A note on blow-up of a nonlinear integral equation

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Abstract

Let us deal with the positive solutions of

$$\frac{\partial u(t)}{\partial t} = k(t)\Delta_\alpha u(t) + h(t)u^{1+\beta}(t), \;\; u(0,x) = \varphi(x) \geq 0, \; x \in \mathbb{R}^d,$$

where Δ_α is the fractional Laplacian, $0 < \alpha \leq 2$, and $\beta > 0$ is a constant. We prove that under certain regularity condition on φ, h and k any non-trivial positive solution blows up in finite time. In this way we answer, in particular, the question raised in [4] for the critical case.

1 Introduction

In this paper we study positive solutions for the semilinear equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \kappa(t)\Delta_\alpha u(t,x) + h(t)u^{1+\beta}(t,x), \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where we denote by $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ the fractional power of the Laplacian Δ, $0 < \alpha \leq 2$, and $\beta > 0$ is a constant. Also we suppose that the initial value $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a bounded continuous function, positive and not identically zero,

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and $\kappa, h : [0,\infty) \to [0,\infty)$ are continuous functions, such that for all t large enough,

$$a_1 t^\rho \le \kappa(0,t) \le a_2 t^\rho, \;\; b_1 t^{\sigma-1} \le h(t) \le b_2 t^{\sigma-1}, \tag{1.2}$$

where $\kappa(s,t) = \int_s^t \kappa(r)dr$, $0 \le s \le t$. Here a_1, a_2, b_1, b_2, ρ and σ are positive constants.

In applied mathematics it is well known the importance of the study of equations like (1.1). In fact, for example, they arise in fields like molecular biology, hydrodynamics and statistical physics [7]. Also, notice that generators of the form $\kappa(t)\Delta_\alpha$ arise in models of anomalous growth of certain fractal interfaces [6].

When $\alpha = 2$, $\kappa \equiv 1$ and $h \equiv 1$, Fujita [2] showed that $d = \frac{2}{\beta}$ is the critical dimension for blow-up of (1.1); this means that if $d > \frac{2}{\beta}$, then (1.1) admits a global solution for all sufficiently small initial condition, but if $d < \frac{2}{\beta}$, then for any non-vanishing initial condition the solution is infinite for all t large enough. In the critical case, $d = \frac{2}{\beta}$, Sugitani [8] showed that (1.1) also blows up in finite time.

Kolkovska et al. [4] proved (when $h \equiv 1$) that any combination of positive parameters d, α, β, ρ obeying $0 < \frac{d\beta\rho}{\alpha} < 1$ yields finite time blow-up of any non-trivial positive solution of equation (1.1). In that paper they raised the question if such result remains true if $\frac{d\beta\rho}{\alpha} = 1$. They call it the critical dimension, it is because it was proved in Corollary 3.2.2 of [5] that $\frac{d\beta\rho}{\alpha} > 1$ implies existence of non-trivial global solutions of (1.1) for all sufficiently small initial values.

Here we prove that when $\sigma \ge \frac{d\beta\rho}{\alpha}$, then the solution of (1.1) blows up in finite time (see Theorem 2). In this way, our result is a generalization of the works done by López-Mimbela and Pérez [5], Kolkovska et al. [4], Guedda and Kirane [3], Birkner et al. [1] and Sugitani [8]. Our method, like in [3], is based on the construction of a convenient subsolution of the solution of (1.1), this subsolution has the property that blows up in finite time. Such procedure was initiated by Sugitani [8]. It is worth while to notice that the solutions are understood in the mild sense.

2 Preliminary facts and local existence

Let us denote by $\langle\cdot,\cdot\cdot\rangle$ the usual scalar product on $\mathbb{R}^d$ and $||\cdot|| := \langle\cdot,\cdot\rangle^{1/2}$ the Euclidean norm.

We know [5] that $p(\kappa(0,t),x)$ is the fundamental solution of (1.1), where $p(t,x)$ is given by

$$\int_{\mathbb{R}^d} e^{i\langle z,x\rangle} p(t,x)dx = e^{-t||z||^\alpha}. \tag{2.1}$$

Moreover $p(t,x)$ has the following properties.

Proposition 1. *Let $p(t,x)$ be defined by* (2.1)*, then*

(a) $p(ts,x) = t^{-d/\alpha} p(s, t^{-1/\alpha}x)$,

(b) if $t \ge s$, then $p(t,x) \ge \left(\frac{s}{t}\right)^{d/\alpha} p(s,x)$,

(c) if $p(t,0) \leq 1$ and $\tau \geq 2$, then $p\left(t, \frac{x-y}{\tau}\right) \geq p(t,x)\, p(t,y)$.

Proof. For the proof see, for example, Section 2 in [8]. ∎

Note that the above property (a) implies that $p(t,0) = t^{-d/\alpha} p(1,0)$ is decreasing in t and $\lim_{t\to\infty} p(t,0) = 0$.

