Multiplicity of Solutions for Doubly Resonant Neumann Problems

Michael E. Filippakis* Nikolaos S. Papageorgiou

Abstract

In this paper,we examine semilinear Neumann problems which at $\pm\infty$ are resonant with respect to two successive eigenvalues (double resonance situation). Using variational methods based on the critical point theory together with Morse theory, we prove two multiplicity results. In the first we obtain two nontrivial solutions and in the second three, two of which have constant sign (one positive, the other negative).

1 Introduction

Let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we consider the following Neumann elliptic problem:

$$\left\{ \begin{array}{l} -\triangle x(z) = f(z, x(z)) \text{ a.e. in } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\} \tag{1.1}$$

Suppose $f(z, x) = \lambda_k x + g(z, x)$ with $\lim\limits_{|x|\to\infty} \frac{g(z,x)}{x} = 0$ uniformly for a.e. $z \in Z$ and λ_k is an eigenvalue of the negative Neumann Laplacian. Then problem (1.1) is said to be resonant at infinity with respect to λ_k. If this happens for two successive distinct eigenvalues $\lambda_k < \lambda_{k+1}$, then we say that the problem is "doubly resonant".

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The goal of this paper is to prove multiplicity results under conditions of double resonance between two successive eigenvalues of the negative Neumann Laplacian. The doubly resonant situation was investigated in the past only in the context of the Dirichlet problem. In this direction, we mention the works of Berestycki-de Figueiredo [4] (who coined the term double resonance), Cac [6], Robinson [24], Su [25] and Zou [30]. To the best of our knowledge, there is no analogous study for the Neumann problem. Certain resonant Neumann problems, were studied by Iannacci-Nkashama [13], [14], Kuo [15], Mawhin-Ward-Willem [19], Rabinowitz [23]. Iannacci-Nkashama [14] and Kuo [15] used variants of the well-known Landesman-Lazer conditions (LL-conditions for short), which were first introduced in the pioneering ”resonant” work of Landesman-Lazer [16]. Iannacci-Nkashama [13] used a sign condition. Mawhin-Ward-Willem [19] used a monotonicity condition and finally Rabinowitz [23] employed a periodicity condition. With the exception of Iannacci-Nkashama [14], all the aforementioned Neuamnn works, treat problems resonant with respect to the principal eigenvalues $\lambda_0 = 0$ and none of them deals with the doubly resonant case. Moreover, all of them prove existence theorems, but do not address the question of existence of multiple nontrivial solutions. Multiplicity results for resonant Neumann problems, were obtained by Filippakis-Papageorgiou [9], Tang [27] and Tang-Wu [28]. However, their hypotheses do not allow for double resonance (neither at zero nor at infinity).

In this paper, we consider the case of double resonance at infinity, with respect to two successive eigenvalues of the negative Neumann Laplacian. Our approach combines variational techniques based on the critical point theory, together with Morse theory. We prove two multiplicity theorems.

The two multiplicity results are the following (for hypotheses H_1 (resp.H_2), we infer to the beginning of Section 3 (resp. Section 4)).

Theorem 1.1. *If hypotheses H_1 hold, then problem (1.1) has at least two nontrivial solutions $x_0, v_0 \in C^1_n(\overline{Z})$.*

Theorem 1.2. *If hypotheses H_2 hold, then problem (1.1) has at least three nontrivial solutions $x_0, v_0, u_0 \in C^1_n(\overline{Z})$ with $x_0(z) > 0 > v_0(z)$ for all $z \in \overline{Z}$.*

2 Mathematical background

We start by recalling some basic elements of critical point theory and of Morse theory, which we shall need in the sequel.

So, let X be a Banach space and X^* its dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say the φ satisfies the Cerami condition (the C-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\{\varphi(x_n)\}_{n\geq 1} \text{ is bounded in } \mathbb{R} \text{ and } (1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

has a strongly convergent subsequence.

The next theorem is the well-known ”mountain pass theorem” and gives a minimax characterization of certain critical values of a C^1-functional.

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$, $x_0, x_1 \in X$, $\|x_1 - x_0\| > r > 0$*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{\|x-x_0\|=r} \varphi(x) = \eta_r,$$

φ satisfies the C-condition and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, then $c \geq \eta_r$ and c is a critical value of φ.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we use the following notation:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \le c\} \text{ (the sublevel set of } \varphi \text{ at } c\text{)},\\ K &= \{x \in X : \varphi'(x) = 0\} \text{ (the critical set of } \varphi\text{)}\\ \text{and } K_c &= \{x \in K : \varphi(x) = c\} \text{ (the critical set of } \varphi \text{ at the level } c\text{)}. \end{aligned}$$

Suppose (Y_1, Y_2) is a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group of the pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated critical point $x \in X$ with $\varphi(x) = c$ are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, \varphi^c \cap U\setminus\{x\}) \text{ for all } k \geq 0,$$

where U is a neighborhood of x such that $K \cap \varphi^c \cap U = \{x\}$ (see Chang [8] and Mawhin-Willem [20]). The excision property of singular homology theory, implies that this definition of critical groups, is independent of the particular choice of the neighborhood U.

Now, suppose that φ satisfies the C-condition and $-\infty < \inf \varphi(K)$. Let $c < \inf \varphi(K)$. Then the critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0$$

(see Bartsch-Li [3]). The deformation theorem, which is valid since by hypothesis φ satisfies the C -condition (see Bartolo-Benci-Fortunato [2]), implies that the above definition of critical groups at infinity, is independent of the particular level $c < \inf \varphi(K)$ used.

If K is finite, then the Morse-type numbers of φ are defined by

$$M_k = \sum_{x\in K} rank C_k(\varphi, x) \text{ for all } k \geq 0.$$

The Betti-type numbers of φ, are defined by

$$\beta_k = rank C_k(\varphi, \infty) \text{ for all } k \geq 0.$$

By Morse theory (see Bartsch-Li [3], Chang [8] and Mawhin-Willem [20]), the "Poincare-Hopf formula" holds

$$\sum_{k\geq 0} (-1)^k M_k = \sum_{k\geq 0} (-1)^k \beta_k, \tag{2.1}$$

if all M_k, β_k are finite and the two series converge.

Recall that if A and B are homotopy equivalent (in particular, if A and B are homeomorphic), then $H_k(X, A) = H_k(X, B)$ for all $k \geq 0$.

In the study of (1.1), we shall use the following two spaces:

$$C_n^1(\overline{Z}) = \{x \in C^1(\overline{Z}) : \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z\}$$
$$\text{and } H_n^1(Z) = \overline{C_n^1(\overline{Z})}^{\|\cdot\|} \quad (\|\cdot\| \text{ denotes the usual norm of } H^1(Z)).$$

The space $C_n^1(\overline{Z})$ is an ordered Banach space, with order cone

$$C_+ = \{x \in C_n^1(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\}.$$

We know that this cone has a nonempty interior given by

$$intC_+ = \{x \in C_+ : x(z) > 0 \text{ for all } z \in \overline{Z}\}.$$

This space seems to be more natural for Neumann problems with homogeneous boundary conditions. However, the main reason for working with this new Sobolev space is Proposition 2.2 below. In general $H_n^1(Z) \neq H^1(Z)$.

