rangle \in M^* : s_x < y < t_x\}$;
(10) $\{\langle y,i\rangle \in M^* : s_x < y \leq t_x\} \cup \{\langle s_x, i\rangle : i \geq k\}$;
(11) $\{\langle y,i\rangle \in M^* : s_x \leq y < t_x\} \cup \{\langle t_x, i\rangle : i \leq l\}$;
(12) $\{\langle y,i\rangle \in M^* : s_x < y < t_x\} \cup \{\langle t_x, i\rangle : i \leq l\}$;
(13) $\{\langle y,i\rangle \in M^* : s_x < y < t_x\} \cup \{\langle s_x, i\rangle : i \geq k\}$.
Then define
$\varphi^*(x^*) = \{\langle y,i\rangle \in M^* : y < r_x\}$ if (1) holds;
$\varphi^*(x^*) = \{\langle y,i\rangle \in M^* : y < t_x\}$ if one of (2) and (3) holds;
$\varphi^*(x^*) = \{\langle y,i\rangle \in M^* : q_x < y\}$ if (4) holds;
$\varphi^*(x^*) = \{\langle y,i\rangle \in M^* : s_x < y\}$ if one of (5) and (6) holds;
$\varphi^*(x^*) = \{\langle y,i\rangle \in M^* : q_x < y < r_x\}$ if (7) holds;
$\varphi^*(x^*) = \{\langle y,i\rangle \in M^* : s_x < y < t_x\}$ if one of (8) to (13) holds.

For $x \in \mathbb{R}$, put $\varphi(\langle x,0\rangle) = \varphi^*(\langle x,0\rangle) \cap (\mathbb{R} \times \{0\})$. Then φ is a neighborhood assignment for the subspace $\mathbb{R} \times \{0\}$ of M^*. Since M is monotonically D and is homeomorphic to the subspace $\mathbb{R} \times \{0\}$ of M^*, there is a closed discrete subset F_φ of M such that $\mathbb{R} \times \{0\} = \varphi(F_\varphi \times \{0\})$ and if ψ is a neighborhood assignment for $\mathbb{R} \times \{0\}$ with $\varphi \prec \psi$ then $F_\varphi \supset F_\psi$.

Put $F^*_{\varphi^*} = \{\langle x,k\rangle \in M^* : x \in F_\varphi\}$. For $x^* = \langle x,k\rangle \in M^* \setminus F^*_{\varphi^*}$, if $x \in \mathbb{P}$, then $\{x^*\} \cap F^*_{\varphi^*} = \emptyset$; if $x \in \mathbb{Q}$, then there are $a_x, b_x \in \mathbb{Q}$ such that $x \in (a_x, b_x)$ and $(a_x, b_x) \cap F_\varphi = \emptyset$ since F_φ is closed in M. So $x^* \in W = (\langle a_x, 0\rangle, \langle b_x, 0\rangle)$ and $W \cap F^*_{\varphi^*} = \emptyset$. Thus $F^*_{\varphi^*}$ is closed in M^*. Let $x^* = \langle x,k\rangle \in F^*_{\varphi^*}$. If $x \in \mathbb{Q}$, then $k = 0$ and there are $c_x, d_x \in \mathbb{Q}$ such that $x \in (c_x, d_x)$ and $(c_x, d_x) \cap F_\varphi = \{x\}$. Put $V_{x^*} = (\langle c_x, 0\rangle, \langle d_x, 0\rangle)$; if $x \in \mathbb{P}$, put $V_{x^*} = \{x^*\}$. Then $V_{x^*} \cap F^*_{\varphi^*} = \{x^*\}$. So $F^*_{\varphi^*}$ is discrete in M^*.