Let us see below that there exists a number $T_\varphi \in (0,\infty]$ and an unique continuous function $u : [0,T_\varphi) \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} p(\kappa(0,t), y-x)\varphi(y)dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(\kappa(s,t), y-x) h(s) u^{1+\beta}(s,y)dyds. \end{aligned} \tag{2.2}$$

For any $0 < T < T_\varphi$ the solution u is bounded on $[0,T] \times \mathbb{R}^d$. If $T_\varphi < \infty$, then $u(t,x) = \infty$ for any $(t,x) \in [T_\varphi, \infty) \times \mathbb{R}^d$. When $T_\varphi = \infty$ we say that u is a global solution, and when $T_\varphi < \infty$ then we say that u blows up in finite time.

Denote by $C_b(\mathbb{R}^d)$ the space of all bounded continuous functions on $\mathbb{R}^d$. Let $R, r > 0$ be a real numbers that we will fix later. Define

$$E_r := \left\{ u : [0,r] \to C_b(\mathbb{R}^d) \mid \|u\|_\infty < \infty \right\},$$

where $\|u\|_\infty$ is the supremum norm. Then E_r is a Banach space and the sets

$$P_r := \{u \in E_r : u \geq 0\} \quad \text{and} \quad B_R = \{u \in E_r : \|u\|_\infty \leq R\},$$

are closed subspaces of E_r.

Theorem 1. *There exists $r = r(\varphi) > 0$ such that the integral equation (2.2) has a local solution in $P_r \cap B_R$.*

Proof. Define the operator $\Psi : P_r \cap B_R \to C_b(\mathbb{R}^d)$, by

$$\begin{aligned} \Psi(u)(t,x) &= \int_{\mathbb{R}^d} p(\kappa(0,t), y-x)\, \varphi(y)\, dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(\kappa(s,t), y-x)\, h(s)\, u^{1+\beta}(s,y)\, dyds. \end{aligned}$$

Let us see that Ψ has a fixed point, which will be the solution of (2.2). For this purpose it is enough to show that Ψ is a contraction. Let $u, \widetilde{u} \in P_r \cap B_R$, then

$$\begin{aligned} &\|\Psi(u) - \Psi(\widetilde{u})\|_\infty \\ &= \left\| \int_0^t \int_{\mathbb{R}^d} p(\kappa(s,t), y-x)\, h(s) \left[u^{1+\beta}(s,y) - \widetilde{u}^{1+\beta}(s,y) \right] dyds \right\|_\infty. \end{aligned}$$

Using the elementary inequality

$$|a^q - b^q| \leq q\,(a \vee b)^{q-1} |a-b|, \quad a, b > 0,\ q \geq 1,$$

we get, by (1.2),

$$\begin{aligned}\|\Psi(u)-\Psi(\widetilde{u})\|_\infty &\leq (1+\beta)R^\beta\int_0^t h(s)\int_{\mathbb{R}^d} p(\kappa(s,t),y-x)\,\|u-\widetilde{u}\|_\infty\,dyds\\ &\leq \frac{b_2(1+\beta)}{\sigma}R^\beta r^\sigma\,\|u-\widetilde{u}\|_\infty\,.\end{aligned}$$

Letting $r>0$ small enough and $R>0$ sufficiently large, we see that $\Psi: P_r\cap B_R\to P_r\cap B_R$ is a contraction. ∎

3 Blow-up condition

Our goal in this section is to prove:

Theorem 2. *If $\sigma\geq\frac{d\beta\rho}{\alpha}$ then all non trivial positive solutions of* (2.2) *blow-up in finite time.*

Let us start by introducing some preliminary results.

Lemma 1. *Let u be a nonnegative solution to* (2.2)*, then there exist some positive constants t_0, c_0, and γ such that*

$$u(t_0,x)\geq c_0p(\gamma,x),\ \ \forall x\in\mathbb{R}^d.$$

Proof. We can choose $t_0>0$ such that $p(\kappa(0,t_0),0)\leq 1$. Therefore by Proposition 1

$$\begin{aligned}p(\kappa(0,t_0),y-x) &= p\left(\kappa(0,t_0),\frac{1}{2}(2y-2x)\right)\\ &\geq p\left(\kappa(0,t_0),2x\right)p\left(\kappa(0,t_0),2y\right)\\ &= p\left(\frac{\kappa(0,t_0)}{2^\alpha}\times 2^\alpha,2x\right)p\left(\kappa(0,t_0),2y\right)\\ &= 2^{-d}p\left(\frac{\kappa(0,t_0)}{2^\alpha},x\right)p\left(\kappa(0,t_0),2y\right).\end{aligned}$$

