Projections and invariant means related to some Banach algebras

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Abstract

Let A be a Banach algebra with a bounded approximate identity. Our first purpose in this paper is to generalize Bekka's results for a certain class of Banach algebras. Let G be an amenable locally compact topological group, and let A be a left Banach G-module. Our second purpose, among the other things, is to define certain weak*-closed subspaces of $\mathcal{B}(A, A^*)$ to consider when their weak*-closed subspaces are the range of a bounded projection on $\mathcal{B}(A, A^*)$. Finally, we explore the link between the projections properties and amenability of semigroup algebras.

1 Introduction

Let G be a locally compact group with left invariant Haar measure and let $L^p(G)$, $1 \leq p \leq \infty$, be the complex Lebesgue spaces associated with it [16]. It was shown by Lau [20] that if G is an amenable locally compact group, then any weak*-closed self-adjoint left translation invariant subalgebra of $L^\infty(G)$ is the range of a continuous projection commuting with left translations. Also, as shown in [19], if G is an infinite locally compact group and X is any closed subspace of $Wap(G)$ containing $C_0(G)$, then X is uncomplemented in $L^\infty(G)$. For a locally compact abelian group G, Gilbert [15] characterized weak*-closed translation invariant complemented subspaces of $L^\infty(G)$ by their spectra. In [27], Wood investigated the ideals in the Fourier algebra of a locally compact group G which are complemented by an invariant projection.

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In our earlier paper [10], we proved that if G is an amenable locally compact group and A is a right Banach G-module, then every weak*-closed translation invariant complemented subspace of A^* is the range of a bounded projection on A^* commuting with left translations. The bounded projections on $L^\infty(G)$ onto a weak*-closed subspace X of $L^\infty(G)$ which commute with translation have been studied by Takahashi in [25] and by Bekka in [3] (see also [26]). Bekka has proved that a weak*-closed left translation invariant subspace X of $L^\infty(G)$ is invariantly complemented if and only if ${}^\perp X$ has a bounded right approximate identity.

In this paper, our first purpose is to generalize some of Bekka's results for a certain class of Banach algebras. Let G be an amenable locally compact group. Let A be a left Banach G-module, and let X be a weak*-closed translation invariant subspace of $\mathcal{B}(A, A^*)$, i.e., $S_x(TL_{x^{-1}}) \in X$ for each $T \in X$ and $x \in G$. In this paper, we prove that if X is the range of a bounded projection on $\mathcal{B}(A, A^*)$, then $\mathcal{M}(A, A^*) \cap X$ is the range of a bounded projection on $\mathcal{M}(A, A^*)$ which commutes with translations.

Let S be a foundation semigroup with identity, and let X be a weak*-closed translation invariant subspace of $M_a(S,\omega)^*$. If $\mathcal{M}(M_a(S,\omega), X)$ is the range of a bounded projection on $\mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$, then X is the range of a bounded projection on $M_a(S,\omega)^*$, see Theorem 4.4. Finally, we study characterizations of amenability in terms of existence properties of left invariant means and in terms of the projections on semigroup algebras.

2 Notation and preliminary results

We introduce our notations briefly; for other ideas used here we refer the reader to [10], [23] and [26]. For any Banach space X, in this paper, the value of an element $x^* \in X^*$ at the element $x \in X$ is denoted by $\langle x^*, x\rangle$. For any Banach algebra A, the second dual A^{**} of A can be given a Banach algebra structure by means of the first Arens product (see [2], [7] and [21]). For $a, b \in A$, f in A^* and $E, F \in A^{**}$, the elements fa, Ef of A^* and EF of A^{**} are defined as follows:

$$\langle fa, b\rangle = \langle f, ab\rangle, \ \ \langle Ff, a\rangle = \langle F, fa\rangle, \ \ \langle EF, f\rangle = \langle E, Ff\rangle.$$

For $a \in A$ the maps L_a and R_a from A^* into itself are defined by $L_a(f) = af$ and $R_a(f) = fa$. We define the subspaces A^*A and AA^* of A^* as

$$A^*A = \{fa;\ f \in A^* \text{ and } a \in A\} \ \text{ and } \ AA^* = \{af;\ a \in A \text{ and } f \in A^*\}.$$

If A has a bounded approximate identity, then by Cohen's factorization theorem, the spaces A^*A and AA^* are closed in A^*.

Let A be a Banach algebra, and let I be a closed left ideal in A. Then

$$I^\perp = \{f \in A^*;\ \langle f, a\rangle = 0 \text{ for all } a \in A\}$$

is a weak*-closed subspace of A^*. For each $a \in A$, $f \in I^\perp$ and $i \in I$, $\langle fa, i\rangle = \langle f, ai\rangle = 0$. This shows that $R_a(f) \in I^\perp$ for all $f \in I^\perp$ and $a \in A$. Conversely, suppose that $X \subseteq A^*$ is a weak*-closed subspace of A^* such that $R_a(f) \in X$ for all $f \in X$ and $a \in A$. ${}^\perp X = \{a \in A;\ \langle f, a\rangle = 0 \text{ for all } f \in X\}$ is a closed left ideal in A. It is easy to verify that the mapping $I \mapsto I^{\perp}$ is a bijection from the set of closed left ideals in A onto the set of weak*-closed subspaces X of A^* such that $R_a(X) \subseteq X$ for all $a \in A$. Indeed, let I and J be closed left ideals in A with $I^{\perp} = J^{\perp}$. By Theorem 4.7 in [24],

$$I = \overline{I} = {}^{\perp}(I^{\perp}) = {}^{\perp}(J^{\perp}) = \overline{J} = J.$$

This shows that $I \mapsto I^{\perp}$ is injective. Next let X be a weak*-closed subspace of A^* such that $R_a(X) \subseteq X$ for all $a \in A$ (X is said to be right translation invariant). By Theorem 4.7 in [24], ${}^{\perp}(X^{\perp}) = \overline{X} = X$, where the closure is taken in the weak*-topology. We have shown that $I \mapsto I^{\perp}$ is surjective.

