An extension of a theorem of E. A. Barbashin to the dichotomy of abstract evolution operators

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Abstract

A necessary and sufficient condition for the uniform exponential dichotomy is pointed out using a discrete-time argument. Thus are extended known results due to Barbashin [1], Datko [7], Lovelady [10], Pazy[13], Preda[14, 15, 16].

1 Introduction and Preliminaries

Let X be a real or complex Banach space and $B(X)$ the Banach algebra of all linear and bounded operators acting on X. We denote by $||\cdot||$ the norms of vectors and operators on X.

Consider now the Cauchy problem

$$\frac{du(t,x)}{dt} = A(t)u(t,x),\ u(0,x) = x \in X,\ t \geq 0$$

with $A(\cdot)$ being locally integrable on R_+.

Intuitively speaking, dichotomy means the existence of a projection-valued function, $P(\cdot)$, such that the solutions which start in $ImP(0)$ decay (in norm) to zero, and the solutions which start in $Im(I-P(0))$ are unbounded.

Regarding the importance of this concept, we can say that dichotomy has an essential contribution in the analysis of the qualitative properties of nonlinear evolution equations such as linearized (in-)stability or the existence of the invariant and center manifolds (see for instance [5], [17])

Received by the editors December 2008.
Communicated by J. Mawhin.
2000 *Mathematics Subject Classification* : 34D05,47D06.
Key words and phrases : evolution families, uniform exponential dichotomy.

Also, it is known that the best way to deal with the asymptotic behavior of the solutions (of above system $\dot{u}(t) = A(t)u(t)$) is to involve the classical notion of an evolution family (of linear and bounded operators). We now recall the definition of an evolution family.

Definition 1. 1. *An operator-valued two variables function* $\Phi : \{(t,s) \in \mathbb{R} \times \mathbb{R} : t \geq s \geq 0\} \mapsto B(X)$ *is called an evolution family if the following properties hold:*

- $e_1)$ $\Phi(t,t) = I$, *for all* $t \geq 0$;
- $e_2)$ $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$, *for all* $t \geq s \geq r \geq 0$;
- $e_3)$ $\Phi(\cdot,s)x$ *is continuous on* $[s,\infty)$, *for all* $s \geq 0$, $x \in X$;
  $\Phi(t,\cdot)x$ *is continuous on* $[0,t)$, *for all* $t \geq 0$, $x \in X$;
- $e_4)$ *there are* $M, \omega > 0$ *such that*

$$\|\Phi(t,s)\| \leq Me^{\omega(t-s)}, \quad \textit{for all} \quad t \geq s \geq 0.$$

Example 1.1. Consider the operator Cauchy problem $(A, 0, I)$ given by

$$\begin{cases} \dot{U}(t) = A(t)U(t) \\ U(0) = I, \text{ as usually } I \text{ denotes the identity on } X \end{cases}$$

If $\sup\limits_{t\geq 0} \int\limits_t^{t+1} A(\tau)d\tau < \infty$ then $\Phi(t,t_0) = U(t)U^{-1}(t_0)$ is an evolution family which has the additional property that (e_2) holds for any $t, s, r \in \mathbb{R}_+$. For details, we refer the reader to [4], [5], [12].

Throughout in this paper we suppose that for every $t_0 \geq 0$ the vector subspace

$$X_1(t_0) = \{x_0 \in X : \ \Phi(\cdot,t_0) \in L^\infty_{[t_0,\infty)}(X)\}$$

is closed in X, where $L^\infty_{[t_0,\infty)}(X)$ is the Banach space of X-valued function f defined a.e. on $[t_0,\infty)$, such that f is strongly measurable and essentially bounded. Also we assume that $X_1(t_0)$ admits a complement $X_2(t_0)$ and we will denote by $P(t_0)$ a projection (that is $P(t_0) \in B(X)$, $P^2(t_0) = P(t_0)$) such that $Ker\ P(t_0) = X_2(t_0)$ and also we denote by $Q(t_0) = I - P(t_0)$.

