Radon inversion problem for holomorphic functions on strictly pseudoconvex domains

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Abstract

Let $p > 0$ and let $\Omega \subset \mathbb{C}^d$ be a bounded, strictly pseudoconvex domain with boundary of class C^2. We consider a family of directions in the form of a continuous function $\gamma : \partial\Omega \times [0,1] \ni (z,t) \to \gamma(z,t) \in \overline{\Omega}$ satisfying some natural properties. Then for a given lower semicontinuous, strictly positive function H on $\partial\Omega$ we construct a holomorphic function $f \in \mathbb{O}(\Omega)$ such that $H(z) = \int_0^1 |f(\gamma(z,t))|^p\, dt$ for η-almost all $z \in \partial\Omega$ where η is a given probability measure on $\partial\Omega$.

1 Introduction

In this paper we intend to investigate the so-called Radon inversion problem, i.e. the problem of reconstructing a function on the basis of known integrals of this function over some subset of submanifolds of its domain.

For a given domain $\Omega \subset \mathbb{C}^n$ and $p > 0$ we consider a family of holomorphic functions on Ω, integrable along the family of real directions in the form of a continuous function $\gamma : \partial\Omega \times [0,1) \ni (z,t) \to \gamma(z,t) \in \Omega$. In particular we can define the Radon operator by

$$\mathfrak{R} : \mathbb{O}(\Omega) \times \partial\Omega \ni (f,\xi) \to \mathfrak{R}(f,\xi) = \int_0^1 |f \circ \gamma(\xi,t)|^p\, dt$$

and formulate the Radon inversion problem in the following way:

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Let us assume that H is a lower semicontinuous function on $\partial\Omega$. Is it possible to construct a function $f \in \mathbb{O}(\Omega)$ such that $\mathfrak{R}(f,\xi) = H(\xi)$ for $\xi \in \partial\Omega$?

Let us observe that the above problem is similar to the construction of the inner function (see [1, 13, 14, 15]). It is known that a non-constant holomorphic function $f \in \mathbb{O}(\Omega)$ with non-tangential limit in all boundary points equal to 1, does not exist. In fact, all the inner functions constructed in the papers [1, 13, 14, 15] have non-tangential limits well defined only in almost all boundary points (in terms of a proper surface measure). In the Radon inversion problem the role of the non-tangential limit is played bythe value $\mathfrak{R}(f,\xi)$ which is well defined in all boundary points ξ.

We will solve the probability version of the Radon inversion problem. In particular (see Theorem 4.1) for a given probability measure η on $\partial\Omega$, we construct a holomorphic function f such that $\mathfrak{R}(f,\xi) = H(\xi)$ for η-almost all $\xi \in \partial\Omega$. However, the full version still remains an open problem.

As an application we give a description of so called exceptional sets (Theorem 4.8)

$$E^p_\Omega(f) := \{\xi \in \partial\Omega : \mathfrak{R}(f,\xi) = \infty\}. \tag{1.1}$$

For more information about exceptional sets we refer the reader to e.g. [2, 3, 4, 5, 6, 9, 10, 11].

We also solve the Dirichlet problem for plurisubharmonic and real analytic functions (Theorem 4.4).

1.1 Geometric notions.

In this paper we assume, in general, that $\Omega \subset \mathbb{C}^d$ is a bounded, strictly convex domain with boundary of class C^2 and a defining function ρ. Only the last section will be devoted to strictly pseudoconvex domains. We consider the natural scalar product $\langle \circ, \circ \rangle$. As usual, by $B(\xi; r)$ we denote the open ball with center ξ and radius r, i.e. $B(\xi;r) := \left\{z \in \mathbb{C}^d : \|\xi - z\| < r\right\}$. Note that there exists $\pi_d > 0$ such that $\mathfrak{L}^{2d}(B(\xi,r)) = \pi_d r^{2d}$ for $\xi \in \mathbb{C}^d$ and $r > 0$, where $\mathfrak{L}^{2d}$ is the $2d$-dimensional Lebesgue measure. Assume that $0 \in \Omega \subset B(0,R)$ for some $R > 0$.

A subset $A \subset \mathbb{C}^d$ is called α-separated if $\|z_1 - z_2\| > \alpha$ for all distinct elements z_1 and z_2 of A. It is clear that for $\alpha > 0$ each α-separated subset of $\partial\Omega$ is finite.

If $g : \mathbb{C}^d \to \mathbb{C}$ is a function of class C^2 then we denote $g_\xi = \left(\frac{\partial g}{\partial z_1}(\xi), ..., \frac{\partial g}{\partial z_d}(\xi)\right)$ and

$$H_g(P,w) := \frac{1}{2}\sum_{j,k=1}^{d} \frac{\partial^2 g}{\partial z_j \partial z_k}(P)\, w_j w_k + \frac{1}{2}\sum_{j,k=1}^{d} \frac{\partial^2 g}{\partial \overline{z}_j \partial \overline{z}_k}(P)\, \overline{w}_j \overline{w}_k + \sum_{j,k=1}^{d} \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(P)\, w_j \overline{w}_k.$$

Definition 1.1. Let X be a compact subset of $\partial\Omega$. We say that a continuous function $\gamma : X \times [0,1] \ni (z,t) \to \gamma(z,t) \in \overline{\Omega}$ defines a set of real directions on Ω if γ has the following properties:

1. $\gamma(X \times [0,1)) \subset \Omega$.