3 Projections on Banach algebras

In [28], Wood proved that if A is an operator amenable Banach algebra, and I a closed ideal, then $I^{\perp}$ is completely complemented if and only if I has a bounded approximate identity. Bekka [3] proved that if X is a weak*-closed translation invariant subspace of $L^{\infty}(G)$, then X is complemented in $L^{\infty}(G)$ if and only if ${}^{\perp}X$ has a right bounded approximate identity. Our first result is a generalization of this fact to a Banach algebra with a bounded approximate identity.

Lemma 3.1. Let A be a Banach algebra with a bounded approximate identity, and let X be a weak*-closed right translation invariant subspace in A^*. Then the following are equivalent:

(1) there exists a bounded projection P of A^* onto X such that $PR_a = R_aP$ for all $a \in A$;

(2) there exists a bounded projection P of A^*A onto $X \cap A^*A$ such that $PR_a = R_aP$ for all $a \in A$;

(3) ${}^{\perp}X$ has a bounded right approximate identity.

Proof. (1) $\Rightarrow$ (2) Let $P : A^* \to X$ be a bounded projection such that $PR_a = R_aP$ for all $a \in A$. We show that P restricted to A^*A is a projection from A^*A onto $X \cap A^*A$. To see that P is a projection of A^*A onto $X \cap A^*A$, it suffices to show that $P(A^*A) \subseteq X \cap A^*A$ and that $f \in X \cap A^*A$ implies $P(f) = f$. Let f be an element of A^*A. Then f is of the form $f = ga$ for some g in A^* and a in A. Hence $P(f) = P(ga) = P(g)a$. Thus we conclude that $P(A^*A) \subseteq X \cap A^*A$. Next, let $f \in X \cap A^*A$. By assumption, we have $P(f) = f$. It is clear that $PR_a = R_aP$ for all $a \in A$. Hence we conclude that P is a bounded projection of A^*A onto $X \cap A^*A$ such that $PR_a = R_aP$ for all $a \in A$.

(2) $\Rightarrow$ (3) Let P be a bounded projection from A^*A onto $X \cap A^*A$ such that $PR_a = R_aP$ for all $a \in A$. Let (e_α) be a bounded approximate identity for A. Then we may suppose that (e_α) converges in the weak*-topology on A^{**}, say to F [7].

Define $P' : A^* \to A^*$ by setting $\langle P'(f), a\rangle = \langle F, P(fa)\rangle$ $(a \in A)$. Then P' is a bounded linear map. For $f \in X$ and $a \in A$, we have

$$\begin{aligned}\langle P'(f), a\rangle &= \langle F, P(fa)\rangle = \langle F, fa\rangle = \lim_\alpha \langle e_\alpha, fa\rangle \\ &= \lim_\alpha \langle f, ae_\alpha\rangle = \langle f, a\rangle,\end{aligned}$$

and so P' is the identity map on X. If $f \in A^*$, then

$$\begin{aligned}\langle P'(f), a\rangle &= \langle F, P(fa)\rangle = \lim_\alpha \langle F, P(fe_\alpha a)\rangle = \lim_\alpha \langle F, P(fe_\alpha)a\rangle \\ &= \lim_\alpha \langle aF, P(fe_\alpha)\rangle = \lim_\alpha \langle a, P(fe_\alpha)\rangle = 0,\end{aligned}$$

for each $a \in {}^\perp X$. Since X is a weak*-closed subspace of A^*, $({}^\perp X)^\perp = X$ and so $P'(f) \in X$ (see Theorem 4.7 in [24]). Consequently P' is an extension of P to A^* as a bounded projection.

Let $f \in ({}^\perp X)^*$, and let f' be any Hahn-Banach extension of f to a continuous functional on A^*, see Theorem A.3.19 in [7]. We consider $E : ({}^\perp X)^* \to \mathbb{C}$ defined by $\langle E, f\rangle = \langle F, f' - P'(f')\rangle$. Let $f'', f' \in A^*$ be two extension of $f \in ({}^\perp X)^*$. For any $a \in {}^\perp X$,

$$\langle f'' - f', a\rangle = \langle f'', a\rangle - \langle f', a\rangle = \langle f, a\rangle - \langle f, a\rangle = 0.$$