Remark 1.1. For any evolution family Φ we have that
(i) $\Phi(t,t_0)\, X_1(t_0) \subset X_1(t)$ (or equivalent $\Phi(t,t_0)P(t_0) = P(t)\Phi(t,t_0)P(t_0)$), for all $t \geq t_0 \geq 0$;
(ii) $\Phi(t,s)P(s)\Phi(s,t_0)P(t_0) = \Phi(t,t_0)P(t_0)$, for all $t \geq s \geq t_0 \geq 0$;
(iii) $\Phi(t,t_0)Q(t_0)x \neq 0$, for all $t \geq t_0 \geq 0$ and $x \in X$ with $Q(t_0)x \neq 0$;

Remark 1.2. If Φ is the evolution family from Example 1.1. then $X_1(t_0) = U(t_0)X_1(0), X_2(t_0) = U(t_0)X_2(0)$ and $P(t_0) = U(t_0)P(0)U^{-1}(t_0)$, for all $t_0 \geq 0$. Thus, in the case of evolution families generated by differential system the splitting at any moment $t_0 \geq 0$ can be obtained by the splitting at the moment zero.

We will assume in what follows that the projection-valued function $P(\cdot)$ is strongly continuous and bounded on R_+. Also, we will say that $P(\cdot)$ is a *dichotomy projection family* if in addition it satisfies

- $\Phi(t, t_0)P(t_0) = P(t)\Phi(t, t_0)$, for all $t \geq t_0 \geq 0$
- $\Phi(t, t_0) : KerP(t_0) \to KerP(t)$ is an isomorphism for all $t \geq t_0 \geq 0$;

Definition 1.2. *An evolution family Φ is said to be uniformly exponentially dichotomic (u.e.d) if there exist P a projection family and $N_1, N_2, \nu > 0$ such that*

- $d_1)$ $||\Phi(t, t_0)P(t_0)x|| \leq N_1 e^{-\nu(t-t_0)}||P(t_0)x||$, *for all $x \in X$ and all $t \geq t_0 \geq 0$.*
- $d_2)$ $||\Phi(t, t_0)Q(t_0)x|| \geq N_2 e^{\nu(t-t_0)}||Q(t_0)x||$, *for all $x \in X$ and all $t \geq t_0 \geq 0$.*

Looking for an extension of the classical Lyapunov theorem (see for instance [2]) to abstract Hilbert spaces, R. Datko establishes in [6] an auxiliary result which has come into widespread usage in the study of the asymptotic behavior of one-parameter semigroups of linear operators. His result is classical now and it says that the semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially stable if and only if, for each vector x from a general Hilbert space X, the function $t \to \|T(t)x\|$ lies in $L^2(\mathbb{R}_+, \mathbb{R}_+)$ (where $\mathbb{R}_+ = [0, \infty)$). Later, A.Pazy (see for instance [13]) shows that the result remains valid even if $L^2(\mathbb{R}_+, \mathbb{R}_+)$ is replaced by any $L^p(\mathbb{R}_+, \mathbb{R}_+)$, where $p \in [1, \infty)$ and X is a general Banach space. In 1973, R.Datko [7] generalize the results above, and he states that an evolution family $\{\Phi(t, s)\}_{t\geq s\geq 0}$, on a Banach space X, is uniformly exponentially stable if and only if there is $p \in [1, \infty)$ such that $\sup\limits_{s\geq 0} \int\limits_s^{\infty} \|\Phi(t, s)x\|^p dt < \infty$, for each $x \in X$. Also, a nonlinear version of Datko-Pazy's theorem is obtained in [8] by Ichikawa in 1984. It is worth to mention here that a version of Datko's result could be already found in the monograph of Krein and Daleckij (see Theorem 6.2., page 133 from [5]) for evolution families generated by differential systems (see Example 1.1.). Also, a discrete-time version, for the case of C_0-semigroups was provided by Zabczyk [19] in 1974.