And (2.2) rendering

$$\begin{aligned}u(t_0,x) &\geq \int_{\mathbb{R}^d}p(\kappa(0,t_0),y-x)\varphi(y)dy\\ &\geq \int_{\mathbb{R}^d}2^{-d}p\left(\kappa(0,t_0),2y\right)\varphi(y)dyp\left(\frac{\kappa(0,t_0)}{2^\alpha},x\right).\end{aligned}$$

Obtaining the required inequality. ∎

Let $v(t+t_0,x)$, $t\geq 0$, $x\in\mathbb{R}^d$, be a solution of

$$\begin{aligned}&v(t+t_0,x)=c_0p(\kappa(t_0,t+t_0)+\gamma,x)\\ &+\int_0^t\int_{\mathbb{R}^d}p(\kappa(s+t_0,t+t_0),y-x)h(s+t_0)v^{1+\beta}(s+t_0,y)dyds,\end{aligned}\tag{3.1}$$

where c_0, γ and t_0 are given in Lemma 1. Define

$$\bar{v}(t+t_0) = \int_{\mathbb{R}^d} p(\kappa(0,t+t_0),x)v(t+t_0,x)dx, \quad t \geq 0.$$

We say that $\bar{v}$ blows up in finite time if there exists some $t_1 > 0$ such that $\bar{v}(t+t_0) = \infty$ for all $t \geq t_1$.

Lemma 2. *Let $v(\cdot+t_0,x)$ be a nonnegative solution of* (3.1)*. If $\bar{v}(\cdot+t_0)$ blows up in finite time, then $v(\cdot+t_0,x)$ does.*

Proof. Let $c_1 := (6a_2/a_1)^{1/\rho}$. We can choose $t_1 > 0$ such that $p(\kappa(0,t_1),0) \leq 1$. If $t_1 \leq t$ and $t \leq s+t_0 \leq l(t)$, where

$$l(t) := t\left(\frac{6}{2^\alpha+1}\right)^{1/\rho} + \left(\frac{a_1}{a_2(2^\alpha+1)}\right)^{1/\rho} t_0,$$

then

$$\tau := \left(\frac{\kappa(s+t_0,c_1t+t_0)}{\kappa(0,s+t_0)}\right)^{1/\alpha} \geq 2$$

and

$$p(\kappa(0,s+t_0),0) \leq p(\kappa(0,t),0) \leq p(\kappa(0,t_1),0) \leq 1.$$

Therefore, by Proposition 1,

$$\begin{aligned} p\left(\kappa\left(s+t_0,c_1t+t_0\right),x-y\right) &= p\left(\kappa(0,s+t_0)\times\frac{\kappa\left(s+t_0,c_1t+t_0\right)}{\kappa(0,s+t_0)},x-y\right) \\ &= \tau^{-d}p\left(\kappa(0,s+t_0),\frac{1}{\tau}(x-y)\right) \\ &\geq \tau^{-d}p\left(\kappa(0,s+t_0),x\right)p\left(\kappa(0,s+t_0),y\right). \end{aligned}$$

From (3.1) and Jensen's inequality we have

$$\begin{aligned} & v(c_1t+t_0,x) \\ & \geq \int_t^{l(t)} h(s+t_0)\left(\int_{\mathbb{R}^d} p\left(\kappa\left(s+t_0,c_1t+t_0\right),x-y\right)v(s+t_0,y)dy\right)^{1+\beta} ds \\ & \geq \int_t^{l(t)} h(s+t_0)\left(\tau^{-d}p\left(\kappa(0,s+t_0),x\right)\bar{v}(s+t_0)\right)^{1+\beta} ds = \infty. \end{aligned}$$

So $v\left(t+t_0,x\right) = \infty$ for any $t \geq c_1t_1$ and $x \in \mathbb{R}^d$. ■

Proposition 2. *If $\sigma \geq \frac{d\beta\rho}{\alpha}$ then all non-trivial positive solutions of* (3.1) *blow-up in finite time.*

Proof. Let $v(\cdot+t_0,x)$ be a nonnegative solution of (3.1). Multiplying both sides of (3.1) by $p\left(\kappa(0,t+t_0),x\right)$, and integrating with respect to x, we have

$$\begin{aligned} & \bar{v}(t+t_0) = c_0p(\kappa(0,t+t_0)+\kappa(t_0,t+t_0)+\gamma,0) \\ & + \int_0^t h(s+t_0)\int_{\mathbb{R}^d} p\left(\kappa(0,t+t_0)+\kappa(s+t_0,t+t_0),y\right)v^{1+\beta}(s+t_0,y)dyds. \end{aligned}$$

