On Singletonness of Remotal and Uniquely Remotal Sets

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Abstract

A bounded subset T of a metric space (X, ρ) is said to be remotal (uniquely remotal) if for each $x \in X$ there exists at least one (exactly one) $t \in T$ such that $\rho(x, t) = \sup\{\rho(x, y) : y \in T\}$. Such a point t is called a farthest point to x in T. In this paper, we discuss properties of remotal and uniquely remotal sets and, conditions under which remotal and uniquely remotal sets are singleton. The underlying spaces are convex metric spaces or externally convex metric spaces.

1 Introduction

One of the most interesting and hitherto unsolved problems in the theory of farthest points, known as the farthest point problem, is: If every point of a normed linear space X admits a unique farthest point in a given bounded subset T, then must T be a singleton? There are some partial affirmative answers to this problem and there are many special cases in which the answer is negative (see e.g. [2], [6], [7], [8] and references cited therein). The problem is not solved in general even in Hilbert spaces. This problem is so closely related to the problem of convexity of Chebyshev sets in a Hilbert space (which too is an open problem in the theory of nearest points) that a solution of one will lead to a solution of the other (Ficken's Theorem - see Klee [5]). In this paper, we discuss properties of remotal and uniquely remotal sets and, conditions under which remotal and uniquely

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remotal sets are singleton thereby giving some partial affirmative answers to the farthest point problem (f.p.p.). The underlying spaces are convex metric spaces or externally convex metric spaces. The results proved in the paper generalize and extend some of the results proved in [1], [2], [9] and [10].

Before proceeding to our main result, we recall few definitions.

2 Definitions and Notations

Definition 1. Let (X, ρ) be a metric space, T a non-empty bounded subset of X and

$$r(T, x) = \sup\{\rho(x, y) : y \in T\}.$$

Then the set- valued map $F_T : X \to 2^T \equiv$ the collection of all subsets of T, defined by

$$F_T(x) = \{t \in T : \rho(x, t) = r(T, x)\}$$

is called a **farthest point map** (f. p. m.) and elements of $F_T(x)$ are called **farthest points** or **remotal points** of $x \in X$ in T. The set is said to be

(a) **remotal** if $F_T(x)$ is non-empty for each $x \in X$,
(b) **uniquely remotal** if $F_T(x)$ is exactly singleton for each $x \in X$.

For uniquely remotal sets T, the f. p. m. F_T is single-valued and is denoted by q_T.

Definition 2. For a remotal set T, the farthest point map F_T is said to be **sectionally continuous** (see *[10]*) at x_o if for any sequence $\{x_n\}$ with $\{x_n\} \to x_o$ there exist $F_T(x_n)$, $F_T(x_o)$ such that $F_T(x_n) \to F_T(x_o)$.

Definition 3. The number $r(T) = \inf\{r(T, x) : x \in X\}$ is called the Chebyshev radius of T. A centre or Chebyshev centre of T is a point c, if it exists, such that $r(T, c) = r(T)$.

Definition 4. A point c in a subset F of a metric space (X, ρ) for which $r(T, c) = \inf\{r(T, x) : x \in F\}$ is called a **relative Chebyshev centre** of subset T of X with respect to F.

Definition 5. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$\rho(u, W(x, y, \lambda)) \leqslant \lambda\rho(u, x) + (1 - \lambda)\rho(u, y)$$

holds for all $u \in X$. A metric space (X, ρ) together with a convex structure is called a **convex metric space***[12]*.

Example 1.*[12]* Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leqslant a_i \leqslant b_i \leqslant 1$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and λ $(0 \leqslant \lambda \leqslant 1)$, define a mapping W by $W(I_i, I_j, \lambda) = [\lambda a_i + (1 - \lambda)a_j, \lambda b_i + (1 - \lambda)b_j]$ and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i, I_j) = \sup_{a \in I}\left\{\left|\inf_{b \in I_i}\{|a - b|\} - \inf_{c \in I_j}\{|a - c|\}\right|\right\}.$$

The metric space (X, d) along with the convex structure W is convex metric space.