Let $f_0 : Z \times \mathbb{R} \to \mathbb{R}$ be Caratheodory function (i.e., measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$), with subcritical growth, i.e.,

$$|f_0(z,x)| \leq a_0(z) + c_0|x|^{r-1} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(Z)_+$, $c_0 > 0$ and $1 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 1,2 \end{cases}$. We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : H_n^1(Z) \to \mathbb{R}$ defined by

$$\varphi_0(x) = \frac{1}{2}\|Dx\|_2^2 - \int_Z F_0(z, x(z))dz \text{ for all } x \in H_n^1(Z).$$

Proposition 2.2. *If $x_0 \in H_n^1(Z)$ is a local $C_n^1(\overline{Z})$-minimizer of φ_0, i.e., there exists $r_0 > 0$ such that*

$$\varphi_0(x_0) \leq \varphi_0(x_0 + h) \text{ for all } h \in C_n^1(\overline{Z}),\ \|h\|_{C_n^1(\overline{Z})} \leq r_0,$$

then $x_0 \in C_n^1(\overline{Z})$ and it is also a local $H_n^1(Z)$-minimizer of φ_0, i.e., there exists $r_1 > 0$ such that

$$\varphi_0(x_0) \leq \varphi_0(x_0 + h) \text{ for all } h \in H_n^1(Z),\ \|h\| \leq r_1.$$

Remark 2.3. *This result for Dirichlet spaces was first proved by Brezis-Nirenberg [5] for $p = 2$ and later generalized to all $1 < p < \infty$ (i.e., to the spaces $W_0^{1,p}(Z)$) by Garcia Azorero-Manfredi-Peral Alonso [10]. The corresponding result for Neumann spaces (i.e., for $W_n^{1,p}(Z)$), was proved by Barletta-Papageorgiou [1] (for $2 \leq p < \infty$) and by Motreanu-Motreanu-Papageorgiou [21] (for $1 < p < \infty$).*

Let $X = H$ be a Hilbert space, $x \in H$, U a neighborhood of x and $\varphi \in C^2(U)$. If $x \in H$ is a critical point of φ, its "Morse index" is defined as the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite.

Finally, let us recall some basic facts about the spectrum of $(-\triangle, H^1_n(Z))$. We shall do this in the more general context of weighted eigenvalue problems. So, let $m \in L^\infty(Z)_+$ $m \neq 0$ (the weight function) and consider the following weighted linear eigenvalue problem:

$$\left\{ \begin{array}{l} -\triangle u(z) = \widehat{\lambda} m(z) u(z) \text{ a.e. in } Z, \\ \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial Z, \ \ \widehat{\lambda} \in \mathbb{R}. \end{array} \right\} \tag{2.2}$$

It is easy to see that $\widehat{\lambda} \geq 0$ is a necessary condition for problem (2.2) to have a nontrivial solution. In fact $\widehat{\lambda}_0 = \widehat{\lambda}_0(m) = 0$ is an eigenvalue of (2.2) with corresponding eigenspace $\mathbb{R}$ (the space of constant functions). Moreover, (2.2) has a sequence $\{\widehat{\lambda}_k(m)\}_{k\geq 0}$ of distinct eigenvalues such that $\widehat{\lambda}_k(m) \to +\infty$ as $k \to +\infty$. If $m \equiv 1$, then we write $\lambda_k = \widehat{\lambda}_k(1)$.

For every integer $k \geq 0$, let $E(\widehat{\lambda}_k(m))$ be the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(m)$ of (2.2). We know that $E(\widehat{\lambda}_k(m)) \subseteq C^1_n(\overline{Z})$ (regularity theory) and it has the unique continuation property, namely, if $u \in E(\widehat{\lambda}_k(m))$ vanishes on a set of positive measure, then $u(z) = 0$ for all $z \in \overline{Z}$. We set

$$\overline{H}_k = \mathop{\oplus}_{i=0}^{k} E(\widehat{\lambda}_i(m)) \ \text{ and } \ \widehat{H}_k = \overline{H}_k^{\perp} = \overline{\mathop{\oplus}_{i\geq k+1} E(\widehat{\lambda}_i(m))}.$$

Then we have the following variational characterizations of the eigenvalues:

$$0 = \widehat{\lambda}_0(m) = \min[\frac{\|Du\|_2^2}{\int_Z mu^2 dz} : u \in H^1_n(Z), \ u \neq 0] \tag{2.3}$$

and for $k \geq 1$

$$\begin{aligned} \widehat{\lambda}_k(m) &= \max[\frac{\|D\overline{u}\|_2^2}{\int_Z m\overline{u}^2 dz} : \overline{u} \in \overline{H}_k, \ \overline{u} \neq 0] \\ &= \min[\frac{\|D\widehat{u}\|_2^2}{\int_Z m\widehat{u}^2 dz} : \widehat{u} \in \widehat{H}_{k-1}, \ \widehat{u} \neq 0]. \end{aligned} \tag{2.4}$$

The minimum in (2.3) is attained on $E(\widehat{\lambda}_0(m)) = \mathbb{R}$. The maximum and the minimum in (2.4) are realized on $E(\widehat{\lambda}_k(m))$, $k \geq 1$. Then (2.3), (2.4) and the unique continuation property imply the following monotonicity property of the eigenvalues with respect to the weight function:

"If $m_1, m_2 \in L^\infty(Z)_+$, $m_1(z) \leq m_2(z)$ a.e. on Z and $m_1 \neq m_2$, then $\widehat{\lambda}_k(m_2) < \widehat{\lambda}_k(m_1)$ for all $k \geq 1$.

Note that $\widehat{\lambda}_0(m) = 0$ is the only eigenvalue with eigenfunctions of constant sign. All other eigenvalues have nodal (i.e., sign changing) eigenfunctions.

Finally in what follows, for every $x \in \mathbb{R}$, we use the notation

$$x^+ = \max\{x, 0\} \ \text{ and } \ x^- = \max\{-x, 0\}.$$

3 Existence of two solutions

In this section we establish the existence of two nontrivial smooth solutions for problem (1.1) under double resonance conditions:

The hypotheses on the nonlinearity $f(z,x)$, are the following:

H_1: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to f(z,x)$ is C^1;

(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|f'_x(z,x)| \leq a(z) + c|x|^p$$

with $a \in L^\infty(Z)_+$, $c > 0$ and $0 < p < \frac{4}{N-2} = 2^* - 2$ if $N \geq 3$ and $0 < p < \infty$ if $N = 1,2$;

(iv) there exists an integer $k \geq 0$ such that

$$\lambda_k \leq \liminf_{|x|\to\infty} \frac{f(z,x)}{x} \leq \limsup_{|x|\to\infty} \frac{f(z,x)}{x} \leq \lambda_{k+1} \text{ uniformly for a.a. } z \in Z;$$

(v) suppose that $\|x_n\| \to \infty$

[i] if $x_n = x_n^0 + \widehat{x}_n$ with $x_n^0 \in E(\lambda_k)$, $\widehat{x}_n \in V_k = E(\lambda_k)^\perp$ and $\frac{\|x_n^0\|}{\|x_n\|} \to 1$, then there exist $\gamma_0 > 0$ and $n_0 \geq 1$ such that

$$\int_Z (f(z,x_n(z)) - \lambda_k x_n(z)) x_n^0(z) dz \geq \gamma_0 > 0 \text{ for all } n \geq n_0;$$

[ii] if $x_n = x_n^0 + \widehat{x}_n$ with $x_n^0 \in E(\lambda_{k+1})$, $\widehat{x}_n \in V_{k+1} = E(\lambda_{k+1})^\perp$ and $\frac{\|x_n^0\|}{\|x_n\|} \to 1$, then there exist $\gamma_1 > 0$ and $n_1 \geq 1$ such that

$$\int_Z (\lambda_{k+1} x_n(z) - f(z,x_n(z))) x_n^0(z) dz \geq \gamma_1 > 0 \text{ for all } n \geq n_1;$$

(vi) there exist $\delta_0 > 0$ and $\xi_0 \in \mathbb{R}\backslash\{0\}$, such that

$$F(z,x) \leq 0 \text{ for a.a. } z \in Z, \text{ all } |x| \leq \delta_0$$
$$\text{and } \int_Z F(z,\xi_0)dz \geq 0, \text{ where } F(z,x) = \int_0^x f(z,s)ds.$$