Hence $f'' - f' \in ({}^\perp X)^\perp = X$, and so $P'(f'' - f') = f'' - f'$. This shows that $\langle E, f' - P'(f')\rangle = \langle E, f'' - P'(f'')\rangle$, so that $E : ({}^\perp X)^* \to \mathbb{C}$ is well-defined. It is clear that $E \in ({}^\perp X)^{**}$. For every $a \in {}^\perp X$ and $f \in ({}^\perp X)^*$, we have

$$\begin{aligned}\langle F, (fa)' - P'((fa)')\rangle &= \langle F, f'a - P'(f'a)\rangle = \lim_\alpha \langle F, f'a - P'(f'e_\alpha a)\rangle \\ &= \lim_\alpha \langle F, f'a - P'(f'e_\alpha)a\rangle = \langle F, f'a\rangle.\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\langle Ef, a\rangle &= \langle E, fa\rangle = \langle F, (fa)' - P'((fa)')\rangle = \langle F, f'a\rangle \\ &= \langle f', a\rangle = \langle f, a\rangle.\end{aligned}$$

One verifies easily that E is a right identity for $({}^\perp X)^{**}$. Hence ${}^\perp X$ has a bounded right approximate identity, see [[4], p.146].

(3) $\Rightarrow$ (1) By Proposition 6.4 in [9], (3) implies (1). □

Throughout S denotes a locally compact Hausdorff topological semigroup. A positive and continuous function ω on S satisfying $\omega(xy) \le \omega(x)\omega(y)$ $(x, y \in S)$, $\omega(e) = 1$ will be called a *weight* function. Let $M(S, \omega)$ be the Banach space of all complex regular Borel measures μ on S such that $\|\mu\|_\omega = \int \omega(t)d|\mu|(t) < \infty$ [8]. We can identify $M(S, \omega)$ with the dual of the Banach space $C_0(S, \omega^{-1})$; the latter being the Banach space of all continuous function ϕ on S such that $\frac{\phi}{\omega} \in C_0(S)$, with the norm in $C_0(S, \omega^{-1})$ defined by $\|\phi\|_\omega = \sup\big\{\big|\frac{\phi(x)}{\omega(x)}\big|;\ x \in S\big\}$. Under convolution product

$$\langle \mu * \nu, \phi\rangle = \int\int \phi(xy)d\mu(x)d\nu(y) \qquad (\mu, \nu \in M(S, \omega),\ \phi \in C_0(S, \omega^{-1})),$$

$M(S,\omega)$ becomes a Banach algebra.

Recall that $M_a(S)$ denotes the space of all measures $\mu \in M(S)$ for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous. We denote by $M_a(S,\omega)$ the space of all $\mu \in M(S,\omega)$ such that $\omega\mu \in M_a(S)$. A *foundation* semigroup is a locally compact semigroup such that $\bigcup\{supp(\mu);\ \mu \in M_a(S)\}$ is dense in S. A trivial example is a topological group and in this case $M_a(S) = L^1(S)$ (for more information on foundation semigroups, see [1], [8] and [10]). It is well known that $M_a(S,\omega)$ is a closed two sided L-ideal of $M(S,\omega)$ [8]. Weighted hypergroup algebras have been studied by Ghahramani and Medghalchi in [13] and [14].

Lemma 3.2. Let S be a foundation topological semigroup with identity, and let X be a weak*-closed subspace of $M_a(S,\omega)^*$ such that $X\delta_x \subseteq X$ for every $x \in S$. Then the following conditions hold:

(1) X is topologically right invariant, i.e., $f\mu \in X$ for all $f \in M_a(S,\omega)^*$ and $\mu \in M_a(S,\omega)$;

(2) let $P : M_a(S,\omega)^*M_a(S,\omega) \to X$ be a bounded linear map. Then $P(f\mu) = P(f)\mu$ for all $f \in M_a(S,\omega)^*M_a(S,\omega)$ and $\mu \in M_a(S,\omega)$ if and only if $P(f\delta_x) = P(f)\delta_x$ for all $x \in S$ and $f \in M_a(S,\omega)^*M_a(S,\omega)$.

Proof. (1) Assume that there exist $f \in X$ and $\mu \in M_a(S,\omega)$ such that $f\mu \notin X$. Without loss of generality, we may assume that $\mu \geq 0$ and $\|\mu\|_\omega = 1$. Since X is a weak*-closed subspace in $M_a(S,\omega)^*$, by Hahn-Banach Theorem [24], there exist $\nu \in M_a(S,\omega)$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$Re\langle \nu, f\delta_x\rangle < \gamma_1 < \gamma_2 < Re\langle \nu, f\mu\rangle,$$

where $x \in S$. By Lemma 3.4 in [12],

$$\begin{aligned} Re\langle f\mu,\nu\rangle &= Re\langle f,\mu*\nu\rangle = Re\int\langle f,\delta_x*\nu\rangle d\mu(x) \\ &\leq \gamma_1 < \gamma_2 < Re\langle f\mu,\nu\rangle. \end{aligned}$$

We would come to a contradiction. Therefore $f\mu \in X$.