Let $y^* = \langle y,k\rangle \in M^* \setminus F^*_{\varphi^*}$. Since $\varphi(F_\varphi \times \{0\}) = \mathbb{R} \times \{0\}$, there is $\langle x,0\rangle \in F_\varphi \times \{0\} \subset F^*_{\varphi^*}$ such that $\langle y,0\rangle \in \varphi(\langle x,0\rangle) = \varphi^*(\langle x,0\rangle) \cap (\mathbb{R} \times \{0\})$. Assume $x \in \mathbb{P}$. By the definition of φ^*, $\varphi^*(\langle x,0\rangle) = \{\langle x,0\rangle\}$ and hence $x = y$, contradicting $y^* \notin F^*_{\varphi^*}$. So $x \in \mathbb{Q}$. By the definition of $\varphi^*(\langle x,0\rangle)$, $\varphi(\langle x,0\rangle)$ is one of the sets $(\leftarrow, r_x) \times \{0\}$, $(q_x, \rightarrow) \times \{0\}$, $(s_x, t_x) \times \{0\}$ and $\mathbb{R} \times \{0\}$, where $x < r_x$, $q_x < x$ and $s_x < x < t_x$. Hence $y^* = \langle y,k\rangle \in \varphi^*(\langle x,0\rangle)$. So $M^* = \cup\{\varphi^*(x^*) : x^* \in F^*_{\varphi^*}\}$. Put $F^*_{\varphi'} = F^*_{\varphi^*}$, then $\cup\{\varphi'(y^*) : y^* \in F^*_{\varphi'}\} = M^*$ since $\varphi^* \prec \varphi'$. If ψ' is a neighborhood assignment for M^* with $\varphi \prec \psi$, then obviously $F^*_{\varphi'} \supset F^*_{\psi'}$. Thus M^* is monotonically D. ∎

Theorem 3. *The space $\ell(M)$ is not a monotone D-space.*

Proof. Assume that $\ell(M)$ is monotonically D. Define a mapping $f : \ell(M) \to \mathbb{R}$, where $\mathbb{R}$ is the Euclidean space, as follows: for each $\langle x,i\rangle \in \ell(M)$, $f(\langle x,i\rangle) = x$. Then f is continuous. In fact, for an open interval (a,b) of $\mathbb{R}$ and $\langle x,i\rangle \in f^{-1}((a,b))$, since $x \in (a,b)$, there are $q_x, r_x \in \mathbb{Q}$ such that $x \in (q_x, r_x) \subset (a,b)$. Thus $\langle x,i\rangle \in (\langle q_x,0\rangle, \langle r_x,0\rangle) \subset f^{-1}((a,b))$. So $f^{-1}((a,b))$ is open in $\ell(M)$. To show that f is closed, let F' be a closed subset of $\ell(M)$ and $x \notin f(F')$. If $x \in \mathbb{Q}$, then $f^{-1}(x) = \{\langle x,0\rangle\}$ and $\langle x,0\rangle \notin F'$. So there is an open interval $G = (\langle c_x,0\rangle, \langle d_x,0\rangle)$ of $\ell(M)$ with $\langle x,0\rangle \in G$ and $G \cap F' = \emptyset$, where $c_x, d_x \in \mathbb{Q}$.

Thus $x \in (c_x, d_x)$ and $(c_x, d_x) \cap f(F') = \emptyset$. If $x \in \mathbb{P}$, then $f^{-1}(x) = \{\langle x, -1\rangle, \langle x, 0\rangle, \langle x, 1\rangle\}$ and $f^{-1}(x) \cap F' = \emptyset$. Since F' is closed in $\ell(M)$, we can take $a_x, b_x \in \mathbb{Q}$ such that $\langle x, -1\rangle \in (\langle a_x, 0\rangle, \langle x, 0\rangle)$ with $(\langle a_x, 0\rangle, \langle x, 0\rangle) \cap F' = \emptyset$ and $\langle x, 1\rangle \in (\langle x, 0\rangle, \langle b_x, 0\rangle)$ with $(\langle x, 0\rangle, \langle b_x, 0\rangle) \cap F' = \emptyset$. Thus $x \in (a_x, b_x)$ and $(a_x, b_x) \cap f(F') = \emptyset$. Hence $f(F')$ is closed. Since $\ell(M)$ is monotonically D, its closed continuous image $\mathbb{R}$ is monotonically D (see Theorem 1.7 of [14]). Because that the monotone D-property is closed hereditary, the subspace $[0,1]$ of $\mathbb{R}$ is monotonically D. However Theorem 2.3 of [14] shows that $[0,1]$ is not monotonically D. A contradiction. ∎

Recall that a space X is meta-Lindelöf if every open cover of X has a point-countable open refinement.

Example 4. There is a monotone D-space which is not a meta-Lindelöf space.