Definition 6. A convex metric space (X, ρ) is called an M-**space**[4] if for every two points $x, y \in X$ with $d(x, y) = \lambda$, and for every $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that

$$B[x, r] \cap B[y, \lambda - r] = \{z_r\},$$

where $B[x, r] = \{y \in X : \rho(x, y) \leqslant r\}$.

We denote by $[x, y]$ the line segment joining x and y i.e. $[x, y] = \{z \in X : \rho(x, z) + \rho(z, y) = \rho(x, y)\}$; $[x, y, -[= \{z \in X : \rho(x, y) + \rho(y, z) = \rho(x, z)\}$ denotes a half ray starting from x and passing through y i.e. it is the union of line segments $[x, z]$ where $[x, y] \subseteq [x, z]$.

Definition 7. A metric space (X, ρ) is called **externally convex**[4] if for all distinct points x, y such that $\rho(x, y) = \lambda$, and $r > \lambda$ there exists a unique z of X such that $\rho(x, y) + \rho(y, z) = \rho(x, z) = r$.

Example 2.[4] Consider metric space (X, d) consisting of points on lines $y = 1$ and $y = 2$ in the cartesian plane with $x \geqslant 0$. Let the distance $d(x, y)$ for $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be given by $|x_1 - x_2|$ if $y_1 = y_2$ and $1 + |x_1| + |x_2|$ if $y_1 \neq y_2$. This satisfies the condition of external convexity. It may be noted that the space X is not a normed linear space as it is not a linear space. Every strictly convex normed linear space is an M-space but converse is not true. Every strictly convex metric space is an M-space but converse is not true as is clear from the following example.

Example 3.[4] Let (X, d) be a closed ball of $S_{2,r}$ of radius ρ with $\pi r/4 < \rho < \pi r/2$. Since X is convex and contains no diametral point pairs of the $S_{2,r}$, (X, d) is an M-space. Here $S_{2,r}$ is the 2-dimensional spherical space of radius r. Its elements are all the ordered 3-tuples $x = (x_1, x_2, x_3)$ of real numbers with

$$\sum_{i=1}^{3} x_i^2 = r^2,$$

distance 'd' is defined for each pair of elements x, y to be the smallest non-negative number xy such that

$$\cos\left(\frac{xy}{r}\right) = \frac{\sum_{i=1}^{3} x_i y_i}{r^2}.$$

The space (X, d) is an M-space.

If we consider the points x, y and z of X such that $d(x, y) = d(x, z) = \frac{\pi r}{2}$ then all points w of X between y and z have property that $d(x, w) = \frac{\pi r}{2}$, contradicting the strict convexity.

If (X, ρ) is a convex metric space then for each two distinct points $x, y \in X$ and for every λ, $0 < \lambda < 1$, there exists at least one point $z \in X$ such that $z = W(x, y, \lambda)$. For M-space such a z is always unique.

Definition 8. In a convex metric space (respectively in an M-space)(X, ρ) we say that $\mathbf{P}(\mathbf{x}, \mathbf{d})$ is true for some $d > 0$, $0 < d < 1$(see [1]) if $y \in F_T(x) \Rightarrow y \in F_T(W(x, y, t))$, $0 < t \leqslant d$ (respectively if $y \in F_T(x)$, $y' \in [x, y]$ such that $\rho(y', y) = (1 - t)\rho(x, y)$, $\rho(y', x) = t\rho(x, y)$, for $0 < t \leqslant d$ imply $y \in F_T(y')$).

Example 4. Let $X = R \setminus \{0\}$ with usual metric and $K = [-1,1] \setminus \{0\}$. Property $P(x,d)$ is true for $x = 1, -1$ and $d = 1/2$. $F_K(-1) = 1$, $F_K(x) = 1 \equiv y$ for all $x \in [-1,0[$, $F_K(ty + (1-t)x) = 1, 0 < t \leqslant d$. $F_K(1) = -1$, $F_K(x) = -1 \equiv y$ for all $x \in]0,-1]$, $F_K(ty + (1-t)x) = -1$.

Note. $F_K(-1) = 1$ but $F_K(x) \neq 1$ for all $x \in [-1,1]$ i.e. for all $x \in [-1, F_K(-1)]$. $F_K(1) = -1$ but $F_K(x) \neq -1$ for all $x \in [F_K(1), 1]$.