2. $\gamma(X \times \{1\}) \subset \partial\Omega$.

3. $\frac{\partial\gamma}{\partial t}(\circ,\circ)$ is a continuous function on $X \times [0,1]$.

4. There exist constants $c_1, c_2 > 0$ such that $c_1 \|z-\xi\| \le \|\gamma(z,1)-\gamma(\xi,1)\| \le c_2 \|z-\xi\|$ for $z,\xi \in X$.

5. $\gamma(\xi,\circ)$ is tangential to $\partial\Omega$ at $\gamma(\xi,1)$ i.e. $\mathrm{Re}\left\langle \frac{\partial\gamma}{\partial t}(\xi,1), \overline{\rho_{\gamma(\xi,1)}}\right\rangle > 0$ for $\xi \in X$.

2 Preliminary calculations

We need the following result.

Lemma 2.1. *There exist constants $c_1, c_2 > 0$ such that for $z,\xi \in \partial\Omega$ one has:*

$$c_1 \|z-\xi\|^2 \le \mathrm{Re}\left\langle \xi - z, \overline{\rho_\xi}\right\rangle \le c_2 \|z-\xi\|^2. \tag{2.1}$$

Proof. It suffices to use the same arguments as in the proof [12, Lemma 2.1]. ∎

In order to control the values of the functions constructed we need some information about α-separated sets.

Lemma 2.2. *Suppose that $A = \{\xi_1, .., \xi_s\}$ is a $2\alpha t$-separated subset of $\partial\Omega$. For $z \in \partial\Omega$ let*

$$A_k(z) := \{\xi \in A : \alpha k t \le \|z-\xi\| \le \alpha(k+1)t\}.$$

Then the set $A_k(z)$ has at most $(k+2)^{2d}$ elements. The set A_0 has at most 1 element and $s \le \max\left\{1, \left(\frac{2R}{\alpha t}\right)^{2d}\right\}$.

Proof. Putting $\rho(z,\xi) = \|z-\xi\|$, it suffices to use the same arguments as in the proof [12, Lemma 2.2]. ∎

Lemma 2.3. *If $A \subset \partial\Omega$ is αt-separated, then for each $\beta > \alpha$ there exists an integer $K = K(\alpha,\beta)$ such that A can be partitioned into K disjoint βt-separated sets.*

Proof. see [12, Lemma 2.3] ∎

3 Basic results for strictly convex domains

Let $p > 0$. Assume that Ω is a bounded strictly convex domain, X is a compact subset of $\partial\Omega$ and $\gamma : X \times [0,1] \to \overline{\Omega}$ defines a set of real directions on Ω.

In particular there exist constants $c_2 \ge c_1 > 0$ such that

$$c_1 \|z-w\| \le \|\gamma(z,1)-\gamma(w,1)\| \le c_2 \|z-w\| \tag{3.1}$$

for $z,w \in X$. Due to Lemma 2.1 there exist constants $c_3, c_4 > 0$ such that for $z,\xi \in \partial\Omega$

$$-c_3 \|z-\xi\|^2 \le \mathrm{Re}\left\langle z-\xi, \overline{\rho_\xi}\right\rangle \le -c_4 \|z-\xi\|^2. \tag{3.2}$$

Lemma 3.1. *Denoting*

$$F_{m,\xi}(z) := \left(m\mathrm{Re}\left\langle \frac{\partial\gamma}{\partial t}(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle \right)^{\frac{1}{p}} \exp\left(\frac{m}{p} \left\langle z - \gamma(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle \right)$$

where $q = \sup_{\xi\in X}\left\{1, \mathrm{Re}\left\langle \frac{\partial\gamma}{\partial t}(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle\right\}$, *if* $0 < b_1 < 1 < b_2$ *then there exist* $\alpha, \beta_1, \beta_2, N_0, r_0 > 0$ *such that for* $m \geq N_0$, $z, \xi \in X$ *one has the following properties:*

1. *if* $\|z-\xi\| \leq r_0$ *then* $b_1 e^{-m\beta_1\|z-\xi\|^2} - e^{-m\alpha} \leq \int_0^1 \left|F_{m,\xi}\circ\gamma(z,t)\right|^p dt \leq b_2 e^{-m\beta_2\|z-\xi\|^2} + e^{-m\alpha}$;
2. *if* $(0 \leq t \leq 1-r_0) \vee (\|z-\xi\| \geq r_0)$ *then* $\left|F_{m,\xi}\circ\gamma(z,t)\right|^p \leq e^{-m\alpha}$.

Proof. There exists a constant $1 > r_0 > 0$ such that

$$0 < \frac{1}{b_2}\mathrm{Re}\left\langle \frac{\partial\gamma}{\partial t}(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle \leq \mathrm{Re}\left\langle \frac{\partial\gamma}{\partial t}(z,t), \overline{\rho_{\gamma(\xi,1)}} \right\rangle \leq \frac{1}{b_1}\mathrm{Re}\left\langle \frac{\partial\gamma}{\partial t}(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle \tag{3.3}$$

for $t \in [1-r_0, 1]$ and $z, \xi \in X$ so that $\|z-\xi\| \leq r_0$. Moreover there exists $\alpha > 0$ such that

$$\mathrm{Re}\left\langle \gamma(z,t) - \gamma(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle \leq -2\alpha$$

for $(z,\xi,t) \in \{(x,y,s) \in X \times X \times [0,1] : \|x-y\| \geq r_0 \vee s \leq 1-r_0\}$.