Proof. Let $N = \mathbb{Z}^+$ and $\mathscr{N} = \{N_s \subset N : |N_s| = \omega,\ s \in S\}$, where $S \cap N = \emptyset$, be infinite such that $N_s \cap N_{s'}$ is finite if $s \neq s'$ and that $\mathscr{N}$ is maximal with respect to the last property, that is, $\mathscr{N}$ is the maximal almost disjoint family of N. Define a topology τ on $X = N \cup S$ by the neighborhood system $\{\mathscr{B}(x) : x \in X\}$, where $\mathscr{B}(x) = \{\{x\}\}$ if $x \in N$ and $\mathscr{B}(x) = \{\{s\} \cup (N_s \setminus F) : F \subset N, |F| < \omega\}$ if $x = s \in S$. Put $\Psi(N) = (X, \tau)$. Since the set of all isolated points of $\Psi(N)$ is N and the subspace S of $\Psi(N)$ is discrete, $\Psi(N)$ is a monotone D-space ([14]) (so a D-space). However $\Psi(N)$ is not meta-Lindelöf ([2]). ∎

Let X be a space, $A \subset X$ and $\mathscr{U}$ be a family of subsets of X, put $st(A, \mathscr{U}) = st^1(A, \mathscr{U}) = \cup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}$. Inductively $st^{n+1}(A, \mathscr{U}) = \cup\{U \in \mathscr{U} : U \cap st^n(A, \mathscr{U}) \neq \emptyset\}$. A space X is ω-star Lindelöf ([8]) if for every open cover $\mathscr{U}$ of X, there is $n \in \mathbb{Z}^+$ and a countable $B \subset X$ such that $st^n(B, \mathscr{U}) = X$.

Theorem 5. *The Michael line M is not an ω-star Lindelöf space.*

Proof. Let $Q = \{q_1, q_2, ..., q_i, ...\}$ and for each $q_i \in Q$, the open interval I_i containing q_i be with the length less than $\frac{1}{2^i}$. Then $\mathscr{U} = \{I_i : i \in \mathbb{Z}^+\} \cup \{\{p\} : p \in \mathbb{P}\}$ is an open cover of M. For any countable subset B of $\mathbb{R}$, $T = \mathbb{R} \setminus (\cup\{I_i : i \in \mathbb{Z}^+\} \cup B)$ is uncountable. Take $t_0 \in T$, then for any $n \in \mathbb{Z}^+$, $t_0 \notin st^n(B, \mathscr{U})$. So M is not an ω-star Lindelöf space. ∎

A space X is ω_1-compact if every closed discrete subset has cardinality $< \omega_1$.

Remark 6. (1) *An ω_1-compact D-space X is Lindelöf:* for the D-space X, $l(X) = e(X)$ ([9]). By ω_1-compactness of X, $e(X) = \omega$ and thus $l(X) = \omega$.

(2) *A space is Lindelöf if and only if it is ω_1-compact and meta-Lindelöf:* note that every point-countable open cover of the ω_1-compact space has a countable subcover (Lemma 7.5 of [11]).

(3) *The Michael line M cannot be the following: strongly n-star-Lindelöf, n-star-Lindelöf, ω_1-compact or Lindelöf:* by Theorem 5 and Fig. 4 of [8].

Since M is a meta-Lindelöf D-space without Lindeöfness, the ω_1-compactness condition in (1) and (2) cannot be removed.

The Alexandroff duplicate space $\mathscr{A}(X)$ for the space X is the set $X \times \{0,1\}$ equipped with the topology as follows: points in $X \times \{1\}$ are isolated and each point $\langle x, 0\rangle$ in $X \times \{0\}$ has the basic neighborhoods as the form: $(U \times \{0,1\}) \setminus \{\langle x, 1\rangle\}$, where U is an open neighborhood of x in X. The following Lemma is obvious.

Lemma 7. *Let X be a space. Then if F is a closed set in X, $F \times \{0,1\}$ is closed in $\mathscr{A}(X)$; if D is a discrete set in X, $D \times \{0,1\}$ is discrete in $\mathscr{A}(X)$.*

Theorem 8. *Let X be a space. Then X is a D-space if and only if $\mathscr{A}(X)$ is a D-space; X is monotonically D if and only if $\mathscr{A}(X)$ is monotonically D.*