3 Farthest Point Problem in Externally Convex M-spaces

The following result may be useful towards a solution of the f.p.p. in externally convex M-spaces:

Theorem 1. *Let T be a uniquely remotal subset of an externally convex M-space (X,ρ). A necessary and sufficient condition that T be a singleton is that*

$$\rho(x, q_T(x)) = \rho(y, q_T(y)) \text{ implies } q_T(x) = q_T(y). \tag{3.1}$$

Proof. If T is a singleton then $q_T(x) = q_T(y)$ for all $x, y \in X$ and so (3.1) is automatically satisfied. Now, suppose (3.1) is true. We show that for any $x, y \in X$, $q_T(x) = q_T(y)$ and hence T is a singleton.

Let $x_o,\ y_o \in X,\ x_o \neq y_o$. Without loss of generality, we assume that $0 < \rho(x_o, q_T(x_o)) < \rho(y_o, q_T(y_o))$. Take $z_o \in [q_T(x_o), x_o, -[\setminus [q_T(x_o), x_o]$ such that $\rho(z_o, q_T(x_o)) = \rho(y_o, q_T(y_o))$. Consider $\rho(y_o, q_T(y_o)) = \rho(z_o, q_T(x_o)) \leqslant \rho(z_o, q_T(z_o)) \leqslant \rho(z_o, x_o) + \rho(x_o, q_T(z_o)) \leqslant \rho(z_o, x_o) + \rho(x_o, q_T(x_o)) = \rho(z_o, q_T(x_o)) = \rho(y_o, q_T(y_o))$.
Hence $\rho(y_o, q_T(y_o)) = \rho(z_o, q_T(x_o)) = \rho(z_o, q_T(z_o))$ and so using (3.1) and unique remotality of T, we get $q_T(y_o) = q_T(z_o) = q_T(x_o)$.
Note. For normed linear spaces, above theorem was proved by Niknam [9].

The f.p.m. has played an important role in giving partial affirmative answers to the f.p.p.(see e.g.[3], [6] and [11]). Blatter [3] proved that if the f.p.m.of a remotal (uniquely remotal) subset T of a Banach space X is lower- semi -continuous (continuous) then T is a singleton. Panda and Kapoor [11] used the idea of Chebyshev centres and obtained Blatter's result for spaces admitting centres. In fact sectional continuity of F_T suffices for giving positive solution to the f.p.p. Moreover, concerning uniquely remotal sets the following conditions have been used:

Let T be a uniquely remotal subset of an M-space (X,ρ), then a point $x \in X$ is said to have property ε_x if

$$u \in [x, F_T(x)] \Rightarrow u \in E_x,$$

where $E_x = \{y \in X : \rho(y, q_T(y)) \geqslant \rho(x, q_T(x))\}$.

It is easy to see that ε_x and $P(x,d)$ are mutually exclusive and $P(x,d)$ implies sectional continuity of q_T at x along the ray containing $[x, q_T(x)]$(see [2]).

Concerning relative Chebyshev centre of uniquely remotal sets, we have

Proposition 1. Let T be a uniquely remotal subset of an externally convex M-space (X,ρ). Suppose x_o is the only relative Chebyshev centre of T w.r.t. E_{x_o} i.e. $x \neq x_o \Rightarrow \rho(x, q_T(x)) \neq \rho(x_o, q_T(x_o))$, then $X = E_{x_o} \cup [x_o, q_T(x_o)]$.

Proof. Let E'_{x_o} be the complement of E_{x_o} in X, then

$$E'_{x_o} = \{x \in X : \rho(x, q_T(x)) < \rho(x_o, q_T(x_o))\}.$$

It is enough to show that $E'_{x_o} \subset [x_o, q_T(x_o)]$. Let $x_1 \in E'_{x_o}$ i.e. $\rho(x_1, q_T(x_1)) < \rho(x_o, q_T(x_o))$. If $x_1 = q_T(x_1)$ then T is a singleton and therefore $x_1 = q_T(x_o) \in [x_o, q_T(x_o)]$. So we can assume that $x_o \neq q_T(x_o)$ and $x_1 \neq q_T(x_1)$.