Remark 3.1. *Hypothesis $H_1(iv)$ implies that at $\pm\infty$, we have double resonance with respect to the successive eigenvalues $\lambda_k < \lambda_{k+1}$. Hypothesis $H_1(v)$, is a generalization of the well-known LL-sufficiency conditions for the solvability of resonant problems, first introduced in the work of Landesman-Lazer [16]. Analogous conditions can be found in the study of resonant Dirichlet problems, see Landesman-Robinson-Rumbos [17], Robinson [24] and Su [25]. Also we note that in this case the growth hypothesis $H_1(iii)$ is stated in terms of $f'_x(z,\cdot) = \frac{\partial}{\partial x} f(z,x)$, because we want to corresponding eu;er functional of the problem to be C^2. Indeed, hypotheses $H_1(ii),(iii)$ imply that the integral functional $u \to \int_\Omega F(z,u(z))dz$ is C^2. Finally, we should point out that our hypotheses near the origin (see $H_1(vi)$) are minimal and particular they do not necessarily dictate a linear growth there for $f(z,\cdot)$ as in [24], [25].*

Example 3.2. *Consider the following nonlinearity $f(x)$ (for the sake of simplicity, we drop the z-dependence)*

$$f(x) = \lambda_k x + g(x), \text{ with } g \in C^1(\mathbb{R}).$$

Then we have

$$F(x) = \frac{1}{2}\lambda_k x^2 + G(x), \text{ with } G(x) = \int_0^x g(s)ds \text{ for all } x \in \mathbb{R}.$$

Suppose that near the origin, we have

$$G(x) = x^4 - sinx$$

and for $|x|$ large, we have

$$G(x) = c|x|^{\frac{3}{2}}, \; c > 0.$$

Then such a nonlinearity $f(\cdot)$ satisfies hypotheses H_1. The generalized LL-condition (hypothesis $H_1(v)$), can be verified using Lemma 2.1 of Su-Tang [26].

We consider the Euler functional $\varphi : H_n^1(Z) \to \mathbb{R}$ for problem (1.1), defined by

$$\varphi(x) = \frac{1}{2}\|Dx\|_2^2 - \int_Z F(z, x(z))dz \text{ for all } x \in H_n^1(Z).$$

Hypotheses H_1 imply that $\varphi \in C^2(H_n^1(Z))$. In the sequel, by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(H_n^1(Z)^*, H_n^1(Z))$. Then

$$\langle \varphi'(x), y \rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N} dz - \int_Z f(z, x) y dz$$
$$\text{and } \langle \varphi''(x)u, v \rangle = \int_Z (Du, Dv)_{\mathbb{R}^N} dz - \int_Z f'_x(z, x) uv dz \text{ for all } x, y, u, v \in H_n^1(Z).$$

Since we are dealing with the Neumann problem, Poincare's inequality is not valid (hence $\|Du\|_p$ is not equivalent to the Sobolev norm) and this makes the verification of the C-condition more difficult. In fact the possibility of resonance at λ_k and λ_{k+1} adds to the above difficulties.

Proposition 3.3. *If hypotheses H_1 hold, then φ satisfies the C-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq H_n^1(Z)$ be a sequence such that

$$|\varphi(x_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \text{ and } (1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } H_n^1(Z)^*. \tag{3.1}$$

We shall show that the sequence $\{x_n\}_{n\geq 1} \subseteq H_n^1(Z)$ is bounded. We argue indirectly. So, suppose that the sequence $\{x_n\}_{n\geq 1} \subseteq H_n^1(Z)$ is unbounded. We may assume that $\|x_n\| \to \infty$. Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_n^1(Z), \; y_n \to y \text{ in } L^2(Z), \; y_n(z) \to y(z) \text{ a.e. on } Z$$
$$\text{and } |y_n(z)| \leq k(z) \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \text{ with } k \in L^2(Z)_+$$

(recall that $H_n^1(Z)$ is embedded compactly in $L^2(Z)$). By virtue of hypothesis $H_1(iii),(iv)$, we have

$$|f(z,x)| \leq a_1(z) + c_1|x| \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}, \tag{3.2}$$

with $a_1 \in L^\infty(Z)_+$, $c_1 > 0$. From (3.2) it follows that

$$\frac{|f(z,x_n(z))|}{\|x_n\|} \leq \frac{a_1(z)}{\|x_n\|} + c_1|y_n(z)| \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \tag{3.3}$$

$$\Rightarrow \{h_n(\cdot) = \frac{f(\cdot, x_n(\cdot))}{\|x_n\|}\}_{n\geq 1} \subseteq L^2(Z) \text{ is bounded.}$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$h_n \xrightarrow{w} h \text{ in } L^2(Z) \text{ as } n \to \infty.$$

For every $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$D_{\varepsilon,n}^+ = \{z \in Z : x_n(z) > 0,\ \lambda_k - \varepsilon \leq \frac{f(z,x_n(z))}{x_n(z)} \leq \lambda_{k+1} + \varepsilon\}$$

$$\text{and } D_{\varepsilon,n}^- = \{z \in Z : x_n(z) < 0,\ \lambda_k - \varepsilon \leq \frac{f(z,x_n(z))}{x_n(z)} \leq \lambda_{k+1} + \varepsilon\}$$

Note that

$$x_n(z) \to +\infty \text{ a.e. on } \{y > 0\} \text{ and } x_n(z) \to -\infty \text{ a.e. on } \{y < 0\}.$$

So, by virtue of hypothesis $H_1(iv)$, we have

$$\chi_{D_{\varepsilon,n}^+}(z) \to 1 \text{ a.e. on } \{y > 0\} \text{ and } \chi_{D_{\varepsilon,n}^-}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

Then the dominated convergence theorem implies that

$$\|(1 - \chi_{D_{\varepsilon,n}^+})h_n\|_{L^2(y>0)} \to 0 \text{ and } \|(1 - \chi_{D_{\varepsilon,n}^-})h_n\|_{L^2(y<0)} \to 0$$

hence

$$\chi_{D_{\varepsilon,n}^+}h_n \xrightarrow{w} h \text{ in } L^2(y > 0) \text{ and } \chi_{D_{\varepsilon,n}^-}h_n \xrightarrow{w} h \text{ in } L^2(y < 0). \tag{3.4}$$

From the definition of the sets $D_{\varepsilon,n}^+$ and $D_{\varepsilon,n}^-$, we have

$$(\lambda_k - \varepsilon)y_n(z) \leq \frac{f(z,x_n(z))}{x_n(z)}y_n(z) = h_n(z) \leq (\lambda_{k+1} + \varepsilon)y_n(z) \text{ a.e. on } D_{\varepsilon,n}^+$$

$$\text{and } (\lambda_k - \varepsilon)y_n(z) \geq \frac{f(z,x_n(z))}{x_n(z)}y_n(z) = h_n(z) \geq (\lambda_{k+1} + \varepsilon)y_n(z) \text{ a.e. on } D_{\varepsilon,n}^-.$$

We pass to the limit as $n \to \infty$, use (3.4) together with Mazur's lemma and let $\varepsilon \downarrow 0$. We obtain

$$\lambda_k y(z) \leq h(z) \leq \lambda_{k+1} y(z) \text{ a.e. on } \{y > 0\} \tag{3.5}$$

$$\text{and } \lambda_k y(z) \geq h(z) \geq \lambda_{k+1} y(z) \text{ a.e. on } \{y < 0\}. \tag{3.6}$$

Moreover, it is clear from (3.2) that

$$h(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{3.7}$$

Combining (3.5), (3.6), (3.7), we infer that

$$h(z) = g(z)y(z) \text{ a.e. on } Z, \tag{3.8}$$

with $g \in L^\infty(Z)_+$ such that $\lambda_k \le g(z) \le \lambda_{k+1}$ a.e. on Z.
Let $A \in \mathcal{L}(H_n^1(Z), H_n^1(Z)^*)$ be defined by

$$\langle A(x), y\rangle = \int_Z (Dx, Dy)_{\mathbb{R}^{\mathbb{N}}} dz \text{ for all } x, y \in H_n^1(Z).$$