Let N_0 be such that

$$e^{-m\alpha} \geq mqe^{-2m\alpha}$$

for $m \geq N_0$. In particular $\left|F_{m,\xi}\circ\gamma(z,t)\right|^p \leq mqe^{-2m\alpha} \leq e^{-m\alpha}$ for $m \geq N_0$ and $(0 \leq t \leq 1-r_0) \vee (\|z-\xi\| \geq r_0)$.

Now assume that $\|z-\xi\| < r_0$. Due to (3.1), (3.2) and (3.3) we may estimate for $\beta_1 := c_2^2 c_3$, $\beta_2 := c_1^2 c_4$ and $m \geq N_0$:

$$\begin{aligned}
\int_0^1 \left|F_{m,\xi}\circ\gamma(z,t)\right|^p dt &\geq \int_{1-r_0}^1 \left|F_{m,\xi}\circ\gamma(z,t)\right|^p dt \\
&\geq b_1 e^{m\left\langle \gamma(z,1)-\gamma(\xi,1), \overline{\rho_{\gamma(\xi,1)}}\right\rangle} - b_1 e^{m\left\langle \gamma(z,1-r_0)-\gamma(\xi,1), \overline{\rho_{\gamma(\xi,1)}}\right\rangle} \\
&\geq b_1 e^{-mc_3\|\gamma(z,1)-\gamma(\xi,1)\|^2} - e^{-m\alpha} \geq b_1 e^{-m\beta_1\|z-\xi\|^2} - e^{-m\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \left|F_{m,\xi}\circ\gamma(z,t)\right|^p dt &\leq \int_{1-r_0}^1 \left|F_{m,\xi}\circ\gamma(z,t)\right|^p dt + e^{-m\alpha} \\
&\leq b_2 e^{m\left\langle \gamma(z,1)-\gamma(\xi,1), \overline{\rho_{\gamma(\xi,1)}}\right\rangle} + e^{-m\alpha} \leq b_2 e^{-m\beta_2\|z-\xi\|^2} + e^{-m\alpha}.
\end{aligned}$$

■

Lemma 3.2. *Assume that* Ω *is a bounded domain,* X *is a compact subset of* $\overline{\Omega}$ *and* $\gamma : X \times [0,1] \to \overline{\Omega}$ *is a continuous function such that* $\gamma(X \times [0,1)) \subset \Omega$. *Let* f *be a continuous complex function on* $\overline{\Omega}$ *and* $\varepsilon, \delta \in (0,1)$. *If* $\{g_m\}_{m\in\mathbb{N}}$ *is a sequence of* *continuous complex functions on* $\overline{\Omega}$ *such that* $\lim_{m\to\infty} g_m(z) = 0$ *for* $z \in \Omega$*, then there exists* $m_0 \in \mathbb{N}$ *such that*

$$\begin{aligned}\int_0^1 |(f+g_m)\circ\gamma(z,t)|^p\,dt &\geq -\varepsilon + \int_0^1 |f\circ\gamma(z,t)|^p\,dt + \delta^p \int_0^1 |g_m\circ\gamma(z,t)|^p\,dt\\ \int_0^1 \underbrace{|(f+g_m)\circ\gamma(z,t)|^p}_{L_m(z,t)}\,dt &\leq \varepsilon + \int_0^1 |f\circ\gamma(z,t)|^p\,dt + \delta^{-p} \int_0^1 |g_m\circ\gamma(z,t)|^p\,dt\end{aligned}$$

for $m > m_0$*,* $z \in X$*.*

Proof. Let $M := \sup_{z\in\overline{\Omega}} |f(z)|$ and $r \in (\frac{1}{2}, 1)$ be such that $\frac{(1-r)2M^p}{(1-\delta)^p} \leq \frac{\varepsilon}{4}$. We may consider a continuous function $\Psi : X \times \overline{\mathbb{D}} \ni (z,\lambda) \to \int_0^r |f\circ\gamma(z,t)+\lambda|^p\,dt$. There exists $\alpha \in (0, \sqrt[p]{\frac{\varepsilon}{4}})$ such that $|\Psi(z,0) - \Psi(z,\lambda)| \leq \frac{\varepsilon}{4}$ for $z \in X$, and $|\lambda| \leq \alpha$. As $\lim_{m\to\infty} g_m(z) = 0$ for $z \in \Omega$, there exists m_0 such that $|g_m\circ\gamma(z,t)| \leq \alpha$ for $m > m_0$, $0 \leq t \leq r$ and $z \in X$. In particular for $m > m_0$ and $z \in X$ we can estimate:

$$\begin{aligned}\int_0^r L_m(z,t)dt &\geq -\frac{\varepsilon}{4} + \int_0^r |f\circ\gamma(z,t)|^p\,dt\\ &\geq -\frac{\varepsilon}{2} + \int_0^r |f\circ\gamma(z,t)|^p\,dt + \delta^p \int_0^r |g_m\circ\gamma(z,t)|^p\,dt\end{aligned}$$

and

$$\begin{aligned}\int_0^r L_m(z,t)dt &\leq \frac{\varepsilon}{4} + \int_0^r |f\circ\gamma(z,t)|^p\,dt\\ &\leq \frac{\varepsilon}{2} + \int_0^r |f\circ\gamma(z,t)|^p\,dt + \delta^{-p} \int_0^r |g_m\circ\gamma(z,t)|^p\,dt.\end{aligned}$$

