# On a Beck-Putnam-Rehder Theorem

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**Abstract**

We give a simple and easy proof of a result obtained by W. Rehder who generalized, in his turn, a famous result by Beck and Putnam. We then give a generalization to unbounded operators.

## 1 Introduction

W. A. Beck and C. R. Putnam proved in [1] the following result:

**Theorem 1.** *Let $A$ be a bounded operator on a Hilbert space. Let $N$ be a normal operator such that $AN = N^*A$. If, whenever $z$ is not real, either $z$ or its conjugate $\overline{z}$ does not lie in the spectrum of $N$, then $AN = NA$.*

W. Rehder [5] gave a generalization of this theorem and proved:

**Theorem 2.** *Assume that $A$ is a bounded operator on a Hilbert space. Let $N$ and $M$ be two normal operators such that $AN = MA$. Assume further that*

1. *$\sigma(M^*) \subset \sigma(N)$, and*
2. *whenever $z$ is not real, either $z$ or its conjugate $\overline{z}$ does not lie in the spectrum of $N$.*

*Then $AN = M^*A$.*

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Rehder's proof was interesting, however, not a straightforward one. In this short paper, we reduce Theorem 2 to Theorem 1 by means of an operator's matrix trick with also a slight weakening of Rehder Theorem's hypotheses. The last result in this paper is a generalization of Theorem 3 (see below) to unbounded operators.

Finally we assume the reader is familiar with notions and results on linear operators on a Hilbert space (see e.g. [2, 3, 4]).

## 2 Main Results

The first main result is the following theorem:

**Theorem 3.** *Assume that $A$ is a bounded operator on a Hilbert space. Let $N$ and $M$ be two normal operators such that $AN = MA$. Assume further that, whenever $z$ is not real, either $z$ or its conjugate $\overline{z}$ does not lie in $\sigma(M^*) \cup \sigma(N)$. Then $AN = M^*A$.*

*Proof.* Let $M$ and $N$ be two bounded normal operators and let $A$ be a bounded operator all defined on a Hilbert space $\mathcal{H}$. Define on $\mathcal{H} \oplus \mathcal{H}$ the following operators

$$\widetilde{N} = \begin{pmatrix} M^* & 0 \\ 0 & N \end{pmatrix} \text{ and } \widetilde{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

Since $M$ and $N$ are normal, so is $\widetilde{N}$ (as one can easily check). Besides one has

$$\widetilde{A}\widetilde{N} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M^* & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} 0 & AN \\ 0 & 0 \end{pmatrix}$$

and

$$\widetilde{N}^*\widetilde{A} = \begin{pmatrix} M & 0 \\ 0 & N^* \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & MA \\ 0 & 0 \end{pmatrix}.$$

Since $AN = MA$, $\widetilde{A}\widetilde{N} = \widetilde{N}^*\widetilde{A}$. We also know that $\sigma(\widetilde{N}) = \sigma(M^*) \cup \sigma(N)$. But by hypothesis whenever $z$ is not real either $z \notin \sigma(M^*) \cup \sigma(N)$ or $\overline{z} \notin \sigma(M^*) \cup \sigma(N)$ and hence whenever $z$ is not real either $z \notin \sigma(\widetilde{N})$ or $\overline{z} \notin \sigma(\widetilde{N})$. Thus Theorem 1 applies and gives

$$\widetilde{A}\widetilde{N} = \widetilde{N}\widetilde{A} \text{ or } AN = M^*A.$$ ∎

Theorem 2 has become now a corollary to the previous theorem. We have

**Corollary 1.** *Assume that $A$ is a bounded operator on a Hilbert space. Let $N$ and $M$ be two normal operators such that $AN = MA$. Assume further that*

1. *$\sigma(M^*) \subset \sigma(N)$, and*
2. *whenever $z$ is not real, either $z$ or its conjugate $\overline{z}$ does not lie in the spectrum of $N$.*

*Then $AN = M^*A$.*

*Proof.* The proof follows from the observation

$$\sigma(M^*) \subset \sigma(N) \Longrightarrow \sigma(\widetilde{N}) = \sigma(M^*) \cup \sigma(N) = \sigma(N).$$ ∎

Theorem 3 has an analog for unbounded operators which is the following theorem:

**Theorem 4.** *Assume that $A$ is a bounded operator on a Hilbert space. Let $N$ and $M$ be two unbounded normal operators (with domains $D(N)$ and $D(M)$ respectively) such that $AN \subset MA$. Assume further that, whenever $z$ is not real, either $z$ or its conjugate $\overline{z}$ does not lie in $\sigma(M^*) \cup \sigma(N)$. Then $AN \subset M^*A$.*

*Proof.* Let $m$ and $n$ be both in $\mathbb{N}$. Consider the two closed balls

$$B_n = \{z \in \mathbb{C} : |z| \leq n\} \text{ and } B_m = \{z \in \mathbb{C} : |z| \leq m\}.$$

Let $P_{B_n}(N)$ and $P_{B_m}(M)$ be the spectral projections associated with $N$ and $M$ respectively. Then $N_n := NP_{B_n}(N)$ and $M_m := MP_{B_m}(M)$ are bounded normal operators.

As $AN \subset MA$, then $AN$ and $MA$ coincide on $D(AN) = D(N)$. Since $\text{ran}P_{B_n}(N) \subset D(N)$, one can say that $ANP_{B_n}(N) = MAP_{B_n}(N)$.

Now known properties about the spectral measures and theorem yield

$$[P_{B_m}(M)AP_{B_n}(N)]N_n = M_m[P_{B_m}(M)AP_{B_n}(N)].$$

Let $z$ be a non-real number. Since $z \notin \sigma(M^*) \cup \sigma(N)$ or $\overline{z} \notin \sigma(M^*) \cup \sigma(N)$, $z \notin \sigma(M_m^*) \cup \sigma(N_n)$ or $\overline{z} \notin \sigma(M_m^*) \cup \sigma(N_n)$ for all $n$ and all $m$. Hence Theorem 3 applies and yields

$$[P_{B_m}(M)AP_{B_n}(N)]NP_{B_n}(N) = P_{B_m}(M)M^*[P_{B_m}(M)AP_{B_n}(N)].$$

Thus

$$P_{B_m}(M)ANP_{B_n}(N)f = P_{B_m}(M)M^*AP_{B_n}(N)f, \ \forall f \in D(N) = D(N^*)$$

(and hence $Af \in D(M) = D(M^*)$). Sending both $n$ and $m$ to $\infty$ we get $P_{B_n}(N) \to I$ and $P_{B_m}(M) \to I$ respectively (and both in strong operator topology).

Therefore

$$ANf = M^*Af, \ \forall f \in D(N) \ (\subset D(M^*A) = D(MA)).$$

Thus $AN \subset M^*A$. ∎

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## References

[1] W. A. Beck, C. R. Putnam, A Note on Normal Operators and Their Adjoints, *J. London Math. Soc.*, **31** (1956), 213-216.

[2] J. B. Conway, *A Course in Functional Analysis*, Springer, 1990 (2nd edition).

[3] I. Gohberg, S. Goldberg, M. A. Kaashoek, *Basic Classes of Linear Operators*, Birkhäuser Verlag, Basel, 2003.

[4] W. Rudin, *Functional Analysis*, McGraw-Hill, 1991 (2nd edition).

[5] W. Rehder, On the Adjoints of Normal Operators, *Arch. Math. (Basel)*, **37/2** (1981) 169-172.

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