A first extension of Datko-Pazy theorem to the general case of exponential dichotomy is due to Popescu and Preda in [14], where is analyzed the case of differential systems. Later Preda and Megan generalize the Datko-Pazy theorem for dichotomy, firstly for C_0-semigroups [15] and later for evolution families [16].

It is worth to mention here that in Datko's integral characterization, the integrand is the first parameter of the evolution family. Integral characterizations with the second parameter as integrand are obtained firstly by Barbashin in 60's. Thus in [1], E.A. Barbashin proved that the differential system $(A, 0, I)$ is uniformly exponentially stable if and only if there is $K > 0$ such that $\int_0^t ||U(t)U^{-1}(\tau)||d\tau \leq K$, for all $t \geq 0$. This result is extended to the more general case of the exponential dichotomy of differential systems in [4], [10], [16]. More precisely it is established that the differential system $(A, 0, I)$ is uniformly exponentially dichotomic if and only if there is $K > 0$ with $(\int_0^t ||U(t)P(0)U^{-1}(\tau)||^p \, d\tau)^{\frac{1}{p}} + (\int_t^{\infty} ||U(t)Q(0)U^{-1}(\tau)||^p \, d\tau)^{\frac{1}{p}} \leq K$, for all $t \geq 0$.

Analyzing the technique of proof of the above results we can distinguish that Datko's line of results is connected with the Lyapunov method for the study of the asymptotic behaviour of differential systems and Barbashin's line of results is related to Perron's method (test functions). The present approach is a double extension of Barbashin's result, first from differential systems to abstract evolution families and second (which is the fact the main extension) from stability to the general case of uniform exponential dichotomy. Moreover, the technique of proof allows us to use a discrete-time argument which is much more convenient for verifying the hypothesis and for other computational reasons. Also, it is obtained as an auxiliary result, a version of Datko's theorem for the exponential instability of abstract evolution families.

2 The main result

Lemma 2.1.*Let $f,\ g : \mathbb{R}_+ \to \mathbb{R}_+$ with g continuous on R_+. If*

i) $f(t) \le g(t-t_0)f(t_0)$, for each $t \ge t_0 \ge 0$;

ii) there exists $\delta > 0$ with $g(\delta) < 1$.

Then there exist $N,\ \nu > 0$, independently of f, such that $f(t) \le Ne^{-\nu(t-t_0)}f(t_0)$, for each $t \ge t_0 \ge 0$.

Proof. See for instance [11].

Lemma 2.2. Let $f,\ g : \mathbb{R}_+ \to \mathbb{R}_+$, g continuous with $g(t) > 0$, for each $t \ge 0$. If

i) $f(t) \ge g(t-t_0)f(t_0)$, for each $t \ge t_0 \ge 0$;

ii) there exists $\delta > 0$ with $g(\delta) > 1$.

Then there exist $N,\ \nu > 0$ such that $f(t) \ge Ne^{\nu(t-t_0)}f(t_0)$, for each $t \ge t_0 \ge 0$.

Proof: Let $t \ge t_0 \ge 0$. We denote by $n = [\frac{t-t_0}{\delta}]$, where $[s]$ denotes the greatest integer less than or equal with s. It follows that $t_0 + n\delta \le t < t_0 + (n+1)\delta$ and

$$f(t) \ge g(t - n\delta - t_0)f(t_0 + \delta) \ge \inf_{s\in[0,\delta]} g(s)f(t_0 + n\delta) \ge$$

$$\ge \beta g(\delta)f(t_0 + (n+1)\delta) \ge \beta g^n(\delta)f(t_0),$$

where $\beta = \inf_{s\in[0,\delta]} g(s)$. Denoting $g(\delta) = e^{\nu\delta}$ we get that $\nu\delta = \ln g(\delta)$ and $\nu = \frac{1}{\delta}\ln g(\delta) > 0$. Thus we have that

$$f(t) \ge \beta e^{\nu n\delta} f(t_0) \ge \beta e^{\nu(n+1)\delta}\, e^{-\nu\delta} f(t_0) \ge$$

$$\ge \beta e^{-\nu\delta}\, e^{\nu(t-t_0)} f(t_0) \text{ for each } t \ge t_0 \ge 0$$

and

$$f(t) \ge Ne^{\nu(t-t_0)}f(t_0), \text{ for each } t \ge t_0 \ge 0,$$

where $N = \beta e^{-\nu\delta} = \inf_{s\in[0,\delta]} g(s)\frac{1}{g(\delta)}$, $\nu = \frac{1}{\delta}\ln g(\delta)$.