If $t \in A_{1,m,z} := \{t \in [r,1] : |(f+g_m)\circ\gamma(z,t)| \leq \delta\,|g_m\circ\gamma(z,t)|\}$ then $|g_m\circ\gamma(z,t)| \leq \frac{|f\circ\gamma(z,t)|}{1-\delta} \leq \frac{M}{1-\delta}$. In particular we may estimate

$$\begin{aligned}\int_r^1 L_m(z,t)dt &\geq \int_{[r,1]\setminus A_{1,m,z}} \delta^p\,|g_m\circ\gamma(z,t)|^p\,dt \geq \int_r^1 |f\circ\gamma(z,t)|^p\,dt +\\ &\quad +\delta^p \int_r^1 |g_m\circ\gamma(z,t)|^p\,dt - \int_r^1 M^p dt - \int_r^1 \frac{M^p\delta^p}{(1-\delta)^p}dt\\ &\geq -\frac{\varepsilon}{2} + \int_r^1 |f\circ\gamma(z,t)|^p\,dt + \delta^p \int_r^1 |g_m\circ\gamma(z,t)|^p\,dt.\end{aligned}$$

If $t \in A_{2,m,z} := \left\{t \in [r,1] : |f\circ\gamma(z,t)| + |g_m\circ\gamma(z,t)| \geq \delta^{-1}\,|g_m\circ\gamma(z,t)|\right\}$ then $|g_m\circ\gamma(z,t)| \leq \frac{|f\circ\gamma(z,t)|}{\delta^{-1}-1} \leq \frac{\delta M}{1-\delta}$. In particular

$$\begin{aligned}\int_r^1 L_m(z,t)dt &\leq \int_{[r,1]\setminus A_{2,m,z}} \delta^{-p}\,|g_m\circ\gamma(z,t)|^p\,dt + \int_r^1 \frac{M^p}{(1-\delta)^p}dt\\ &\leq \int_r^1 |f\circ\gamma(z,t)|^p\,dt + \delta^{-p} \int_r^1 |g_m\circ\gamma(z,t)|^p\,dt +\\ &\quad -\int_r^1 M^p dt - \int_r^1 \frac{2M^p}{(1-\delta)^p}dt\\ &\leq -\frac{\varepsilon}{2} + \int_r^1 |f\circ\gamma(z,t)|^p\,dt + \delta^{-p} \int_r^1 |g_m\circ\gamma(z,t)|^p\,dt. \quad \blacksquare\end{aligned}$$

Lemma 3.3. *There exist constants $C > c > 0$ such that if T is a compact subset of $\overline{\Omega} \setminus X$, $\varepsilon \in (0,1)$ and H is a continuous strictly positive function on X, then we can choose $N_1 > 0$ such that for $m \geq N_1$ and each $\frac{C}{\sqrt{m}}$-separated subset A of X, the holomorphic function $g_{m,A} := \sum_{\xi \in A} (H(\xi))^{\frac{1}{p}} F_{m,\xi}$ satisfies*

1. $|g_{m,A}(w)| \leq \varepsilon$ *for* $w \in T$;
2. $\int_0^1 |g_{m,A}(\gamma(z,t))|^p \, dt < 2H(z)$ *for all* $z \in X$;
3. $\int_0^1 |g_{m,A}(\gamma(z,t))|^p \, dt > \frac{H(z)}{2}$ *for each* $z \in X$ *such that* $\|z - \xi\| \leq \frac{c}{\sqrt{m}}$ *for some* $\xi \in A$.

Proof. Let us denote $a = \min\left\{1, \frac{1}{p}\right\}$. We may assume that $\|H\|_\infty = 1$. Let $0 < \delta < b_1 < 1 < b_2$ be such that

$$(1+\delta)^a (b_2 + \delta)^a + 3\delta^a < 2^a \tag{3.4}$$

$$(1-\delta)^a \left(b_1 e^{-\frac{1}{16}} - \delta\right)^a - 3\delta^a > 2^{-a}. \tag{3.5}$$

Now we can choose $\alpha, \beta_1, \beta_2, N_0, r_0 > 0$ from Lemma 3.1. Let $c = \frac{1}{4\sqrt{\beta_1}}$. There exists $C > 0$ such that $C > c$ and for $k \in \mathbb{N} \setminus \{0\}$ we have

$$b_2^a (k+2)^{2d} e^{-\frac{aC^2\beta_2 k^2}{4}} \leq 2^{-k}.$$

Due to Lemma 2.2 we have $\#A \leq \left(\frac{4R\sqrt{m}}{C}\right)^{2d}$.

Let $t := \sup_{w \in T, \xi \in X} \frac{1}{p} \left\langle w - \gamma(\xi,1), \overline{\rho_{\gamma(\xi,1)}} \right\rangle$. As $t < 0$, for $w \in T$, sufficiently large N_1 and $m \geq N_1$, we may estimate

$$|g_{m,A}(w)| \leq \sum_{\xi \in A} (mq)^{\frac{1}{p}} e^{mt} \leq \left(\frac{4R\sqrt{m}}{C}\right)^{2d} (mq)^{\frac{1}{p}} e^{mt} \leq \varepsilon$$

and conclude that property (1) holds.