Choose $y_o \in [q_T(x_1), x_1, -[$ such that

$$\rho(y_o, q_T(x_1)) = \rho(x_o, q_T(x_o)). \tag{3.2}$$

Then

$$\rho(x_o, q_T(x_o)) = \rho(y_o, q_T(x_1)) \leqslant \rho(y_o, q_T(y_o)).$$

$$\rho(y_o, x_1) = \rho(y_o, q_T(x_1)) - \rho(x_1, q_T(x_1)).$$

By (3.2), we have

$$\rho(y_o, x_1) = \rho(x_o, q_T(x_o)) - \rho(x_1, q_T(x_1)). \tag{3.3}$$

Consider
$\rho(x_o, q_T(x_o)) = \rho(y_o, q_T(x_1)) \leqslant \rho(y_o, q_T(y_o)) \leqslant \rho(y_o, x_1) + \rho(x_1, q_T(y_o)) \leqslant \rho(y_o, x_1) + \rho(x_1, q_T(x_1)) = \rho(x_o, q_T(x_o))$.

Therefore $\rho(x_o, q_T(x_o)) = \rho(y_o, q_T(x_1)) = \rho(y_o, q_T(y_o))$. This gives $x_o = y_o$ and $q_T(x_1) = q_T(x_o)$ as x_o is the only relative Chebyshev centre of T w.r.t. E_{x_o} and T is uniquely remotal. Now by (3.3) we have

$$\rho(x_o, x_1) = \rho(x_o, q_T(x_o)) - \rho(x_1, q_T(x_1)) \text{ i.e.}$$
$$\rho(x_o, x_1) + \rho(x_1, q_T(x_1)) = \rho(x_o, q_T(x_o))$$

Therefore $\rho(x_o, x_1) + \rho(x_1, q_T(x_o)) = \rho(x_o, q_T(x_o))$, which implies that $x_1 \in [x_o, q_T(x_o)]$ and hence $E'_{x_o} \subset [x_o, q_T(x_o)]$.

Concerning centre of uniquely remotal sets, we have

Proposition 2. Let T be a uniquely remotal subset of an externally convex M-space (X,ρ), then x_o is a centre of T iff $[x_o, q_T(x_o)] \subset E_{x_o}$ i.e x_o satisfies property ε_{x_o}.

Proof. If x_o is a centre of T then $\rho(x_o, q_T(x_o)) = \inf \rho(x, q_T(x))$ i.e. $\rho(x, q_T(x)) \geqslant \rho(x_o, q_T(x_o))$ for all $x \in X$. This implies $X = E_{x_o}$ as $E_{x_o} = \{x \in X : \rho(x, q_T(x)) \geqslant \rho(x_o, q_T(x_o))\}$ and so we have $[x_o, q_T(x_o)] \subset E_{x_o} = X$.

Conversely, if $[x_o, q_T(x_o)] \subset E_{x_o}$. Then by Proposition 1, we have $X = E_{x_o} \cup [x_o, q_T(x_o)] = E_{x_o}$. This gives $\rho(x, q_T(x)) \geqslant \rho(x_o, q_T(x_o))$ for all $x \in X$. i.e. $\rho(x_o, q_T(x_o)) = \inf \rho(x, q_T(x))$ i.e. x_o is a centre of T.

Note. For normed linear spaces, these results were proved by Niknam [10].

4 Property $P(x,d)$ and Singletonness of Remotal Sets

The following result shows that for a remotal set, property $P(c,d)$ can not be true if c is centre of the set.

Proposition 3. If T is remotal subset of a convex metric space (X,ρ) then $P(c,d)$ cannot be true if c is a centre of the set T.

Proof. Suppose $P(c,d)$ is true where c is a centre of T i.e. $r(T,c) = \inf\{r(T,x) : x \in X\}$, so $r(T,c) \leqslant r(T,x)$ for all $x \in X$. Let $y \in F_T(c)$ and property $P(c,d)$ is true. Then for all $y' \in [c,y]$ such that $\rho(y',y) = (1-t)\rho(c,y)$, $\rho(y',c) = t\rho(c,y)$, for $0 < t \leqslant d$, we have $y \in F_T(y')$. But this implies $r(T,y') = \rho(y',y) < \rho(c,y) = r(T,c)$, which is not true as $r(T,c) = \inf\{r(T,x) : x \in X\}$.
Note. For normed linear spaces, this result was proved by Baronti[2].