Proposition 2.1. Let Φ be an evolution family with the property that $\Phi(t,t_0)P(t_0) = P(t)\Phi(t,t_0)$ for every $t \ge t_0 \ge 0$. If there exist $p > 0$ and $L > 0$ such that

$$\Big(\int_{t_0}^{\infty} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \le \frac{L}{||Q(t_0)x||}, \text{ for all } t_0 \ge 0, x \in X, \text{ with } Q(t_0)x \ne 0$$

then there are $N, \nu > 0$ such that $\|\Phi(t,t_0)Q(t_0)x\| \geq Ne^{\nu(t-s)}\|\Phi(s,t_0)Q(t_0)x\|$, for all $t \geq s \geq t_0 \geq 0,\ x \in X$.

Proof. Let $t \geq t_0 \geq 0, \tau \in [t, t+1], x \in X$ with $Q(t_0)x \neq 0$. Then

$$||\Phi(\tau,t_0)Q(t_0)x|| \leq Me^{\omega}||\Phi(t,t_0)Q(t_0)x||$$

Thus

$$\frac{1}{||\Phi(t,t_0)x||} \leq Me^{\omega}\Big(\int_t^{t+1} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}}$$

$$\leq Me^{\omega}\Big(\int_{t_0}^{\infty} \frac{d\tau}{||\Phi(\tau,t_0)x||^p}\Big)^{\frac{1}{p}} \leq \frac{MLe^{\omega}}{||Q(t_0)x||}.$$

Hence

$$||\Phi(t,t_0)Q(t_0)x|| \geq \frac{||Q(t_0)x||}{MLe^{\omega}}, \text{ for all} t \geq t_0 \geq 0, \text{ and } x \in X.$$

Let $t \geq t_0 \geq 0, x \in X$ with $Q(t_0)x \neq 0$ and $\tau \in [t_0, t]$. Then

$$||\Phi(t,t_0)Q(t_0)x|| = ||\Phi(t,\tau)\Phi(\tau,t_0)Q(t_0)x|| = ||\Phi(t,\tau)Q(\tau)\Phi(\tau,t_0)Q(t_0)x||$$

$$\geq \frac{||\Phi(\tau,t_0)Q(t_0)x||}{MLe^{\omega}} \text{ and by integrating it follows that}$$

$$\frac{(t-t_0)^{\frac{1}{p}}}{||\Phi(t,t_0)Q(t_0)x||} \leq MLe^{\omega}\Big(\int_{t_0}^{t} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq$$

$$\leq MLe^{\omega}\Big(\int_{t_0}^{\infty} \frac{d\tau}{||\Phi(\tau,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq \frac{ML^2e^{\omega}}{||Q(t_0)x||},$$

for all $t \geq t_0 \geq 0, x \in X$ with $Q(t_0)x \neq 0$. So, we get

$$||\Phi(t,t_0)Q(t_0)x|| \geq \frac{(t-t_0)^{\frac{1}{p}}}{ML^2e^{\omega}}||Q(t_0)x||, \text{ for all } t \geq t_0 \geq 0, \text{ and } x \in X$$

which implies that

$$||\Phi(t,t_0)Q(t_0)x|| \geq \frac{(t-\tau)^{\frac{1}{p}}}{ML^2e^{\omega}}||\Phi(\tau,t_0)Q(t_0)x||, \text{ for all } t \geq \tau \geq t_0.$$

Applying Lemma 2.2. we can find $N, \nu > 0$ such that

$$||\Phi(t,t_0)x|| \geq Ne^{\nu(t-\tau)}||\Phi(\tau,t_0)Q(t_0)x||, \text{ for all } t \geq \tau \geq t_0, \text{ and } x \in X.$$

Remark 2.1. Note that if in the above result we take $\Phi(t,t_0)$ to be one-to-one for all $t \geq t_0 \geq 0$ and $Q(t_0) = I$ for all $t_0 \geq 0$, then we can get a version of Datko's theorem (see [7]) for the instability of evolution families.