For $z \in X$ let us denote

$$A_k(z) := \left\{\xi \in A : \frac{Ck}{2\sqrt{m}} \leq \|z - \xi\| \leq \frac{C(k+1)}{2\sqrt{m}}\right\}.$$

Let now $s > 0$ be so small that $\|\eta - \xi\| \leq s \implies (1-\delta)H(\eta) \leq H(\xi) \leq (1+\delta)H(\eta)$. We may assume that N_1 is large enough that $s \geq \frac{C}{2\sqrt{N_1}} + \frac{c}{\sqrt{N_1}}$ and $e^{-aN_1\alpha} \leq \delta$. Observe that we may estimate

$$b_2^a \sum_{k : C(k+1) \geq 2s\sqrt{m}} (k+2)^{2d} e^{-\frac{aC^2\beta_2 k^2}{4}} \leq \sum_{k \geq \left[\frac{2s\sqrt{m}}{C} - 1\right]} 2^{-k} \leq 2^{-\frac{2s\sqrt{m}}{C} + 1}.$$

Now if $z \in X$ and $A_0(z) = \emptyset$, then, due to (3.4), Lemma 2.2 and Lemma 3.1, we may estimate, for N_1 large enough and $m \geq N_1$

$$
\begin{aligned}
\left(\int_0^1 |g_{m,A}(\gamma(z,t))|^p \, dt\right)^a &\leq \sum_{k=1}^{\infty} \sum_{\xi \in A_k(z)} \left(H(\xi) \int_0^1 \left|F_{m,\xi}(\gamma(z,t))\right|^p dt\right)^a \\
&\leq \sum_{k=1}^{\infty} \sum_{\xi \in A_k(z)} H(\xi)^a \left(b_2^a e^{-\frac{aC^2\beta_2 k^2}{4}} + e^{-am\alpha}\right) \\
&\leq (1+\delta)^a H(z)^a \sum_{k=1}^{\left[\frac{2s\sqrt{m}}{C}\right]} b_2^a (k+2)^{2d} e^{-\frac{aC^2\beta_2 k^2}{4}} + \\
&\quad + 2^{-\frac{2s\sqrt{m}}{C}+1} + \left(\frac{4R\sqrt{m}}{C}\right)^{2d} e^{-am\alpha} \\
&\leq \delta^a (1+\delta)^a H(z)^a + \delta^a H(z)^a \leq 3\delta^a H(z)^a.
\end{aligned}
$$

Due to Lemma 2.2, if $A_0(z) \neq \emptyset$ then $A_0(z) = \{\xi_0\}$ for some $\xi_0 \in \partial\Omega$ where $\|z - \xi\| \leq \frac{C}{2\sqrt{m}} \leq s$. In particular

$$
\begin{aligned}
\left(\int_0^1 |g_{m,A}(\gamma(z,t))|^p \, dt\right)^a &\leq \left(H(\xi_0) \int_0^1 \left|F_{m,\xi_0}(\gamma(z,t))\right|^p dt\right)^a + 3\delta^a H(z)^a \\
&\leq H(\xi_0)^a \left(b_2 + e^{-m\alpha}\right)^a + 3\delta^a H(z)^a \\
&\leq H(z)^a (1+\delta)^a \left(b_2 + \delta\right)^a + 3\delta^a H(z)^a < 2^a H(z)^a
\end{aligned}
$$

for $z \in X$, N_1 large enough and $m \geq N_1$, which gives property (2).

Now let $\xi_1 \in A$ be such that $\|z - \xi_1\| \leq \frac{c}{\sqrt{m}} \leq s$. Due to Lemma 3.1 and (3.5) we may estimate, for N_1 large enough and $m \geq N_1$

$$
\begin{aligned}
\left(\int_0^1 |g_{m,A}(\gamma(z,t))|^p \, dt\right)^a &\geq \left(H(\xi_0) \int_0^1 \left|F_{m,\xi_1}(\gamma(z,t))\right|^p dt\right)^a - 3\delta^a H(z)^a \\
&\geq H(\xi_1)^a \left(b_1 e^{-\frac{1}{16}} - e^{-m\alpha}\right)^a - 3\delta^a H(z)^a \\
&\geq H(z)^a (1-\delta)^a \left(b_1 e^{-\frac{1}{16}} - \delta\right)^a - 3\delta^a H(z)^a > \frac{H(z)^a}{2^a}
\end{aligned}
$$

which gives property (3). ∎

Now we are ready to prove the following result:

Theorem 3.4. *There exists a natural number K such that, if $\varepsilon \in (0,1)$, T is a compact subset of $\overline{\Omega} \setminus X$ and H is a continuous, strictly positive function on X, then there exist holomorphic entire functions $f_1, ..., f_K$ such that $\|f_j\|_T \leq \varepsilon$, and one has for $z \in X$ the following inequality*

$$\frac{H(z)}{4} < \max_{j=1,\dots,K} \int_0^1 \left|f_j(\gamma(z,t))\right|^p dt < H(z).$$

Proof. Let $C > c > 0$ be the constants from Lemma 3.3. Due to Lemma 2.3 there exists a natural number K such that each $\frac{c}{\sqrt{m}}$-separated subset of X can be partitioned into K disjoint $\frac{C}{\sqrt{m}}$-separated sets. Let A be a maximal $\frac{c}{\sqrt{m}}$-separated subset of X. It can be partitioned into $A_1, ..., A_K$ disjoint $\frac{C}{\sqrt{m}}$-separated sets. Now due to Lemma 3.3 there exists m and holomorphic, entire functions $f_j := g_{m,A_j}$ such that $\|f_j\|_T \le \varepsilon$ and

1. $\int_0^1 |f_j(\gamma(z,t))|^p \, dt < H(z)$ for all $z \in X$;
2. $\int_0^1 |f_j(\gamma(z,t))|^p \, dt > \frac{H(z)}{4}$ for each $z \in X$ such that $\|z - \xi\| \le \frac{c}{\sqrt{m}}$ for some $\xi \in A_j$.