Clearly A is monotone, hence maximal monotone. Also let $N : H_n^1(Z) \to L^2(Z)$ be defined by

$$N(x)(\cdot) = f(\cdot, x(\cdot)) \text{ for all } x \in H_n^1(Z).$$

We know that

$$\varphi'(x_n) = A(x_n) - N(x_n) \text{ for all } n \ge 1. \tag{3.9}$$

From (3.1), we have

$$\begin{aligned} &|\langle \varphi'(x_n), v\rangle| \le \frac{\varepsilon_n}{1 + \|x_n\|}\|v\| \text{ for all } v \in H_n^1(Z) \text{ with } \varepsilon_n \downarrow 0,\\ \Rightarrow &|\langle A(y_n), v\rangle - \int_Z \frac{N(x_n)}{\|x_n\|} v dz| \le \frac{\varepsilon_n}{1 + \|x_n\|}\frac{\|v\|}{\|x_n\|}\\ &\text{for all } n \ge 1 \text{ (see (3.9)).}\end{aligned} \tag{3.10}$$

In (3.10), we choose $v = y_n - y \in H_n^1(Z)$. Then

$$\int_Z \frac{N(x_n)}{\|x_n\|}(y_n - y) dz \to 0 \text{ as } n \to \infty.$$

So, from (3.10) it follows that

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0. \tag{3.11}$$

Note that $A(y_n) \xrightarrow{w} A(y)$ in $H_n^1(Z)^*$ as $n \to \infty$. So, from (3.11), we have

$$\begin{aligned} &\langle A(y_n), y_n\rangle \to \langle A(y), y\rangle,\\ \Rightarrow &\|Dy_n\|_2 \to \|Dy\|_2.\end{aligned}$$

On the other hand, we also have $Dy_n \xrightarrow{w} Dy$ in $L^2(Z, \mathbb{R}^{\mathbb{N}})$. Hence, from the Kadec-Klee property of Hilbert spaces, we have

$$\begin{aligned} &Dy_n \to Dy \text{ in } L^2(Z, \mathbb{R}^{\mathbb{N}}),\\ \Rightarrow &y_n \to y \text{ in } H_n^1(Z).\end{aligned}$$

Therefore, $\|y\| = 1$. Passing to the limit as $n \to \infty$ in (3.10), we obtain

$$\begin{aligned} &\langle A(y), v\rangle = \int_Z gyvdz \text{ for all } v \in H_n^1(Z) \text{ (see (3.8))},\\ \Rightarrow &A(y) = gy. \end{aligned}$$

From this equation, using Green's identity, we obtain

$$\left\{ \begin{array}{l} -\triangle y(z) = g(z)y(z) \text{ a.e. in } Z \\ \frac{\partial y}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\} \tag{3.12}$$

Standard regularity theory, implies that $y \in C_n^1(\overline{Z})\backslash\{0\}$. We consider three distinct cases, depending on the position of g in the spectral interval $[\lambda_k, \lambda_{k+1}]$.

Case 1: $g(z) = \lambda_k$ a.e. on Z.

Then, from (3.12) we infer that $y \in E(\lambda_k)\backslash\{0\}$. Hence

$$\frac{\|x_n^0\|}{\|x_n\|} \to 1 \text{ as } n \to \infty, \tag{3.13}$$

where $x_n = x_n^0 + \widehat{x}_n$, with $x_n^0 \in E(\lambda_k)$ and $\widehat{x}_n \in V_k = E(\lambda_k)^\perp$, $n \geq 1$. From (3.1), we have

$$\begin{aligned} &|\langle A(x_n), x_n^0\rangle - \int_Z N(x)x_n^0(z)dz| \leq \frac{\varepsilon_n}{1+\|x_n\|}\|x_n^0\| \leq \varepsilon_n,\\ \Rightarrow &|\|Dx_n^0\|_2^2 - \int_Z N(x_n)x_n^0 dz| \leq \varepsilon_n,\\ \Rightarrow &|\int_Z (\lambda_k x_n(z) - f(z, x_n(z)))x_n^0(z)dz| \leq \varepsilon_n \text{ for all } n \geq 1. \end{aligned} \tag{3.14}$$

Note that in the last two implications we have used the orthogonality of the spaces $E(\lambda_k)$ and V_k. But then, because of (3.13), we see that (3.14) contradicts hypothesis $H_1(v)[i]$.

Case 2: $g(z) = \lambda_{k+1}$ a.e. on Z.

This case is treated similarly as Case 1, using this time hypothesis $H_1(v)[ii]$.

Case 3: $\lambda_k \leq g(z) \leq \lambda_{k+1}$ a.e. on Z and the two inequalities are strict on sets (not necessarily the same) of positive measure.

From the monotone dependence of the eigenvalues $\{\widehat{\lambda}_n(g)\}_{n\geq 0}$ on the weight function $g \in L^\infty(Z)_+$, we have

$$\widehat{\lambda}_k(g) < \widehat{\lambda}_k(\lambda_k) = 1 \text{ and } 1 = \widehat{\lambda}_{k+1}(\lambda_{k+1}) < \widehat{\lambda}_{k+1}(g). \tag{3.15}$$

From (3.15), it follows that 1 is not an eigenvalue of $(-\triangle, H_n^1(Z), g)$ and so in (3.12) we must have $y = 0$, a contradiction to the fact that $\|y\| = 1$.

So, in all three cases we have reached a contradiction. This means that $\{x_n\}_{n\geq 1} \subseteq H_n^1(Z)$ is bounded. Therefore, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H_n^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z). \tag{3.16}$$

Then

$$|\langle A(x_n), x_n - x\rangle - \int_Z N(x_n)(x_n - x)dz| \leq \varepsilon_n \|x_n - x\|,$$
$$\Rightarrow \lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0 \quad \text{(see (3.16))}.$$

From this as before, using the Kadec-Klee property of Hilbert spaces, we conclude that $x_n \to x$ in $H^1_n(Z)$. This proves that φ satisfies the C-condition. ∎

Proposition 3.4. *If hypotheses H_1 hold, then the origin is a local minimizer of φ.*

Proof. Let $\delta_0 > 0$ be as in hypothesis $H_1(vi)$ and consider $u \in C^1_n(\overline{Z})$ with $\|u\|_{C^1_n(\overline{Z})} \leq \delta_0$. Then by virtue of hypothesis $H_1(vi)$, we have

$$F(z, u(z)) \leq 0 \quad \text{for a.a. } z \in Z. \tag{3.17}$$

So, for any $u \in C^1_n(\overline{Z})$ with $\|u\|_{C^1_n(\overline{Z})} \leq \delta_0$, we have

$$\varphi(u) = \frac{1}{2}\|Du\|_2^2 - \int_Z F(z, u(z))dz \geq 0 = \varphi(0) \quad \text{(see (3.17))}.$$

Therefore the origin is a local $C^1_n(\overline{Z})$-minimizer of φ. Invoking Proposition 2.2, we conclude that the origin is a local $H^1_n(Z)$-minimizer of φ. ∎

We may assume that $x = 0$ is an isolated critical point and local minimizer of φ, or otherwise we have a whole sequence of distinct nontrivial solutions of (1.1) and we are done. Then because of Proposition 3.4, we have (see Chang [8], p.33 and Mawhin-Willem [20], p.175).

Proposition 3.5. *If hypotheses H_1 hold, then $C_m(\varphi, 0) = \delta_{m,0}\mathbb{Z}$ for all $m \geq 0$ ($\delta_{m,\cdot}$ denotes the Kronecker function).*

Since we have assumed without any loss of generality that $x = 0$ is an isolated critical point and local minimizer of φ, we can find $\rho > 0$ small, $\rho < \|\xi_0\|$, such that

$$0 = \varphi(0) < \varphi(y) \quad \text{and} \quad \varphi'(y) \neq 0 \quad \text{for all } y \in \overline{B}_\rho \backslash \{0\}, \tag{3.18}$$

with $\overline{B}_\rho(0) = \{y \in H^1_n(Z) : \|y\| \leq \rho\}$.