Proof. Sufficiency: let ψ be a neighborhood assignment for $\mathscr{A}(X)$. If X is a D-space, for each $\langle x, 0\rangle \in \mathscr{A}(X)$, take an open U_x in X containing x with $(U_x \times \{0,1\} \setminus \{\langle x,1\rangle\}) \subset \psi(\langle x, 0\rangle)$. Then for the neighborhood assignment $\{U_x : x \in X\}$ for X there is a closed discrete subset F of X such that $X = \cup\{U_x : x \in F\}$. By Lemma 7 $F' = F \times \{0,1\}$ is a closed discrete subset of $\mathscr{A}(X)$ and $\mathscr{A}(X) = \cup\{\psi(z) : z \in F'\}$. So $\mathscr{A}(X)$ is a D-space. If X is monotonically D, for each $x \in X$, put

$$V_x = \{x\} \cup \{y \in X : y \neq x \textit{ and } \{\langle y, 0\rangle, \langle y, 1\rangle\} \subset \psi(\langle x, 0\rangle)\}.$$

Take an open $U_x \subset X$ containing x with $(U_x \times \{0,1\} \setminus \{\langle x,1\rangle\}) \subset \psi(\langle x, 0\rangle)$, then $U_x \subset V_x$ and thus $x \in V_x^{\circ}$. Put $\psi_X(x) = V_x^{\circ}$. Then the neighborhood assignment ψ_X for X satisfying that $(\psi_X(x) \times \{0,1\}) \setminus \{\langle x,1\rangle\} \subset \psi(\langle x, 0\rangle)$. So there is a closed discrete subset F_{ψ_X} of X such that $X = \cup\{\psi_X(x) : x \in F_{\psi_X}\}$. For the closed discrete subset $F_\psi = F_{\psi_X} \times \{0,1\}$ of $\mathscr{A}(X)$, it holds that $\mathscr{A}(X) = \cup\{\psi(z) : z \in F_\psi\}$. The rest proof of the sufficiency is obvious.

Necessity: note that the D-property and the monotone D-property are closed hereditary and the closed subspace $X \times \{0\}$ is homeomorphic to X. ∎

In the following corollary, M, $\mathbb{R}$, S, P, C and $[0, \omega_1]$ are the Michael line, the Euclidean space, the Sorgenfrey line (the real line with the half-open intervals of the form $[a, b)$ as a basis for the topology), the Niemytzki plane, the Cantor set and the usual ordinal space respectively.

Corollary 9. *$\mathscr{A}(M)$ is a monotone D-space; $\mathscr{A}(\mathbb{R})$, $\mathscr{A}(S)$, $\mathscr{A}(P)$, $\mathscr{A}(C)$ and $\mathscr{A}([0, \omega_1])$ are D-spaces, but not monotone D-spaces.*

Proof. M is monotonically D ([14]). Clearly $\mathbb{R}$, S, P, C and $[0, \omega_1]$ are D-spaces. By [14], S, C, $[0, \omega_1]$ and $[0,1]$ are not monotonically D. Since $\mathbb{R}$ has a closed subspace $[0,1]$ and P has a closed subspace $[0,1] \times \{1\}$ homeomorphic to $[0,1]$, $\mathbb{R}$ and P are not monotonically D. Hence by Theorem 8, the conclusion of the corollary is true. ∎

Remark 10. (1) *For the Michael line M, $\mathscr{A}(M)$ has a point-countable base*: put $\mathscr{B}_q = \{((a,b) \times \{0,1\}) \setminus \{\langle q, 1\rangle\} : a, b \in \mathbb{Q}, a < q < b\}$, $q \in \mathbb{Q}$. Then $\mathscr{B} = \cup\{\mathscr{B}_q : q \in Q\} \cup \{\{\langle x, 1\rangle\} : x \in \mathbb{R}\} \cup \{\{\langle p, 0\rangle\} : p \in \mathbb{P}\}$ is a point-countable base for $\mathscr{A}(M)$.

In general, a Moore space may not be monotonically D. For a first countable T_2-space X, let $x \in X$ and $\{B_n(x) : n < \omega\}$ be fixed basis of x with $B_{n+1}(x) \subset B_n(x)$, $n < \omega$. Define a topology ν on $\mathscr{M}(X) = X \cup (X \times \omega)$: points of $X \times \omega$

are isolated; a basic neighborhood of $x \in X$ is the form $C_m(x) = \{x\} \cup \{\langle y, n\rangle : (n \geq m) \wedge (y \in B_n(x))\}$, $m < \omega$. Then $(\mathscr{M}(X), \nu)$ is a Moore space ([15]).

(2) *The Moore space* $(\mathscr{M}(X), \nu)$ *is monotonically* D: since the subspace X of all non-isolated points of $(\mathscr{M}(X), \nu)$ is discrete, by Theorem 1.7 of [14] $(\mathscr{M}(X), \nu)$ is monotonically D.

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