As A is a maximal $\frac{c}{\sqrt{m}}$-separated subset of X there exists, for $z \in X$, $j_0 \in \{1, ..., K\}$ and $\xi_{j_0} \in A_{j_0}$ such that $\|z - \xi_{j_0}\| \le \frac{c}{\sqrt{m}}$. In particular

$$\frac{H(z)}{4} < \int_0^1 |f_{j_0}(\gamma(z,t))|^p \, dt \le \max_{j=1,\dots,K} \int_0^1 |f_j(\gamma(z,t))|^p \, dt < H(z). \qquad \blacksquare$$

4 Consequences of Theorem 3.4 for strictly pseudoconvex domains

In this section we assume that Ω is a bounded, strictly pseudoconvex domain with boundary of class C^2, X is a compact subset of $\partial\Omega$ and $\gamma : X \times [0,1] \to \overline{\Omega}$ defines a set of real directions on Ω.

As a first application of Theorem 3.4 we give the following result.

Theorem 4.1. *It is possible to choose a neighbourhood W of $\overline{\Omega}$ and a natural number K such that, if $\varepsilon \in (0,1)$, T is a compact subset of $\overline{\Omega} \setminus X$ and H is a continuous, strictly positive function on X, then there exist holomorphic functions $f_1, ..., f_K$ on W such that $\|f_j\|_T \le \varepsilon$, and one has for $z \in X$ the following inequality*

$$\frac{H(z)}{4} < \max_{j=1,\dots,K} \int_0^1 |f_j(\gamma(z,t))|^p \, dt < H(z).$$

Proof. By Fornaess' embedding theorem [7], there exists a neighbourhood W of $\overline{\Omega}$, a strictly convex, bounded domain $\widetilde{\Omega} \subset \mathbb{C}^N$ with boundary of class C^2 and a holomorphic mapping $\psi : U \to \mathbb{C}^N$, such that ψ maps W biholomorphically onto some complex submanifold $\psi(W)$ of $\mathbb{C}^N$, such that

1. $\psi(\Omega) \subset \widetilde{\Omega}$;
2. $\psi(\partial\Omega) \subset \partial\widetilde{\Omega}$;
3. $\psi(W \setminus \overline{\Omega}) \subset \mathbb{C}^N \setminus \overline{\widetilde{\Omega}}$;
4. $\psi(W)$ intersects $\partial\widetilde{\Omega}$ transversally.

Let $\widetilde{X} = \psi(X)$. Observe that

$$\widetilde{\gamma} : \widetilde{X} \times [0,1] \ni (z,t) \to \psi(\gamma(\psi^{-1}(z),t)) \in \overline{\widetilde{\Omega}}$$

defines a set of real directions on $\widetilde{\Omega}$. Let K be the natural number from Theorem 3.4 used for the domain $\widetilde{\Omega}$. Now due to Theorem 3.4 there exist entire holomorphic functions $\widetilde{f}_1, ..., \widetilde{f}_K$ on $\mathbb{C}^N$ such that $\left\|\widetilde{f}_j\right\|_{\psi(T)} \le \varepsilon$, and we have for $z \in \widetilde{X}$ the following inequality

$$\frac{H(\psi^{-1}(z))}{4} < \max_{j=1,...,K} \int_0^1 \left|\widetilde{f}_j(\widetilde{\gamma}(z,t))\right|^p dt < H(\psi^{-1}(z)).$$

In particular the functions $f_j = \widetilde{f}_j \circ \psi$ have the required properties. ∎

¿From this moment on we assume that K and W are as in Theorem 4.1.

Lemma 4.2. *Let $g_1, ..., g_K$ be continuous complex functions on $\overline{\Omega}$, T be a compact subset of $\overline{\Omega} \setminus X$, $\varepsilon > 0$ and u be a strictly positive, continuous function on X. Then there exist functions $f_1, ..., f_K$ holomorphic on W such that*

1. $|f_j(z)| \le \varepsilon$ *for* $z \in T$;
2. $u(z) - \varepsilon < \sum_{j=1}^K \int_0^1 |(f_j + g_j)(\gamma(z,t))|^p dt - \sum_{j=1}^K \int_0^1 |g_j(\gamma(z,t))|^p dt < u(z)$ *for* $z \in X$.