Proposition 3.6. *If hypotheses H_1 hold, then $\varphi(0) < \inf[\varphi(y) : \|y\| = \rho] = c_\rho$, with $\rho > 0$ as in (3.18).*

Proof. We proceed by contradiction. So, suppose we can find $\{y_n\}_{n\geq 1} \subseteq H^1_n(Z)$ such that

$$\|y_n\| = \rho \quad \text{and} \quad \varphi(y_n) \downarrow \varphi(0) = 0. \tag{3.19}$$

We may assume that

$$y_n \xrightarrow{w} y \ \text{ in } \ H^1_n(Z) \ \text{ and } \ y_n \to y \ \text{ in } \ L^2(Z) \ \text{ as } \ n \to \infty. \tag{3.20}$$

Exploiting the compact embedding of $H^1_n(Z)$ into $L^2(Z)$ and the sequential weak lower semicontinuity of the norm functional in a Banach space, we have

$$\varphi(y) \le \liminf_{n\to\infty} \varphi(y_n) = 0 \text{ (see (3.19))}$$
$$\text{and } \|y\| \le \rho.$$

Because of (3.18), we must have $y = 0$. From the mean value theorem, we have

$$\varphi(y_n) - \varphi(\frac{1}{2}y_n) = \langle v^*_n, \frac{1}{2}y_n\rangle \text{ for all } n \ge 1, \tag{3.21}$$

with

$$v^*_n = A(\frac{1+t_n}{2}y_n) - N(\frac{1+t_n}{2}y_n),\ t_n \in (0,1),\ n \ge 1. \tag{3.22}$$

We may assume that $t_n \to t \in [0,1]$. Since $\varphi(y_n) \to 0$ (see (3.19)) and $0 = \varphi(0) \le \liminf\limits_{n\to\infty} \varphi(\frac{1}{2}y_n)$, from (3.21) and (3.22), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle A(\frac{1+t_n}{2}y_n), y_n\rangle \le 0 \text{ (recall } y = 0),\\
\Rightarrow &\limsup_{n\to\infty}\langle A(\frac{1+t_n}{2}y_n), \frac{1+t_n}{2}y_n\rangle \le 0,\\
\Rightarrow &(\frac{1+t_n}{2})^2\|Dy_n\|_2^2 \to 0 \text{ as } n \to \infty,\\
\Rightarrow &y_n \to 0 \text{ in } H^1_n(Z) \text{ as } n \to \infty,
\end{aligned}$$

a contradiction to the fact that $\|y_n\| = \rho$ for all $n \ge 1$. ■

Now we are ready to produce the first nontrivial solution for problem (1.1).

Proposition 3.7. *If hypotheses H_1 hold, then problem (1.1) has a nontrivial solution $x_0 \in C^1_n(\overline{Z})$.*

Proof. From Proposition 3.6, for $\rho < \|\xi_0\|$, we have

$$0 = \varphi(0) < \inf[\varphi(y) : \|y\| = \rho]. \tag{3.23}$$

Also, by hypothesis $H_1(vi)$, we have

$$\varphi(\xi_0) = -\int_Z F(z, \xi_0)dz \le 0 = \varphi(0). \tag{3.24}$$

From (3.23),(3.24) and Proposition 3.3, we see that we can apply Theorem 2.1 (the mountain pass theorem) and obtain $x_0 \in H^1_n(Z)$ such that

$$\begin{aligned}
&0 = \varphi(0) < c_\rho \le \varphi(x_0) \text{ and } \varphi'(x_0) = 0,\\
\Rightarrow &x_0 \ne 0 \text{ and } A(x_0) = N(x_0),\\
\Rightarrow &-\triangle x_0(z) = f(z, x_0(z)) \text{ a.e. in } Z,\ \frac{\partial x_0}{\partial n} = 0 \text{ on } \partial Z.
\end{aligned}$$

So, $x_0 \in H^1_n(Z)$ is a nontrivial solution of problem (1.1) and the regularity theory implies $x_0 \in C^1_n(\overline{Z})$. ■

Proposition 3.8. *If hypotheses H_1 hold and $x_0 \in C_n^1(\overline{Z})$ is the nontrivial solution of problem (1.1) obtained in Proposition 3.7, then $C_m(\varphi, x_0) = \delta_{m,1}\mathbb{Z}$ for all $m \geq 0$.*

Proof. We shall show that we can apply Corollary 8.5, p.195 of Mawhin-Willem [20]. To this end, it suffices to check that, if the Morse index of x_0 is equal to 0, then its nullity is less than 2. So, we may assume that

$$\langle \varphi''(x_0)u, u\rangle \geq 0 \text{ for all } u \in H_n^1(Z) \tag{3.25}$$

(i.e., the Morse index of x_0 is equal to 0). Note that $u \in ker\varphi''(x_0)$ if and only if

$$-\triangle u(z) = \widehat{m}(z)u(z) \text{ a.e. in } Z, \frac{\partial u}{\partial n} = 0 \text{ on } \partial Z, \tag{3.26}$$

where $\widehat{m}(z) = f_x'(z, x_0(z))$.

If $\widehat{m}^+ = 0$, then clearly the only solution of (3.26) is $u = 0$ and so we are done.

If $\widehat{m}^+ \neq 0$, then we define

$$\lambda^*(m) = \inf[\|Du\|_2^2 : u \in H_n^1(Z), \int_Z \widehat{m}u^2 dz = 1]. \tag{3.27}$$

From (3.25), we have

$$\begin{aligned} &\|Du\|_2^2 \geq \int_Z \widehat{m}u^2 dz \text{ for all } u \in H_n^1(Z), \\ \Rightarrow &\lambda^*(\widehat{m}) \geq 1 \text{ (see (3.27)).} \end{aligned} \tag{3.28}$$

If $\int_Z \widehat{m} \geq 0$, the from Proposition 2.2 of Godoy-Gossez-Paczka [12], we have $\lambda^*(m) = 0$, which contradicts (3.28).

So, we must have $\int_Z \widehat{m}dz < 0$. Then according in Proposition 2.7 of Goday-Gossez-Paczka [12], we have $dimker\varphi''(x_0) \leq 1$. Therefore, we can apply Corollary 8.5, p.195 of Mawhin-Willem [20] and infer that $C_m(\varphi, x_0) = \delta_{m,1}\mathbb{Z}$ for all $m \geq 0$. ∎

To compute the critical groups at φ at infinity, we shall need the following slight modification of Lemma 2.4 of Perera-Schechter [22]. The new formulation is suitable for problems in which the Euler functional satisfies the C-condition.

Lemma 3.9. *If H is a Hilbert space, $(t, x) \to \varphi_t(x)$ is a function belonging in $C([0,1] \times X)$ such that $x \to \partial_t\varphi_t(x)$ and $x \to \varphi_t'(x)$ are both locally Lipschitz on H and there exists $R > 0$ such that*

$$\inf[(1 + \|u\|)\|\varphi_t'(u)\|_* : t \in [0,1], \|u\| > R] > 0 \tag{3.29}$$

$$\textit{and } \inf[\varphi_t(u) : t \in [0,1], \|u\| \leq \mathbb{R}] > -\infty, \tag{3.30}$$

then $C_m(\varphi_0, \infty) = C_m(\varphi_1, \infty)$ for all $m \geq 0$.

Proof. We choose $\eta < \inf[\varphi_t(u) : t \in [0,1], \|u\| \leq \mathbb{R}]$ such that

$$\varphi_0^\eta \neq \emptyset \text{ or } \varphi_1^\eta \neq \emptyset.$$

If no such $\eta \in \mathbb{R}$ can be found, then $C_m(\varphi_0, \infty) = C_m(\varphi_1, \infty) = \delta_{m,0}\mathbb{Z}$ for all $m \geq 0$.