Proposition 2.2. Let Φ be an evolution family. If there exist $p > 0$ and $L > 0$ such that

$$\Big(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\Big)^{\frac{1}{p}} \le L, \text{ for all } t \ge 0$$

then there are $N, \nu > 0$ such that $\|\Phi(t,t_0)P(t_0)\| \le Ne^{-\nu(t-t_0)}$, for all $t \ge t_0 \ge 0$.

Proof. Let $t \ge t_0 + 1$ and $r(t) = M \sup_{\tau \ge 0} ||P(\tau)||e^{\omega t}$. Then

$$||\Phi(t,t_0)P(t_0)||^p \int_{t_0}^t r^{-p}(\tau - t_0)d\tau = \int_{t_0}^t ||\Phi(t,\tau)P(\tau)\Phi(\tau,t_0)P(t_0)||^p r^{-p}(\tau - t_0)d\tau$$

$$\le \int_{t_0}^t ||\Phi(t,\tau)P(\tau)||^p d\tau \le \int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau \le L^p.$$

But

$$\int_{t_0}^t r^{-p}(\tau - t_0)d\tau = \int_0^{t-t_0} r^{-p}(s)ds \ge \int_0^1 r^{-p}(s)ds = \alpha > 0$$

and hence

$$\alpha^{\frac{1}{p}}||\Phi(t,t_0)P(t_0)|| \le L, \text{ for all } t \ge t_0 + 1,$$

which implies that

$$||\Phi(t,t_0)P(t_0)|| \le \frac{L}{\alpha^{\frac{1}{p}}}, \text{ for all } t \ge t_0 + 1.$$

Taking now $t_0 \le t < t_0 + 1$ we have that

$$||\Phi(t,t_0)P(t_0)|| \le Me^{\omega} \sup_{t \ge 0} ||P(t)||.$$

Denoting

$$L' = \max\{\frac{L}{\alpha^{\frac{1}{p}}}, Me^{\omega} \sup_{t \ge 0} ||P(t)||\},$$

we obtain that

$$(\diamond\diamond) \qquad ||\Phi(t,t_0)P(t_0)|| \le L', \text{ for all } t \ge t_0 \ge 0.$$

Taking by this time $t \ge t_0 \ge 0$ and $\tau \in [t_0, t]$ we get that

$$||\Phi(t,t_0)P(t_0)|| \le L'||\Phi(t,\tau)P(\tau)||,$$

which implies that

$$(\diamond\diamond\diamond) \qquad (t-t_0)^{\frac{1}{p}}||\Phi(t,t_0)P(t_0)|| \le L'L, \text{ for all } t \ge t_0 \ge 0.$$

Adding up $(\diamond\diamond)$ with $(\diamond\diamond\diamond)$ we deduce that

$$||\Phi(t,t_0)P(t_0)|| \le \frac{L'(1+L)}{1+(t-t_0)^{\frac{1}{p}}}, \text{ for all } t \ge t_0 \ge 0,$$

and hence

$$||\Phi(t,t_0)P(t_0)|| \leq ||\Phi(t,\tau)P(\tau)||\ ||\Phi(\tau,t_0)P(t_0)|| \leq$$
$$\leq \frac{L'(1+L)}{1+(t-\tau)^{\frac{1}{p}}}||\Phi(\tau,t_0)P(t_0)||,\ \text{ for all } t \geq \tau \geq t_0 \geq 0.$$

Applying Lemma 2.1. we have that there exist $N,\nu > 0$ such that

$$||\Phi(t,t_0)P(t_0)|| \leq Ne^{-\nu(t-t_0)},\ \text{ for all } t \geq t_0 \geq 0.$$

Remark 2.2. The converse statement from Proposition 2.2. is also valid.