Proof. Let $\theta = 1 - \frac{1}{4K}$, $1 - \delta^{2p} = \frac{1-\theta}{4}$ and $g(z) = \sum_{j=1}^K \int_0^1 |g_j(\gamma(z,t))|^p dt$. Let us define a sequence of continuous functions H_j such that, for $z \in \partial\Omega$, we have

$$0 = H_0(z) < ... < H_j(z) < H_{j+1}(z) < ... < \lim_{j\to\infty} H_j(z) = g(z) + u(z).$$

Now we construct sequences $\{f_{j,k}\}_{k\in\mathbb{N}}^{j=1,...,K}$ of holomorphic functions on W such that, if $v_m(z) := \sum_{j=1}^K \int_0^1 \left|\left(g_j + \sum_{k=1}^m f_{j,k}\right)(\gamma(z,t))\right|^p dt$ then

(a) $|f_{j,k}(z)| \le \frac{\varepsilon}{2^k}$ for $z \in T$;

(b) $0 < H_m(z) - v_m(z) < 2\sum_{k=1}^m \left(\frac{1+\theta}{2}\right)^{m-k} (H_k(z) - H_{k-1}(z))$ for $z \in X$ and $m \in \mathbb{N}$.

If $m = 1$ then it is sufficient to select $f_{1,1} = f_{2,1} = ... = f_{K,1} = 0$. Now assume that we have constructed holomorphic functions $\{f_{j,k}\}_{k=1,...,m-1}^{j=1,...,K}$ on W such that (a)-(b) hold. Let us denote

$$\begin{aligned} 2\varepsilon_m &= \frac{1-\theta_0}{4} \inf_{z\in\partial\Omega} (H_{m-1}(z) - v_{m-1}(z)) \\ G_m(z) &= H_m(z) - \varepsilon_m - v_{m-1}(z). \end{aligned}$$

Due to Lemma 3.2 and Theorem 4.1 there exist $f_{1,m}, ..., f_{K,m}$, holomorphic functions on W, such that property (a) holds and:

- $0 < G_m(z) - \sum_{j=1}^{K} \delta^{-p} \int_0^1 |f_{j,m}(\gamma(z,t))|^p \, dt < \theta G_m(z)$;
- $v_m(z) \geq -\varepsilon_m + v_{m-1}(z) + \sum_{j=1}^{K} \delta^{p} \int_0^1 |f_{j,m}(\gamma(z,t))|^p \, dt$;
- $v_m(z) \leq \varepsilon_m + v_{m-1}(z) + \sum_{j=1}^{K} \delta^{-p} \int_0^1 |f_{j,m}(\gamma(z,t))|^p \, dt$.

Now we may estimate

$$H_m(z) > \varepsilon_m + v_{m-1}(z) + \delta^{-p} \sum_{j=1}^{K} \int_0^1 |f_{j,m}(\gamma(z,t))|^p \, dt \geq v_m(z).$$

Moreover

$$\begin{aligned} H_m(z) &< \varepsilon_m + v_{m-1}(z) + \delta^{-p} \sum_{j=1}^{K} \int_0^1 |f_{j,m}(\gamma(z,t))|^p \, dt + \theta G_m(z) \\ &\leq v_m(z) + 2\varepsilon_m + ((\delta^{-p} - \delta^{p})\delta^{p} + \theta) G_m(z) \\ &\leq v_m(z) + \frac{1-\theta}{4}(H_{m-1}(z) - v_{m-1}(z)) + \left(\frac{1-\theta}{4} + \theta\right) G_m(z). \end{aligned}$$

In particular

$$\begin{aligned} H_m(z) - v_m(z) &< \frac{1+\theta}{2}(H_{m-1}(z) - v_{m-1}(z)) + \frac{1+3\theta}{4}(H_m(z) - H_{m-1}(z)) \\ &\leq 2 \sum_{k=1}^{m} \left(\frac{1+\theta}{2}\right)^{m-k} (H_k(z) - H_{k-1}(z)). \end{aligned}$$

Let $M := \sup_{z \in \partial\Omega}(u(z) + g(z))$. There exists m_0 such that $m\left(\frac{1+\theta}{2}\right)^m M < \frac{\varepsilon}{4}$ and $H_m(z) - H_{m-1}(z) < \varepsilon_0 := \frac{\varepsilon(1-\theta)}{8}$ for $m \geq m_0$ and $z \in X$. In particular for $z \in X$ we may estimate

$$\begin{aligned} \sum_{k=1}^{2m} \left(\frac{1+\theta}{2}\right)^{m-k} (H_k(z) - H_{k-1}(z)) &\leq m \left(\frac{1+\theta}{2}\right)^m M + \sum_{k=m_0}^{2m} \left(\frac{1+\theta}{2}\right)^{2m-k} \varepsilon_0 \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}. \end{aligned}$$

Now we may conclude that there exists $m \in \mathbb{N}$ sufficiently large, such that, for $z \in X$, we have

$$v_m(z) > H_m(z) - \sum_{k=1}^{m} \left(\frac{1+\theta}{2}\right)^{m-k} (H_k(z) - H_{k-1}(z)) \geq u(z) + g(z) - \varepsilon.$$

Observe that the functions $f_j = \sum_{k=1}^{m} f_{j,k}$ have the properties (1)-(2). ■

Now we can prove our second application.

Theorem 4.3. *Let $\varepsilon > 0$, u be a lower semi-continuous, strictly positive function on X and T be a compact subset of $\overline{\Omega} \setminus X$. Then there exist holomorphic functions $f_1, ..., f_K$ on Ω such that $\|f_j\|_T \leq \varepsilon$ and $\sum_{j=1}^{K} \int_0^1 |f_j(\gamma(z,t))|^p \, dt = u(z)$ for $z \in X$.*

Proof. Let $\{T_j\}_{j\in\mathbb{N}}$ be a sequence of compact sets such that T_j is contained in the interior of T_{j+1} for each j and $\bigcup_{j\in\mathbb{N}} T_j = \Omega$.