For definiteness, we assume that $\varphi_0^\eta \neq \emptyset$ (the argument is similar if we assume that $\varphi_1^\eta \neq \emptyset$). Take $u \in \varphi_0^\eta$ and consider the following Cauchy problem:

$$\left\{ \begin{array}{l} \dot{h}(t) = -(1 + \|h(t)\|)\frac{\partial_t \varphi_t(h(t))}{\|\varphi_t'(h(t))\|_*}\varphi_t'(h(t)) \text{ a.e. on } \mathbb{R}_+, \\ h(0) = u. \end{array} \right\} \tag{3.31}$$

Since by hypothesis both $x \to \partial_t\varphi_t(x)$ and $x \to \varphi_t'(x)$ are locally Lipschitz, then from the local existence theorem (see, for example, Gasinski-Papageorgiou [11], p.618), we know that (3.31) admits a local flow denoted b $h(t)$ (or $h(t,u)$ to emphasize the initial point u). If by $\langle \cdot, \cdot \rangle_H$ we denote the inner product of H, then

$$\begin{aligned} &\frac{d}{dt}\varphi_t(h(t)) = \langle \varphi_t'(h(t)), \dot{h}(t)\rangle_H + \partial_t\varphi_t(h(t)) \\ &\leq -(1 + \|h(t)\|)\partial_t\varphi_t(h(t)) + \partial_t\varphi_t(h(t)) \ \ \text{(see (3.31))} \\ &\leq 0 \ \text{ a.e. on th maximal interval of existence } T = [0,b] \\ &\Rightarrow \varphi_t(h(t)) \leq \varphi_0(u) \leq \eta \ \text{ for all } \ t \in T, \\ &\Rightarrow \|h(t)\| > \mathbb{R} \ \text{ for all } \ t \in T \text{ (recal the choice of } \eta \in \mathbb{R}), \\ &\Rightarrow \|\varphi_t'(h(t))\| \geq \beta > 0 \ \text{ for all } \ t \in T. \end{aligned} \tag{3.32}$$

Therefore, the flow $h(\cdot)$ is global (i.e., on the whole $\mathbb{R}_+$) and we have that φ_0^η is homeomorphic to a subset of φ_1^η (see (3.32)).

Similarly, if we consider the family $\{\psi_t = \varphi_{t-1}\}_{t\in[0,1]}$, then an analogous argument gives us that $\psi_0^\eta = \varphi_1^\eta$ is homeomorphic to a subset of $\psi_1^\eta = \varphi_0^\eta$. Therefore we conclude that φ_0^η is homotopic to φ_1^η. Hence

$$\begin{aligned} &H_m(H, \varphi_0^\eta) = H_m(H, \varphi_1^\eta) \ \text{ for all } \ m \geq 0, \\ \Rightarrow &C_m(\varphi_0, \infty) = C_m(\varphi_1, \infty) \ \text{ for all } \ m \geq 0. \end{aligned}$$ ■

Using this lemma we can compute the critical groups of φ at infinity.

Proposition 3.10. *If hypotheses H_1 hold, then $C_m(\varphi, \infty) = \delta_{m,d_k}\mathbb{Z}$ for all $m \geq 0$, with $d_k = dim\overline{H}_k$ ($\overline{H}_k = \bigoplus_{i=0}^{k} E(\lambda_i)$).*

Proof. We consider the following one-parameter family of functions

$$\varphi_t(x) = \frac{1}{2}\|Dx\|_2^2 - t\int_Z F(z, x(z))dz - \frac{(1-t)\theta}{2}\|x\|_2^2 \ \text{ for all } \ x \in H_n^1(Z),$$

with $t \in [0,1]$,$\theta \in (\lambda_k, \lambda_{k+1})$.

Claim: We can find $R > 0$ such that

$$\inf[(1 + \|u\|)\|\varphi_t'(u)\|_* : t \in [0,1], \|u\| > R] > 0.$$

We proceed by contradiction. So, suppose that the Claim is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq H_n^1(Z)$ such that

$$t_n \to t \in [0,1], \ \|x_n\| \to \infty \ \text{ and } \ (1 + \|x_n\|)\|\varphi_{t_n}'(x_n)\|_* \to 0.$$

So, we have

$$|\langle A(x_n), v\rangle - t_n \int_Z f(z, x_n) v dz - (1-t_n)\theta \int_Z x_n v dz| \le \frac{\varepsilon_n}{1+\|x_n\|}\|v\| \tag{3.33}$$

for all $v \in H_n^1(Z)$ with $\varepsilon_n \downarrow 0$.

We set $y_n = \frac{x_n}{\|x_n\|}$, $n \ge 1$. Then $\|y_n\| = 1$ for all $n \ge 1$ and so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_n^1(Z) \text{ and } y_n \to y \text{ in } L^2(Z).$$

We multiply (3.33) with $\frac{1}{\|x_n\|}$ and obtain

$$|\langle A(y_n), v\rangle - t_n \int_Z \frac{f(z, x_n)}{\|x_n\|} v dz - (1-t_n)\theta \int_Z y_n v dz| \le \frac{\varepsilon_n}{1+\|x_n\|}\frac{\|v\|}{\|x_n\|} \tag{3.34}$$

for all $v \in H_n^1(Z)$.

We choose $v = y_n - y \in H_n^1(Z)$ and then pass to the limit as $n \to \infty$ in (3.34). Since $\{\frac{f(\cdot, x_n(\cdot))}{\|x_n\|}\}_{n\ge 1} \subseteq L^2(Z)$ is bounded (see the proof of Proposition 3.3), we obtain

$$\begin{aligned} &\lim \langle A(y_n), y_n - y\rangle = 0, \\ \Rightarrow &y_n \to y \text{ in } H_n^1(Z),\ \|y\| = 1,\ y \neq 0. \end{aligned}$$

Also, arguing as in the proof of Proposition 3.3, we can show that

$$h_n = \frac{N(x_n)}{\|x_n\|} = \frac{f(\cdot, x_n(\cdot))}{\|x_n\|} \xrightarrow{w} h \text{ in } L^2(Z)$$
$$\text{and } h = gy,\ \ \lambda_k \le g(z) \le \lambda_{k+1} \text{ a.e. on } Z.$$

So, if we pass to the limit as $n \to \infty$ in (3.34), we obtain

$$\begin{aligned} &\langle A(y), v\rangle = \int_Z (tg + (1-t)\theta) y v dz \text{ for all } v \in H_n^1(Z), \\ \Rightarrow &A(y) = (tg + (1-t)\theta)y, \\ \Rightarrow &-\triangle y(z) = (tg(z) + (1-t)\theta)y(z) \text{ a.e. in } Z,\ \frac{\partial y}{\partial n} = 0 \text{ on } \partial Z. \end{aligned} \tag{3.35}$$

As in the proof of Proposition 3.3, we consider three distinct cases depending on the position of the weight function $g_t = tg + (1-t)\theta \in L^\infty(T)_+$ in the spectral interval $[\lambda_k, \lambda_{k+1}]$.

Case 1: $t = 1$ and $g = \lambda_k$.

In this case (3.35) becomes

$$\begin{aligned} &-\triangle y(z) = \lambda_k y(z) \text{ a.e. in } Z,\ \frac{\partial y}{\partial n} = 0 \text{ on } \partial Z, \\ \Rightarrow &y \in E(\lambda_k)\backslash\{0\}, \\ \Rightarrow &\frac{\|x_n^0\|}{\|x_n\|} \to 1 \text{ as } n \to \infty. \end{aligned}$$

Then arguing as in Case 1 in the proof of Proposition 3.3, we reach a contradiction to the generalized LL-condition (see hypothesis $H_1(v)[i]$).

Case 2: $t = 1$ and $g = \lambda_{k+1}$

This is treated as Case 1 using this time hypothesis $H_1(v)[ii]$.