Concerning singletonness of remotal sets in convex metric space, we have the next result:

Theorem 2. *Let T be a remotal subset of a convex metric space (X,ρ) admitting centres. If there is a centre c in $E(T)$ such that $P(c,d)$ is true with $d > 0$ then T is a singleton. Here $E(T)$ = collection of all Chebyshev centres of T.*

Proof. Let c be any element of $E(T)$ such that $P(c,d)$ is true for $d > 0$. Let $f_T(c)$ be in $F_T(c)$, $\rho(c, f_T(c))$ is Chebyshev radius of T, so $T \subset B[c, \rho(c, f_T(c))]$. Let k be such that $0 < k < min\{1,d\}$. We claim that $\rho(c, f_T(c)) = 0$. If $\rho(c, f_T(c)) > 0$, we have $(1-k)\rho(c, f_T(c)) < \rho(c, f_T(c))$.Then by the definition of Chebyshev radius, $T \not\subset B[W(f_T(c), c, k), (1-k)\rho(c, f_T(c))]$. So, there exists $t_o \in T$ such that $t_o \notin B[W(f_T(c), c, k), (1-k)\rho(c, f_T(c))]$. This gives

$$\rho(W(f_T(c), c, k), t_o) > (1-k)\rho(c, f_T(c)). \tag{4.1}$$

From the fact that P(c,d) is true, we have

$$\rho(W(f_T(c), c, k), t_o) \leqslant \rho(W(f_T(c), c, k), f_T(c)),$$

which implies

$$\begin{aligned}\rho(W(f_T(c), c, k), t_o) &\leqslant k\rho(f_T(c), f_T(c)) + (1-k)\rho(c, f_T(c)) \\ &= (1-k)\rho(c, f_T(c)),\end{aligned}$$

but this contradicts (4.1). Therefore, our supposition is wrong and so $\rho(c, f_T(c)) = 0$ i.e. $c = f_T(c)$ i.e. c is its own farthest point in T and hence T is a singleton.
Note. For Banach spaces, this result was proved in [1].

Concerning singletonness of remotal sets in an M-space, we have the next result:

Theorem 3. *Let T be a remotal set in an M-space (X,ρ) such that for some $\varepsilon > 0$, there exists a $d > 0$ for which $P(y,d)$ is true for all $y \in X$ satisfying $r(T,y) < r(T) + \varepsilon$, then T is a singleton.*

Proof. Suppose that T is not a singleton. Under our assumptions, let $r(T) > 0$, take $\varepsilon' = \min\{\varepsilon, d\, r(T)\}$. Therefore $r(T) + \varepsilon'$ cannot be a lower bound from the definition of $r(T)$. So there exists $c_d \in X$ such that

$$r(T, c_d) < r(T) + \varepsilon' \leqslant r(T) + d\, r(T) \tag{4.2}$$

i.e.

$$r(T, c_d) < r(T)(1 + d). \tag{4.3}$$

For $f_d \in F_T(c_d)$, take $c' \in [f_d, c_d]$ such that $\rho(c', f_d) = (1 - d)\rho(c_d, f_d)$. Since c_d satisfies (4.2), by hypothesis we have, $P(y, d)$ is true for $y = c_d$ i.e.

$$f_d \in F_T(c_d) \text{ implies } f_d \in F_T(c'). \tag{4.4}$$

Consider

$$\rho(c', f_d) = (1 - d)\rho(c_d, f_d) = (1 - d)r(T, c_d) = r(T, c_d) - d\, r(T, c_d) \leqslant \\ r(T, c_d) - d\, r(T) < r(T)(1 + d) - d\, r(T) = r(T) + d\, r(T) - d\, r(T) = r(T).$$

Therefore

$$\rho(c', f_d) < r(T) \text{ as } f_d \in T \tag{4.5}$$

By (4.4), we have $f_d \in F_T(c')$ and so $\rho(c', f_d) = r(T, c')$. Then (4.5) gives $r(T, c') < r(T)$, which is absurd as $r(T) = \inf\{r(T, x) : x \in X\}$. This absurdity proves the result.

Note. For normed linear spaces, Theorem 3 was proved in [2].

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