(2) Let $P : M_a(S,\omega)^*M_a(S,\omega) \to X$ be a bounded linear map with $P(f\mu) = P(f)\mu$ for all $f \in M_a(S,\omega)^*M_a(S,\omega)$ and $\mu \in M_a(S,\omega)$. Let $f \in M_a(S,\omega)^*M_a(S,\omega)$, $x \in S$. By Lemma 3.4 in [18], $M_a(S,\omega)$ has a bounded approximate identity, say (e_α). It is easy to see that $fe_\alpha * \delta_x \to f\delta_x$ in the norm topology. Hence

$$P(f\delta_x) = \lim_\alpha P(fe_\alpha * \delta_x) = \lim_\alpha P(f)e_\alpha * \delta_x.$$

On the other hand, $P(f)e_\alpha * \delta_x$ converges to $P(f)\delta_x$ in the weak*-topology. Thus $P(f\delta_x) = P(f)\delta_x$.

The converse is obvious. □

Let S be a foundation semigroup with identity. Then a function $f \in C(S,\omega)$ is called ω-left uniformly continuous if the mapping $x \mapsto \frac{L_xf}{\omega(x)}$ of S into $C(S,\omega)$

is norm continuous, where $L_x(f)(y) = f(xy)$ for every $x, y \in S$. As known $LUC(S, \omega^{-1}) = M_a(S,\omega)^* M_a(S,\omega)$ (see Proposition 3.5 in [18]). Let S be a locally compact Hausdorff topological group. Bekka in [3] has proved that a weak*-closed left translation invariant subspace X of $L^\infty(S)$ is invariantly complemented if and only if ${}^\perp X$ has a bounded right approximate identity. In the following Theorem, our main purpose is to generalize Bekka's results in [3] for a certain class of weighted semigroup algebras.

Theorem 3.3. Let S be a foundation topological semigroup with identity, and let X be a weak*-closed left translation invariant subspace of $M_a(S,\omega)^*$. Then the following conditions are equivalent:

(1) there exists a bounded projection P of $M_a(S,\omega)^*$ onto X such that $PR_\mu = R_\mu P$ for all $\mu \in M_a(S,\omega)$;

(2) there exists a bounded projection P of $LUC(S,\omega^{-1})$ onto $X \cap LUC(S,\omega^{-1})$ such that $PR_\mu = R_\mu P$ for all $\mu \in M_a(S,\omega)$;

(3) there exists a bounded projection P of $LUC(S,\omega^{-1})$ onto $X \cap LUC(S,\omega^{-1})$ such that $PR_x = R_x P$ for all $x \in S$;

(4) ${}^\perp X$ has a bounded right approximate identity.

Proof. This is immediate from Lemma 3.1 and Lemma 3.2. □

4 Projections and amenability

We recall that a locally compact group G is *amenable* if there is a positive norm one linear functional on $L^\infty(G)$ which is invariant under left translation. Every abelian group is amenable. The discrete free group F_2 on two generators a and b is not amenable [22].

Definition 4.1. An *action* of a semigroup S on a Banach algebra A is a mapping $\sigma : S \times A \to A$ such that:

(1) $\sigma(x, a+b) = \sigma(x,a) + \sigma(x,b)$, $\alpha\sigma(x,a) = \sigma(x,\alpha a)$, $\sigma(xy,a) = \sigma(x, ya)$ and $\sigma(e,a) = a$, where $\alpha \in \mathbb{C}$, $x, y \in S$ and $a, b \in A$;

(2) for all $a \in A$, the map $x \mapsto \sigma(x,a)$ is continuous from S into A;

(3) there exists $k \in \mathbb{R}$ such that $\|\sigma(x,a)\| \leq k\|a\|$ for every $x \in S$ and $a \in A$.

We shall write xa for $\sigma(x,a)$. A left Banach S-module is a pair (S, A), where A is a Banach algebra and σ is an action of S on A.

Let A be a Banach algebra. As is well known, we can define an isometric linear isomorphism from $(A \otimes_p A)^*$ onto $\mathcal{B}(A, A^*)$ by the correspondence between f and Φ_f defined by $\Phi_f(a)(b) = \langle f, a \otimes b \rangle$ for each $a, b \in A$ [4].

Now, let A be a left Banach S-module, and let $x \in S$ and $T \in \mathcal{B}(A, A^*)$ (see [10]). We define $S_x(T) \in \mathcal{B}(A, A^*)$ by $\langle S_x(T)(a), b\rangle = \langle T(a), xb\rangle$. Then S_x is a bounded linear map of $\mathcal{B}(A, A^*)$ into $\mathcal{B}(A, A^*)$. If $x \in S$, we consider $L_x : a \mapsto xa$, $A \to A$. Clearly $L_x \in \mathcal{B}(A, A)$. We consider the space $\mathcal{M}(A, A^*)$ of all $T \in \mathcal{B}(A, A^*)$ for which $S_x(T) = TL_x$ for all $x \in S$ [17].

Theorem 4.2. Let G be a locally compact abelian group, and let A be a left Banach G-module. Let X be a weak*-closed subspace of $\mathcal{B}(A, A^*)$ satisfying $S_x(TL_{x^{-1}}) \in X$ for each $T \in X$ and $x \in G$. If there exists a bounded projection from $\mathcal{B}(A, A^*)$ onto X, then there exists a bounded projection P from $\mathcal{M}(A, A^*)$ onto $\mathcal{M}(A, A^*) \cap X$ such that $C_xP = PC_x$ for all $x \in G$, where $C_x(T) = S_x(TL_{x^{-1}})$.