By Proposition 1, (1.2) and Jensen's inequality we obtain

$$\begin{aligned}
\bar{v}(t+t_0) &\geq c_0 p(1,0)(\kappa(0,t+t_0)+\kappa(t_0,t+t_0)+\gamma)^{-d/\alpha} \\
&\quad + \int_0^t h(s+t_0)\left(\frac{\kappa(0,s+t_0)}{\kappa(0,t+t_0)+\kappa(s+t_0,t+t_0)}\right)^{d/\alpha} \\
&\quad \times \int_{\mathbb{R}^d} p\left(\kappa(0,s+t_0),y\right) v^{1+\beta}(s+t_0,y)dyds \\
&\geq c_0 p(1,0)(2\kappa(0,t+t_0)+\gamma)^{-d/\alpha} \\
&\quad + \int_0^t h(s+t_0)\left(\frac{\kappa(0,s+t_0)}{\kappa(0,t+t_0)+\kappa(s+t_0,t+t_0)}\right)^{d/\alpha} \\
&\quad \times \left(\int_{\mathbb{R}^d} p\left(\kappa(0,s+t_0),y\right) v(s+t_0,y)dy\right)^{1+\beta} ds \\
&\geq c_0 p(1,0)(2\kappa(0,t+t_0)+\gamma)^{-d/\alpha} \\
&\quad + \int_0^t h(s+t_0)\left(\frac{\kappa(0,s+t_0)}{2\kappa(0,t+t_0)}\right)^{d/\alpha} \left(\bar{v}(s+t_0)\right)^{1+\beta} ds \\
&\geq c_0 p(1,0)(2a_2(t+t_0)^\rho+\gamma)^{-d/\alpha} \qquad (3.2) \\
&\quad + \int_0^t b_1(s+t_0)^{\sigma-1}\left(\frac{a_1(s+t_0)^\rho}{2a_2(t+t_0)^\rho}\right)^{d/\alpha} \left(\bar{v}(s+t_0)\right)^{1+\beta} ds.
\end{aligned}$$

Put $\bar{v}_1(t+t_0)=(t+t_0)^{d\rho/\alpha}\bar{v}(t+t_0)$, then by (3.2) we get

$$\bar{v}_1(t+t_0) \geq C_1 + C_2\int_0^t (s+t_0)^{\sigma-1-d\beta\rho/\alpha}\left(\bar{v}_1(s+t_0)\right)^{1+\beta} ds,$$

where $C_1 := c_0 p(1,0)\left(2a_2+\gamma t_0^{-\rho}\right)^{-d/\alpha}$ and $C_2 := b_1\left(a_1/(2a_2)\right)^{d/\alpha}$. Let $\bar{v}_2(\cdot+t_0)$ be the solution of

$$\bar{v}_2(t+t_0) = C_1 + C_2\int_0^t (s+t_0)^{\sigma-1-d\beta\rho/\alpha}\left(\bar{v}_2(s+t_0)\right)^{1+\beta} ds, \quad t\geq 0.$$

Notice that $\bar{v}_2(\cdot+t_0)$ is given by $\bar{v}_2(t_0)=C_1$ and

$$(\bar{v}_2)^\beta(t+t_0) = \frac{(C_1)^\beta}{1-\beta C_2(C_1)^\beta H(t+t_0)}, \quad t>0,$$

where

$$H(t+t_0) = \begin{cases} (\sigma-d\rho\beta/\alpha)^{-1}\left((t+t_0)^{\sigma-d\beta\rho/\alpha}-t_0^{\sigma-d\rho\beta/\alpha}\right), & \sigma-d\beta\rho/\alpha>0, \\ \log(t+t_0)-\log(t_0), & \sigma-d\beta\rho/\alpha=0. \end{cases}$$

Since $\lim_{t\to\infty} H(t+t_0)=\infty$ there exist t_2 such that $\bar{v}_2(t+t_0)=\infty$ for $t\geq t_2$. Hence, by the comparison theorem, we have the result. ■

Proof of Theorem 2. Let $u(t,x)$ be a nonnegative solution of (2.2) and let t_0 given in Lemma 1. Then $u(t+t_0,x)$ satisfies

$$\begin{aligned}
u(t+t_0,x) &= \int_{\mathbb{R}^d} p(\kappa(t_0,t+t_0),y-x)u(t_0,y)dy \\
&\quad + \int_0^t\int_{\mathbb{R}^d} p(\kappa(s+t_0,t+t_0),y-x)h(s+t_0)u^{1+\beta}(s+t_0,y)dyds.
\end{aligned}$$

By Lemma 1 we have

$$\begin{aligned} u(t+t_0,x) &\geq c_0 p(\kappa(t_0,t+t_0)+\gamma,x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p(\kappa(s+t_0,t+t_0),y-x)h(s+t_0)u^{1+\beta}(s+t_0,y)dyds. \end{aligned}$$

By the comparison theorem it is enough that the solution $v(t+t_0,x)$ of the integral equation (3.1) blows up in finite time. By Proposition 2 we are done. ■

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