Case 3: $0 \leq t < 1$ or ($g \neq \lambda_k$ and $g \neq \lambda_{k+1}$)

In this case

$$\begin{aligned} &\lambda_k < g_t(z) < \lambda_{k+1} \ \text{ a.e. on } \ Z, \\ \Rightarrow &\widehat{\lambda}_k(g_t) < \widehat{\lambda}_k(\lambda_k) = 1 \text{ and } 1 = \widehat{\lambda}_{k+1}(\lambda_{k+1}) < \widehat{\lambda}_k(g_t). \end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36), we infer that $y = 0$, a contradiction to the fact that $\|y\| = 1$. This proves the Claim.

Clearly, we also have

$$\inf[\varphi_t(u) : t \in [0,1], \|u\| \leq \mathbb{R}] > -\infty.$$

Therefore, we can apply Lemma 3.9 and obtain

$$C_m(\varphi_0, \infty) = C_m(\varphi_1, \infty) \ \text{ for all } \ m \geq 0. \tag{3.37}$$

Note that $\varphi_0(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\theta}{2}\|x\|_2^2$ and $\varphi_1(x) = \varphi(x)$. Since $\theta \in (\lambda_k, \lambda_{k+1})$, $x = 0$ is the only critical point of φ_0. It is a nondegenerate critical point with Morse index $\mu = d_k$, where $d_k = dim\overline{H}_k$, $\overline{H}_k = \overset{k}{\underset{i=0}{\oplus}} E(\lambda_i)$. So, it follows that

$$\begin{aligned} &C_m(\varphi_0, \infty) = C_m(\varphi_0, 0) = \delta_{m,d_k}\mathbb{Z} \ \text{ for all } \ m \geq 0, \\ \Rightarrow &C_m(\varphi_1, \infty) = C_m(\varphi, \infty) = \delta_{m,d_k}\mathbb{Z} \ \text{ for all } \ m \geq 0 \ \text{ (see (3.37))}. \end{aligned}$$

■

Now we are ready for the first multiplicity theorem for doubly resonant semilinear Neumann problems.

Theorem 3.11. *If hypotheses H_1 hold, then problem (1.1) has at least two nontrivial solutions $x_0, u_0 \in C_n^1(\overline{Z})$.*

Proof. From Proposition 3.7, we already have one nontrivial solution $x_0 \in C_n^1(\overline{Z})$.

Suppose that $\{0, x_0\}$ are the only critical points of φ. Then from Propositions 3.5, 3.8, 3.10 and the Poincare-Hopf formula (see (2.1)), we have

$$(-1)^0 + (-1)^1 = (-1)^{d_k},$$

a contradiction. This means that there is one more critical point $y_0 \in H_n^1(Z)$ of φ, distinct from 0 and x_0. Then y_0 is a solution of (1.1) and $y_0 \in C_n^1(\overline{Z})$ by the regularity theory. ■

4 Existence of three solutions

In this section we strengthen hypotheses H_1 and we produce three nontrivial solutions for problem (1.1).

The new hypotheses on the nonlinearity $f(z, x)$ are the following:

H_2: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and hypotheses $H_2(i),(ii),(iii),(v),(vi)$ are the same as the corresponding hypotheses $H_1(i),(ii),(iii),(v),(vi)$

(iv) there exists an integer $k \geq 1$ such that $d_k = dim\overline{H}_k$ ($\overline{H}_k = \overset{k}{\underset{i=0}{\oplus}} E(\lambda_i)$) is even and

$$\lambda_k \leq \liminf_{|x|\to\infty} \frac{f(z,x)}{x} \leq \limsup_{|x|\to\infty} \frac{f(z,x)}{x} \leq \lambda_{k+1} \text{ uniformly for a,a. } z \in Z;$$

(vii) there exists $\widehat{c} > 0$ such that

$$\frac{f(z,x)}{x} \geq -\widehat{c} \text{ for a.a. } z \in Z, \text{ all } x \neq 0.$$

Remark 4.1. *In hypothesis $H_2(iv)$ since d_k is assumed to be even, k can not be zero. That is why we assume $k \geq 1$.*

We introduce the positive and negative truncations of $f(z,\cdot)$ defined by

$$f_+(z,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ f(z,x) & \text{if } x \geq 0 \end{cases} \text{ and } f_-(z,x) = \begin{cases} f(z,x) & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 0 \end{cases}.$$

We set $F_\pm(z,x) = \int_0^x f_\pm(z,s)ds$ and then for $\varepsilon \in (0,1)$ we introduce the functionals $\varphi^\varepsilon_\pm : H^1_n(Z) \to \mathbb{R}$ defined by

$$\varphi^\varepsilon_\pm(x) = \frac{1}{2}\|Dx\|_2^2 + \frac{\varepsilon}{2}\|x\|_2^2 - \int_Z F_\pm(z,x(z))dz - \frac{\varepsilon}{2}\|x^\pm|_2^2.$$

Note that $\varphi^\varepsilon_\pm \in C^{2-0}(H^1_n(Z))$.

Proposition 4.2. *If hypotheses H_2 hold, then the functionals φ and $\varphi^\varepsilon_\pm$ satisfy the C-condition.*

Proof. That φ satisfies the C-condition follows from Proposition 3.3.

Next we prove that φ^ε_+ satisfies the C-condition. The proof for φ^ε_- is similar. So, we consider a sequence $\{x_n\}_{n\geq 1} \subseteq H^1_n(Z)$ such that

$$\begin{array}{c} |\varphi^\varepsilon_+(x_n)| \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1 \\ \text{and } (1 + \|x_n\|)(\varphi^\varepsilon_+)'(x_n) \to 0 \text{ as } n \to \infty. \end{array} \tag{4.1}$$

From (4.1), we easily see that

$$\|x_n^-\| \to 0 \text{ as } n \to \infty.$$

Suppose $\|x_n^+\| \to \infty$ as $n \to \infty$. We set $y_n = \frac{x_n^+}{\|x_n^+\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1_n(Z) \text{ and } y_n \to y \text{ in } L^2(Z) \text{ as } n \to \infty.$$

Reasoning as in the proof of Proposition 3.3, we show that:

"If $N_+(x_n)(\cdot) = f_+(\cdot, x_n(\cdot))$ and $g_n = \frac{N_+(x_n)}{\|x_n^+\|}$, then $g_n \xrightarrow{w} g$ in $L^2(Z)$ and $g(z) = \xi(z)y^+(z)$, $\lambda_k \leq \xi(z) \leq \lambda_{k+1}$ a.e. on Z and $y_n \to y$ in $H_n^1(Z)$ as $n \to \infty$".

Hence $\|y\| = 1$ (i.e., $y \neq 0$) and $y \geq 0$. Moreover, in the limit as $n \to \infty$, we have

$$-\triangle y(z) = \xi(z)y(z) \;\text{ a.e. on }\; Z,\; \frac{\partial y}{\partial n} = 0 \;\text{ on } \partial Z. \tag{4.2}$$

Since d_k is even, $k \geq 1$ and so from (4.2), we have that y is nodal, a contradiction to the fact that $y \geq 0$, $y \neq 0$. This proves that $\{x_n^+\}_{n\geq 1} \subseteq H_n^1(Z)$ is bounded, hence $\{x_n\}_{n\geq 1} \subseteq H_n^1(Z)$ is bounded. From this, as before, it follows that φ_+^ε satisfies the C-condition.