Proof. We can prove this Theorem by using an argument similar to one of the proof of Lemma 4 in [26]. Let M be a right invariant mean on $L^\infty(G)$. Let Q be a bounded projection of $\mathcal{B}(A, A^*)$ onto X. Take $\phi \in A \otimes_p B$ and $T \in \mathcal{B}(A, A^*)$. The mapping

$$\begin{aligned} f_T^\phi : x \mapsto \langle \Phi^{-1}(C_{x^{-1}}QC_x(T)), \phi\rangle \\ G \to \mathbb{C} \end{aligned}$$

belongs to $L^\infty(G)$. If $T \in \mathcal{B}(A, A^*)$, we consider the mapping $f_T : \phi \mapsto \langle M, f_T^\phi\rangle$. Now we consider $P : \mathcal{M}(A, A^*) \to \mathcal{M}(A, A^*) \cap X$ defined by $\langle P(T)(a), b\rangle = \langle M, f_T^{a\otimes b}\rangle$. We claim that P is a bounded projection of $\mathcal{M}(A, A^*)$ onto $\mathcal{M}(A, A^*) \cap X$ and that $C_xP = PC_x$ for all $x \in G$. Let $\phi \in {}^\perp(\Phi^{-1}(X))$ and $T \in \mathcal{M}(A, A^*)$. Then $\langle \Phi^{-1}(C_{x^{-1}}QC_x(T)), \phi\rangle = 0$ for all $x \in G$, since $C_{x^{-1}}QC_x(T) \in X$. Hence

$$\langle \Phi^{-1}(P(T)), \phi\rangle = \langle f_T, \phi\rangle = \langle M, f_T^\phi\rangle = 0,$$

and so $P(T) \in X$. This shows that $P(\mathcal{M}(A, A^*)) \subseteq X$. Next, let $T \in X \cap \mathcal{M}(A, A^*)$. Then $QC_x(T) = C_x(T)$ for each $x \in G$, and so $\Phi^{-1}(C_{x^{-1}}QC_x(T)) = \Phi^{-1}(T)$. We have

$$\langle \Phi^{-1}(P(T)), \phi\rangle = \langle f_T, \phi\rangle = \langle M, f_T^\phi\rangle = \langle \Phi^{-1}(T), \phi\rangle$$

for each $\phi \in A \otimes_p A$. Hence $P(T) = T$. For every $a \otimes b \in A \otimes_p B$ and $x, y \in G$,

$$\begin{aligned} f_{C_y(T)}^{a\otimes b}(x) &= \langle \Phi^{-1}(C_{x^{-1}}QC_xC_y(T)), a \otimes b\rangle \\ &= \langle \Phi^{-1}(S_{x^{-1}}(Q(S_x(S_y(TL_{y^{-1}})L_{x^{-1}}))L_x)), a \otimes b\rangle \\ &= \langle \Phi^{-1}(S_{x^{-1}}(Q(S_{xy}(TL_{(xy)^{-1}}))L_x)), a \otimes b\rangle \\ &= \langle \Phi^{-1}(S_y(S_{y^{-1}}(S_{x^{-1}}(QC_{xy}(T)L_x)L_y)L_{y^{-1}})), a \otimes b\rangle \\ &= \langle \Phi^{-1}(S_y(S_{(xy)^{-1}}(QC_{xy}(T)L_{xy})L_{y^{-1}})), a \otimes b\rangle \\ &= \langle \Phi^{-1}(S_y(C_{(xy)^{-1}}QC_{xy}(T)L_{y^{-1}})), a \otimes b\rangle = f_T^{y^{-1}a\otimes yb}(xy). \end{aligned}$$

Thus

$$\begin{aligned} \langle PC_y(T)(a), b\rangle &= \langle P(C_y(T))(a), b\rangle = \langle M, f_{C_y(T)}^{a\otimes b}\rangle = \langle M, f_T^{y^{-1}a\otimes yb}\rangle \\ &= \langle P(T)(y^{-1}a), yb\rangle = \langle C_yP(T)(a), b\rangle. \end{aligned}$$

Hence we have $C_yP = PC_y$ for all $y \in G$. If $T \in \mathcal{M}(A, A^*)$, then $S_x(T) = TL_x$ for all $x \in G$. For every $a \in A$ and $x \in G$,

$$\begin{aligned} S_x(P(T))(a) &= S_x(P(T))(L_{x^{-1}}L_xa) = S_x(P(T)L_{x^{-1}})(L_xa) \\ &= C_xP(T)L_x(a) = PC_x(T)L_x(a) \\ &= P(S_x(TL_{x^{-1}}))L_x(a) = P(T)L_x(a). \end{aligned}$$

This proves that $S_x(P(T)) = P(T)L_x$, and so $P(\mathcal{M}(A, A^*)) \subseteq \mathcal{M}(A, A^*)$. Consequently we conclude that P is a bounded projection from $\mathcal{M}(A, A^*)$ onto $\mathcal{M}(A, A^*) \cap X$. This completes the proof. $\square$

Theorem 4.3. Let G be an amenable locally compact group, and let A be a left Banach G-module. Let X be a weak*-closed subspace of A^* such that $S_x(TL_{x^{-1}})(a) \in X$ for all $T \in X$, $a \in A$ and $x \in G$. Let P be a bounded projection of A^* onto X. Then there exists a bounded projection from $\mathcal{B}(A, A^*)$ onto $\mathcal{M}(A, X)$.