Remark 2.3. Note that if in the above result we take $P(\tau) = I$ for all $\tau \geq 0$, then we can get an extension of Barbashin's theorem (see [1]) for the uniform exponential stability of abstract evolution families.

Theorem 2.1. Let Φ be an evolution family with $\Phi(t,t_0)P(t_0) = P(t)\Phi(t,t_0)$, for every $t \geq t_0 \geq 0$. If there exist $p > 0$ and $L > 0$ such that

$$\Big(\sum_{k=[t_0]+1}^{\infty} \frac{1}{||\Phi(k,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq \frac{L}{||Q(t_0)x||},$$
$$\text{for all } t_0 \geq 0, x \in X,\ \text{ with } Q(t_0)x \neq 0$$

then there are $N,\nu > 0$ such that $\|\Phi(t,t_0)Q(t_0)x\| \geq Ne^{\nu(t-t_0)}\|Q(t_0)x\|$, for all $t \geq t_0 \geq 0,\ x \in X.$

Proof. Let $t_0 \geq 0$ and $x \in X$ with $Q(t_0)x \neq 0$. Take now $t \geq t_0 \geq 0$ and denote by $k = [t]$. Then we can find $M \geq 1$ and $\omega \geq 0$ such that

$$||\Phi(k+1,t_0)Q(t_0)x||^p \leq M^p e^{\omega p}||\Phi(t,t_0)Q(t_0)x||^p,$$

which implies that

$$\int_k^{k+1} \frac{1}{||\Phi(t,t_0)Q(t_0)x||^p}dt \leq \frac{M^p e^{\omega p}}{||\Phi(k+1,t_0)Q(t_0)x||^p},$$

and hence

$$\sum_{k=[t_0]+1}^{\infty} \int_k^{k+1} \frac{1}{||\Phi(t,t_0)Q(t_0)x||^p}dt \leq M^p e^{\omega p} \sum_{i=[t_0]+2}^{\infty} \frac{1}{||\Phi(i,t_0)Q(t_0)x||^p}.$$

Thus we can get that

$$\Big(\int_{[t_0]+1}^{\infty} \frac{dt}{||\Phi(t,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq \frac{Me^{\omega p}L}{||Q(t_0)x||},$$
$$\text{for all } t_0 \geq 0 \text{ and } x \in X \text{ with } Q(t_0)x \neq 0.$$

Thus

$$\int_{t_0}^{\infty} \frac{dt}{||\Phi(t,t_0)Q(t_0)x||^p} \leq \frac{M^p e^{\omega p}L^p}{||Q(t_0)x||^p} + \int_{t_0}^{[t_0]+1} \frac{dt}{||\Phi(t,t_0)Q(t_0)x||^p}.$$

Taking into account that

$$\|\Phi([t_0]+1,t_0)Q(t_0)x\| \le Me^{\omega}\|\Phi(t,t_0)Q(t_0)x\|$$

which implies that

$$\frac{1}{||\Phi(t,t_0)Q(t_0)x||} \le \frac{Me^{\omega}}{||\Phi([t_0]+1,t_0)Q(t_0)x||} \le \frac{Me^{\omega}L}{||Q(t_0)x||},$$

for all $t \in [t_0,[t_0]+1]$.

Thus we can obtain that

$$\int_{t_0}^{[t_0]+1} \frac{dt}{||\Phi(t,t_0)Q(t_0)x||^p} \le \frac{M^p e^{\omega p} L^p}{||Q(t_0)x||^p}$$

and hence

$$\int_{t_0}^{\infty} \frac{dt}{||\Phi(t,t_0)Q(t_0)x||^p} \le \frac{2M^p e^{\omega p} L^p}{||Q(t_0)x||^p}$$

for all $t_0 \ge 0$ and $x \in X$ with $Q(t_0)x \ne 0$.