Similarly for φ_-^ε. ∎

Proposition 4.3. *If hypotheses H_2 hold, then $u = 0$ is a local minimizer for the functionals $\varphi_\pm^\varepsilon$.*

Proof. Let $\delta_0 > 0$ be as in hypothesis $H(vi)$. Then for every $u \in C_n^1(\overline{Z})$ with $\|u\|_{C_n^1(\overline{Z})} \leq \delta_0$, we have

$$F_+(z, u(z)) \leq 0 \;\text{ a.e. on }\; Z$$

(recall that $F_+(z,x) = 0$ for a.a. $z \in Z$, all $x \leq 0$). Hence for $u \in C_n^1(\overline{Z})$ with $\|u\|_{C_n^1(\overline{Z})} \leq \delta_0$, we have

$$\varphi_+^\varepsilon(u) = \frac{1}{2}\|Du\|_2^2 + \frac{\varepsilon}{2}(\|u\|_2^2 - \|u^+\|_2^2) - \int_Z F_+(z,u(z))dz \geq 0,$$
$$\Rightarrow u = 0 \text{ is a local } C_n^1(\overline{Z})\text{-minimizer of } \varphi_+^\varepsilon,$$
$$\Rightarrow u = 0 \;\text{ is a local } H_n^1(Z)\text{-minimizer of } \varphi_+^\varepsilon \;\text{ (see Proposition 2.2).}$$

The proof for φ_-^ε, is similar. ∎

As before we may assume that $u = 0$ is an isolated critical point and local minimizer of $\varphi_\pm^\varepsilon$ (otherwise we have a whole sequence of distinct constant sign solutions). Then as in the proof of Proposition 3.6, we obtain:

Proposition 4.4. *If hypotheses H_2 hold, then there exists $\rho > 0$ small such that*

$$0 = \varphi_\pm^\varepsilon(0) < \inf[\varphi_\pm^\varepsilon(y) : \|y\| = \rho] = c_\rho^\pm.$$

Now we are ready for the second multiplicity result for problem (1.1).

Theorem 4.5. *If hypotheses H_2 hold, then problem (1.1) has at least three nontrivial solutions*

$$x_0 \in intC_+,\; v_0 \in -intC_+ \;\textit{and}\; u_0 \in C_n^1(\overline{Z}).$$

Proof. By virtue of hypothesis $H_2(iv)$, we have

$$\liminf_{|x|\to\infty} \frac{2F(z,x)}{x^2} \geq \lambda_k \;\text{ uniformly for a.a }\; z \in Z.$$

Since $k \geq 1$ (recall d_k is even, see Remark), $\lambda_k > 0$ and so for large $x > 0$, $F(z,x) > 0$. Hence, if $\theta > 0$ is large (such that $\|\theta\| > \rho$, then

$$\varphi_+^\varepsilon(\theta) = -\int_Z F(z,\theta)dz < 0 = \varphi_+^\varepsilon(0).$$

Hence Propositions 4.2 and 4.4 permit the application of Theorem 2.1. So, we obtain $x_0 \in H_n^1(Z)$ a critical point of φ_+^ε such that

$$\begin{aligned} 0 = \varphi_+^\varepsilon(0) < c_\rho^+ &\leq \varphi_+^\varepsilon(x_0), \\ \Rightarrow x_0 &\neq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} &(\varphi_+^\varepsilon)'(x_0) = 0 \\ &\Rightarrow A(x_0) + \varepsilon x_0 - N_+(x_0) - \varepsilon(x_0^+) = 0 \\ &\text{with } \ N_+(u)(\cdot) = f_+(\cdot, u(\cdot)) \ \text{ for all } \ u \in H_n^1(Z), \\ &\Rightarrow A(x_0) + \varepsilon x_0 = N_+(x_0) + \varepsilon(x_0^+). \end{aligned} \tag{4.3}$$

We act with the test function $-x_0^- \in H_n^1(\overline{Z})$ and obtain

$$\begin{aligned} &\|Dx_0^-\|_2^2 + \varepsilon\|x_0^-\|_2^2 = 0 \\ &(\text{recall } f_+(z,x) = 0 \text{ for a.a. } z \in Z, \text{ all } x \leq 0), \\ \Rightarrow &\|x_0^-\|^2 = 0, \ \text{i.e. } x_0 \geq 0, \ x_0 \neq 0. \end{aligned}$$

So, (4.3) becomes

$$\begin{aligned} &A(x_0) = N(x_0) \\ &(\text{recall } N(u)(\cdot) = f(\cdot, u(\cdot)) \text{ for all } u \in H_n^1(Z)) \\ \Rightarrow -\triangle x_0(z) = f(z, x_0(z)) \ \text{ a.e. in } \ Z, \ &\frac{\partial x_0}{\partial n} = 0 \text{ on } \partial Z, \end{aligned} \tag{4.4}$$

using Green's identity. Therefore, $x_0 \in H_n^1(Z)$ is a solution of problem (1.1). Moreover, regularity theory implies that $x_0 \in C_+$. Then, by virtue of hypothesis $H_2(vii)$, we have

$$\begin{aligned} &-\triangle x_0(z) \geq -\widehat{c}x_0(z) \ \text{ for a.a. } \ z \in Z \text{ (see (4.4))} \\ &\Rightarrow \triangle x_0(z) \leq \widehat{c}x_0(z) \ \text{ a.e. in } \ Z, \\ &\Rightarrow x_0 \in intC_+, \end{aligned}$$

using the strong maximum principle (see Vazquez [29]).

As in the proof of Proposition 3.8, we have

$$C_m(\varphi_+^\varepsilon, x_0) = \delta_{m,1}\mathbb{Z} \ \text{ for all } \ m \geq 0. \tag{4.5}$$

Since $\varphi_+^\varepsilon|_{C_+} = \varphi|_{C_+}$ and using a result of Liu-Wu [18] (see also Chang [7]), we have

$$\begin{aligned} &C_m(\varphi_+^\varepsilon, x_0) = C_m(\varphi_+^\varepsilon|_{C_n^1(\overline{Z})}, x_0) = C_m(\varphi|_{C_n^1(\overline{Z})}, x_0) = C_m(\varphi, x_0) \ \text{ for all } \ m \geq 1, \\ &\Rightarrow C_m(\varphi, x_0) = \delta_{m,1}\mathbb{Z} \ \text{ for all } \ m \geq 0 \ \text{ (see (4.5))}. \end{aligned} \tag{4.6}$$

Similarly, working this time with $\varphi_{-}^{\varepsilon}$, we obtain $v_0 \in -intC_+$ a solution of (1.1) such that

$$C_m(\varphi, v_0) = \delta_{m,1}\mathbb{Z} \text{ for all } m \geq 0. \tag{4.7}$$

Finally, from Propositions 3.5 and 3.10, we have

$$C_m(\varphi, 0) = \delta_{m,0}\mathbb{Z} \text{ for all } m \geq 0 \tag{4.8}$$
$$C_m(\varphi, \infty) = \delta_{m,d_k}\mathbb{Z} \text{ for all } m \geq 0. \tag{4.9}$$

Suppose $\{0, x_0, v_0\}$ are the only critical points of φ. Then from (4.6), (4.7), (4.8), (4.9) and the Poincare-Hopf formula (see (2.1)), we have

$$(-1)^0 + (-1)^1 + (-1)^1 = (-1)^{d_k}, \tag{4.10}$$
$$\Rightarrow (-1)^1 = (-1)^{d_k}, \tag{4.11}$$

a contradiction, since by hypothesis d_k is even. Therefore, there must be a third nontrivial critical point u_0 of φ, distinct from $\{x_0, v_0\}$. This is a solution of (1.1) and regularity theory implies that $u_0 \in C_n^1(\overline{Z})$. ■

Remark 4.6. *If $N = 1$ (i.e., ordinary differential equation problem), then d_k =even means that $k \geq 0$ is odd. Recall that in this case $dimE(\lambda_k) = 1$ for all $k \geq 0$ and so $dim\overline{H}_k = k + 1$.*

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Department of Mathematics,
Hellenic Military Academy
Vari 16673 Athens, Greece,
email:mfilip@math.ntua.gr

Department of Mathematics,
National Technical University,
Zografou Campus, Athens 15780, Greece
email:npapg@math.ntua.gr