Proof. For $a, b \in A$ and $T \in \mathcal{B}(A, A^*)$, we define $f_T^{a,b} : G \to \mathbb{C}$ by $f_T^{a,b}(x) = \langle P(T)(xa), x^{-1}b\rangle$. We see immediately that $f_T^{a,b} \in L^\infty(G)$. Let M be a right invariant mean on $L^\infty(G)$ [22]. For $T \in \mathcal{B}(A, A^*)$, we define $\langle F_T(a), b\rangle = \langle M, f_T^{a,b}\rangle$. Clearly $F_T \in \mathcal{B}(A, A^*)$. Let Q denote the function on $\mathcal{B}(A, A^*)$ defined by $Q(T) = F_T$. We claim that Q is a bounded projection of $\mathcal{B}(A, A^*)$ onto $\mathcal{M}(A, X)$. For $T \in \mathcal{B}(A, A^*)$, $y \in G$ and $a, b \in A$, we have

$$f_T^{a,yb}(x) = \langle P(T)(xa), x^{-1}yb\rangle = \langle P(T)(xy^{-1}ya), x^{-1}yb\rangle = f_T^{ya,b}(xy^{-1}).$$

Since M is a right invariant mean, we have

$$\langle Q(T)(a), yb\rangle = \langle M, f_T^{a,yb}\rangle = \langle M, f_T^{ya,b}\rangle = \langle Q(T)(ya), b\rangle.$$

Hence $S_y(Q(T)) = Q(T)L_y$, and so $Q(\mathcal{B}(A, A^*)) \subseteq \mathcal{M}(A, A^*)$. Let $T \in \mathcal{B}(A, A^*)$ and $a \in A$. Then $\langle P(T)(xa), x^{-1}b\rangle = 0$ for each $x \in G$ and $b \in {}^\perp X$, since $S_x(P(T)L_{x^{-1}})(a) \in X$. Thus $\langle Q(T)(a), b\rangle = \langle M, f_T^{a,b}\rangle = 0$, and so $Q(T)(a) \in ({}^\perp X)^\perp = X$. This shows that $Q(\mathcal{B}(A, A^*)) \subseteq \mathcal{M}(A, X)$. Next, let $T \in \mathcal{M}(A, X)$. Then $P(T(xa)) = T(xa)$ for each $x \in G$ and $a \in A$. For every $x \in G$,

$$\begin{aligned} f_T^{a,b}(x) &= \langle P(T(xa)), x^{-1}b\rangle = \langle T(xa), x^{-1}b\rangle = \langle S_x(T)(a), x^{-1}b\rangle \\ &= \langle T(a), b\rangle = f_T^{a,b}(e). \end{aligned}$$

Hence

$$\langle Q(T)(a), b\rangle = \langle M, f_T^{a,b}\rangle = \langle M, f_T^{a,b}(e)1\rangle = f_T^{a,b}(e) = \langle T(a), b\rangle.$$

It follows that $Q(T) = T$. This completes our proof. $\square$

Let S be a foundation topological semigroup. If H is a subsemigroup of S, we put

$$X_H = \{f \in M_a(S, \omega)^*;\ f\delta_x = \delta_x f = f \text{ for all } x \in H\}.$$

We can easily see that every X_H is a weak* closed linear subspace of $M_a(S,\omega)^*$ which $\delta_x X_H \subseteq X_H$ and $X_H\delta_x \subseteq X_H$ for all $x \in S$. In the following Theorem, we shall study relation between the weak*-closed subspace X of $M_a(S,\omega)^*$ and $\mathcal{M}(M_a(S,\omega),X)$. Note that $(S, M_a(S,\omega))$ is a left Banach S-module under the natural action $(x,\mu) \mapsto \delta_x * \mu$. If S is a commutative semigroup, then $T \in \mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$ if and only if T commutes with translations, see [12] and [17].

Theorem 4.4. Let S be a foundation semigroup with identity, and let X be a weak*-closed subspace of $M_a(S,\omega)^*$ such that $X\delta_x \subseteq X$ for all $x \in S$:

(1) let $T \in \mathcal{B}(M_a(S,\omega), M_a(S,\omega)^*)$ satisfying $T(\mu * \delta_x) = T(\mu)\delta_x$ for each $x \in S$. Then there is a unique $f \in M_a(S,\omega)^*$ such that $T(\mu) = f\mu$. Moreover $\|T\| = \|f\|$;

(2) $\mathcal{M}(M_a(S,\omega),X)$ is a closed subspace of $\mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$ (in the weak* operator topology).

(3) further suppose that S is commutative. Let Q be a bounded projection from $\mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$ onto $\mathcal{M}(M_a(S,\omega),X)$. Then there exists a bounded projection P of $M_a(S,\omega)^*$ onto X.