By Proposition 2.1. we can find $N,\nu > 0$ such that

$$||\Phi(t,t_0)Q(t_0)x|| \ge Ne^{\nu(t-t_0)}||Q(t_0)x||, \text{ for all } t \ge t_0, \text{ and } x \in X.$$

Theorem 2.2. Let $P : \mathbb{R}_+ \to B(X)$ be a dichotomy projection family. Then Φ is uniformly exponentially dichotomic if and only if there exist $p, L > 0$ such that

$$\Big(\sum_{k=0}^{n} ||\Phi(n,k)P(k)||^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=[t_0]+1}^{\infty} ||\Phi^{-1}(k,t_0)Q(k)||^p\Big)^{\frac{1}{p}} \le L,$$

for all $n \in \mathbb{N}$, $t_0 \ge 0$.

Proof.Sufficiency. Let $t \ge \tau+1$, $\tau \ge 0$, $n=[t]$, $k=[\tau]$. Then $k+1 \le n$ and

$$||\Phi(t,\tau)P(\tau)||^p = ||\Phi(t,n)\Phi(n,k+1)\Phi(k+1,\tau)P(\tau)||^p \le$$
$$\le M^{2p}e^{2\omega p}||\Phi(n,k+1)P(k+1)||^p.$$

Moreover

$$\sum_{k=0}^{n-1}\int_k^{k+1} ||\Phi(t,\tau)P(\tau)||^p d\tau \le M^{2p}e^{2\omega p}\sum_{k=0}^{n-1}||\Phi(n,k+1)P(k+1)||^p =$$

$$= M^{2p}e^{2\omega p}\sum_{i=1}^{n}||\Phi(n,i)P(i)||^p \le M^{2p}e^{2\omega p}L^p,$$

which implies that

$$\int_0^n ||\Phi(t,\tau)P(\tau)||^p d\tau \le M^{2p}e^{2\omega p}L^p,$$

and hence

$$\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau = \int_0^n ||\Phi(t,\tau)P(\tau)||^p d\tau + \int_n^t ||\Phi(t,\tau)P(\tau)||^p d\tau \leq$$

$$\leq M^2 e^{2\omega} L^p + M^p e^{p\omega} (\sup_{t\geq 0} ||P(t)||)^p = k', \ \forall\, t \geq 1.$$

For $t \in [0,1)$ we have that

$$\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau \leq M^p e^{p\omega} (\sup_{t\geq 0} ||P(t)||)^p \leq k',$$

which implies that

$$\Big(\int_0^t ||\Phi(t,\tau)P(\tau)||^p d\tau\Big)^{\frac{1}{p}} \leq (k')^{\frac{1}{p}}, \text{ for all } t \geq 0.$$

Applying Proposition 2.2. we can find $N_1, \nu_1 > 0$ with

$$||\Phi(t,t_0)P(t_0)|| \leq N_1 e^{-\nu_1(t-t_0)}, \text{ for all } t \geq t_0 \geq 0.$$

Let $x \in X$ with $Q(t_0)x \neq 0$. Then

$$||Q(t_0)x|| = ||\Phi^{-1}(k,t_0)\Phi(k,t_0)Q(t_0)x|| \leq$$

$$\leq ||\Phi^{-1}(k,t_0)Q(k)||\ ||\Phi(k,t_0)Q(t_0)x||$$

and hence

$$\Big(\sum_{k=[t_0+1]}^{\infty} \frac{1}{||\Phi(k,t_0)Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=[t_0+1]}^{\infty} \frac{||\Phi^{-1}(k,t_0)Q(k)||^p}{||Q(t_0)x||^p}\Big)^{\frac{1}{p}} \leq$$

$$\leq \frac{L}{||Q(t_0)x||}, \text{ for all } t_0 \geq 0.$$

Using now Theorem 2.1. we have that there exist $N_2, \nu_2 > 0$ such that

$$||\Phi(t,t_0)Q(t_0)x|| \geq N_2 e^{\nu_2(t-t_0)} ||Q(t_0)x||, \text{ for all } t \geq t_0 \geq 0,$$

and hence Φ is uniformly exponentially dichotomic.

Necessity follows easily by Definition 1.2.

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