Proof. (1) For every $\mu,\nu,\eta \in M_a(S,\omega)$, we have

$$\begin{aligned}\langle T(\mu*\nu),\eta\rangle &= \langle T^*(\eta),\mu*\nu\rangle = \int \langle T^*(\eta),\mu*\delta_x\rangle d\nu(x)\\ &= \int \langle T(\mu*\delta_x),\eta\rangle d\nu(x) = \int \langle T(\mu)\delta_x,\eta\rangle d\nu(x)\\ &= \int \langle T(\mu),\delta_x*\eta\rangle d\nu(x) = \langle T(\mu)\nu,\eta\rangle.\end{aligned}$$

Hence $T(\mu*\nu) = T(\mu)\nu$. Now, let (e_α) be a bounded approximate identity of norm 1 in $M_a(S)$ (see Lemma 3.4 in [18]). Without loss of generality, we may assume that $T(e_\alpha) \to f$ in the weak*-topology. It is clear that

$$\langle T(\mu),\nu\rangle = \lim_\alpha \langle T(e_\alpha*\mu),\nu\rangle = \lim_\alpha \langle T(e_\alpha),\mu*\nu\rangle = \langle f,\mu*\nu\rangle = \langle f\mu,\nu\rangle,$$

where $\mu,\nu \in M_a(S,\omega)$. This shows that $T(\mu) = f\mu$. It is easy to see that $\|T\| = \|f\|$. It is obvious that the correspondence between T and f is an isometric isomorphism.

(2) Clearly $\mathcal{M}(M_a(S,\omega),X)$ is a subspace of $\mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$. If (T_α) is a net in $\mathcal{M}(M_a(S,\omega),X)$ that converges weak* to some $T \in \mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$, then $T_\alpha(\mu)$ converges to $T(\mu)$ in the weak*-topology (for any $\mu \in M_a(S,\omega)$). Pick $T(\mu) \notin X$. Since X is weak*-closed, Theorem 3.5 in [24], shows that there is a $\nu \in M_a(S,\omega)$ such that $\langle f,\nu\rangle = 0$ for every $f \in X$, but $\langle T(\mu),\nu\rangle \neq 0$. Hence $\langle T_\alpha(\mu),\nu\rangle = 0$ for every α, but $\langle T(\mu),\nu\rangle \neq 0$ which is a contradiction. This proves that $\mathcal{M}(M_a(S,\omega),X)$ is weak*-closed.

(3) For $f \in M_a(S,\omega)^*$, define $\lambda_f : M_a(S,\omega) \to M_a(S,\omega)^*$ by $\lambda_f(\mu) = f\mu$. As above, the map $\Lambda : M_a(S,\omega)^* \to \mathcal{M}(M_a(S,\omega), M_a(S,\omega)^*)$ given by $\Lambda(f) = \lambda_f$,

is an isometric isomorphism. Now we consider $P : M_a(S,\omega)^* \to M_a(S,\omega)^*$ defined by $P(f) = \Lambda^{-1}(Q(\Lambda(f)))$. To see that P is a projection of $M_a(S,\omega)^*$ onto X, it suffices to show that $P(M_a(S,\omega)) \subseteq X$ and that $f \in X$ implies $P(f) = f$. Fix $f \in M_a(S,\omega)^*$. As above, there exists $f' \in M_a(S,\omega)^*$ with $Q(\lambda_f) = \lambda_{f'}$. By assumption, $Q(\lambda_f)(\mu) \in X$ for every $\mu \in M_a(S,\omega)$. Pick $\nu \in {}^\perp X$, so that $\langle f'\mu, \nu\rangle = \langle Q(\lambda_f)(\mu), \nu\rangle = 0$ for all $\mu \in M_a(S,\omega)$. By Lemma 3.4 in [18], there is an approximate identity (e_α) in $M_a(S,\omega)$. We have

$$\langle f', \nu\rangle = \lim_\alpha \langle f', e_\alpha * \nu\rangle = \lim_\alpha \langle f' e_\alpha, \nu\rangle = 0,$$

and so $f' \in X$. Hence $P(M_a(S,\omega)^*) \subseteq X$. Next, let $f \in X$. By Lemma 3.2, $f\mu \in X$ for every $\mu \in M_a(S,\omega)$. This shows that $\lambda_f \in \mathcal{M}(M_a(S,\omega), X)$. Therefore $P(f) = \Lambda^{-1}(Q(\Lambda(f))) = f$. This completes our proof. □

Let A be a Banach algebra. Given $a \in A$, let I_a be the norm-closure of

$$\{x - xa;\ x \in A\}$$

in A, see [5] and [6]. Let S be a locally compact foundation semigroup with identity, and let $\eta \in M_a(S,\omega)$. It is easy to see that I_η is a left closed ideal in $M_a(S,\omega)$ and its annihilator $I_\eta{}^\perp = \left(\frac{M_a(S,\omega)}{I_\eta}\right)^*$ is the space $\{f \in M_a(S,\omega)^*;\ \eta f = f\}$ which we call the η-harmonic functional on $M_a(S,\omega)$.

Lemma 4.5. Let S be a foundation topological semigroup with identity, and let $\eta \in M_a(S,\omega)$. Then $I_\eta^\perp \cap M_a(S,\omega)^* M_a(S,\omega)$ is weak*-dense in $I_\eta^\perp$.

Proof. Let (e_α) be an approximate identity of norm 1 in $M_a(S,\omega)$, and let $f \in I_\eta^\perp$. Then $fe_\alpha \in I_\eta^\perp \cap M_a(S,\omega)^* M_a(S,\omega)$ and for $\mu \in M_a(S,\omega)$, we have $\langle f, \mu\rangle = \lim_\alpha \langle f, e_\alpha * \mu\rangle$. This shows that (fe_α) converges to f in the weak*-topology.□

For $x \in S$, we observe that $\delta_x(I_\eta^\perp) \subseteq I_\eta^\perp$ if and only if

$$\delta_x(I_\eta^\perp \cap M_a(S,\omega)^* M_a(S,\omega)) \subseteq I_\eta^\perp \cap M_a(S,\omega)^* M_a(S,\omega).$$

This follows from the fact that $I_\eta^\perp \cap M_a(S,\omega)^* M_a(S,\omega)$ is weak* dense in $I_\eta^\perp$ and that $x \mapsto \delta_x f$ from S into $M_a(S,\omega)^*$ is weak*-continuous, where $f \in M_a(S,\omega)^*$. Moreover, if η is central which means that $\delta_x * \eta = \eta * \delta_x$ for all $x \in S$, then $\delta_x(I_\eta^\perp) \subseteq I_\eta^\perp$ for all $x \in S$. It is easy to see that $\delta_x(I_\eta^\perp) \subseteq I_\eta^\perp$ for all $x \in S$ if and only if I_η is a two sided ideal in $M_a(S,\omega)$.

Proposition 4.6. Let $a \in A$ and suppose $\|a\| \leq 1$. Then there exists a projection P from A^* onto $I_a^\perp$.

Proof. Fix a Banach limit LIM on N. For $f \in A^*$ and $x \in A$, put

$$\langle P(f), x\rangle = LIM_n \langle f, xa^n\rangle$$

Then $P : A^* \to A^*$ is a well-defined, contractive linear map.

Clearly $P(f) = f$ for each $f \in I_a^{\perp}$. If $f \in A^*$ is arbitrary and $x \in A$, we have

$$\begin{aligned} \langle P(f), x - xa \rangle &= LIM_n \langle f, xa^n \rangle - \langle f, xa^{n+1} \rangle \\ &= LIM_n \langle f, xa^n \rangle - LIM_m \langle f, xa^m \rangle = 0. \end{aligned}$$

and so by continuity $P(f) \in I_a^{\perp}$, as required. □

Theorem 4.7. Let S be a foundation semigroup, and let η be a probability measure in $M(S)$. If $dimI_\eta^{\perp} = 1$, then S is amenable.

Note that a topological semigroup S is *amenable* if there is a positive norm one linear functional M on $M_a(S)^*$ such that $\langle M, f\delta_x \rangle = \langle M, f \rangle$ for all $f \in M_a(S)^*$ and $x \in S$ [11].

Proof. From the above construction of P_η, there is (P_n) in $co\{\rho_\eta^{*n};\ n \in \mathbb{N}\}$ such that $\|P_n\| \leq 1$ and $P_n(1) = 1$ for $n \in \mathbb{N}$ and such that $P_n \to P_\eta$ as $n \to \infty$. For every $f \in M_a(S)^*$ and $n \in \mathbb{N}$, $P_n(f\delta_x) = P_n(f)\delta_x$ where $x \in S$. It follows that $P_\eta(f\delta_x) = P_\eta(f)\delta_x$. Define $M : M_a(S)^* \to \mathbb{C}$ by $\langle M, f \rangle = \langle P_\eta(f), \eta \rangle$. Since M is positive linear and $\langle M, 1 \rangle = 1$, so M is a mean on $M_a(S)^*$. Next, suppose that $f \in M_a(S)^*$ and $x \in S$. Since $dimI_\eta^{\perp} = 1$, $P_\eta(f) = c_f 1$ for a constant c_f. We have

$$\begin{aligned} \langle M, f\delta_x \rangle &= \langle P_\eta(f\delta_x), \eta \rangle = \langle P_\eta(f)\delta_x, \eta \rangle = \langle P_\eta(f), \delta_x * \eta \rangle \\ &= \langle c_f 1, \delta_x * \eta \rangle = c_f = \langle c_f 1, \eta \rangle \\ &= \langle P_\eta(f), \eta \rangle = \langle M, f \rangle. \end{aligned}$$

Therefore S is amenable. □

Let S be a foundation topological semigroup. If η is a probability measure, then I_η is contained in the ideal $\{\mu \in M_a(S,\omega);\ \mu(S) = 0\}$. If η is a probability measure, then evidently $I_\eta = \{\mu \in M_a(S,\omega);\ \mu(S) = 0\}$ if and only if $dimI_\eta^{\perp} = 1$, in other words, the bounded η-harmonic functions are constant. Indeed, if $I_\eta = \{\mu \in M_a(S,\omega);\ \mu(S) = 0\}$, then the map $T : \frac{M_a(S,\omega)}{I_\eta} \to \mathbb{C}$ given by $T(\mu + I_\eta) = \mu(S)$ is a linear isomorphism. It follows that $dimI_\eta^{\perp} = 1$. Conversely, since

$$I_\eta \subseteq \{\mu \in M_a(S,\omega);\ \mu(S) = 0\} \subsetneqq M_a(S,\omega),$$

this shows that $I_\eta = \{\mu \in M_a(S,\omega);\ \mu(S) = 0\}$.

By Theorem 4.7, $I_\eta = \{\mu \in M_a(S);\ \mu(S) = 0\}$ implies that S is amenable.

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