There exists a sequence H_m of continuous functions on $\partial\Omega$ such that $0 = H_0(z) < H_1(z) < H_2(z) < ... < \lim_{j\to\infty} H_j(z) = u(z)$.

Due to Lemma 4.2 there exists a sequence $\{f_{j,k}\}_{k\in\mathbb{N}}^{j=1,...,K}$ of holomorphic functions on W such that

1. $|f_{j,k}(z)| \le 2^{-k}\varepsilon$ for $z \in T_k \cup T$;
2. $H_m(z) - 2^{-m} < \sum_{j=1}^K \int_0^1 \left|\sum_{k=1}^m f_{j,k}(\gamma(z,t))\right|^p dt < H_m(z)$ for $z \in X$.

Now it suffices to define $f_j = \sum_{k=1}^\infty f_{j,k}$ and to observe that the functions $f_1, ..., f_K$ have the required properties. ∎

Now we can solve the Dirichlet problem for plurisubharmonic functions.

Theorem 4.4. *Let Ω be a bounded, strictly pseudoconvex domain with boundary of class C^2 such that $[0,1)\overline{\Omega} \subset \Omega$. Assume that $[0,1]z$ is transversal to $\partial\Omega$ at $z \in \partial\Omega$. Let u be a continuous, strictly positive function on $\partial\Omega$. Then there exist holomorphic functions $f_1, ..., f_K$ such that $v(z) = \sum_{j=1}^K \int_0^1 |f_j(tz)|^2 dt$ is a plurisubharmonic, real analytic function on Ω and continuous on $\overline{\Omega}$. Moreover $u(z) = v(z)$ for $z \in \partial\Omega$.*

Proof. Observe that $\gamma : \partial\Omega \times [0,1] \ni (z,t) \to zt \in \overline{\Omega}$ is a set of real directions on Ω. Let us define a sequence of continuous functions H_j such that $0 = H_0(z)$ and $H_j(z) - H_{j-1}(z) = 2^{-j}u(z)$. Observe that $\lim_{j\to\infty} H_j(z) = u(z)$. Let $\{T_j\}_{j\in\mathbb{N}}$ be a sequence of compact subsets of Ω such that T_j is contained in the interior of T_{j+1} for each j.

Let $\theta = 1 - \frac{1}{4K}$ and $1 - \delta^4 = \frac{1-\theta}{4}$. Now we construct sequences $\{f_{j,k}\}_{k\in\mathbb{N}}^{j=1,...,K}$ of holomorphic functions on W such that

(a) $|f_{j,k}(z)| \le 2^{-k}$ for $z \in T_k$.

(b) $0 < H_m(z) - v_m(z) < m\left(\frac{1+\theta}{2}\right)^{m-1} u(z)$ for $z \in \partial\Omega$ and $m \in \mathbb{N}$.

(c) $|v_{m+1}(z) - v_m(z)| \le m\left(\frac{1+\theta}{2}\right)^{m-2} \sup_{w\in\partial\Omega} u(w)$ for $z \in \overline{\Omega}$ and $m \in \mathbb{N}$.

where $v_m(z) := \sum_{j=1}^K \int_0^1 \left|\sum_{k=1}^m f_{j,k}(tz)\right|^2 dt$ and $v_0 = 0$. If $m = 1$ then it is sufficient to choose $f_{1,1} = f_{2,1} = ... = f_{K,1} = 0$. Now assume that we have constructed holomorphic functions $\{f_{j,k}\}_{k=1,...,m-1}^{j=1,...,K}$ on W such that (a)-(c) holds. Let us denote

$$\begin{aligned} 2\varepsilon_m &= \frac{1-\theta}{4} \inf_{z\in\partial\Omega} (H_{m-1}(z) - v_{m-1}(z)) \\ G_m(z) &= H_m(z) - \varepsilon_m - v_{m-1}(z). \end{aligned}$$

As $[0,1)\overline{\Omega} \subset \Omega$, due to Lemma 3.2 and Theorem 4.1, there exist $f_{1,m}, ..., f_{K,m}$, holomorphic functions on W, such that property (a) holds and:

- $0 < G_m(z) - \sum_{j=1}^{K} \delta^{-2} \int_0^1 \left|f_{j,m}(tz)\right|^2 dt < \theta G_m(z)$ for $z \in \partial\Omega$;
- $v_m(z) \geq -\varepsilon_m + v_{m-1}(z) + \sum_{j=1}^{K} \delta^{2} \int_0^1 \left|f_{j,m}(tz)\right|^2 dt$ for $z \in \overline{\Omega}$;
- $v_m(z) \leq \varepsilon_m + v_{m-1}(z) + \sum_{j=1}^{K} \delta^{-2} \int_0^1 \left|f_{j,m}(tz)\right|^2 dt$ for $z \in \overline{\Omega}$.

Now we may estimate, for $z \in \partial\Omega$,

$$H_m(z) \quad > \quad \varepsilon_m + v_{m-1}(z) + \delta^{-2} \sum_{j=1}^{K} \int_0^1 \left|f_{j,m}(tz)\right|^2 dt \geq v_m(z).$$

Moreover for $z \in \partial\Omega$ we have

$$\begin{aligned} H_m(z) &< \varepsilon_m + v_{m-1}(z) + \delta^{-2} \sum_{j=1}^{K} \int_0^1 \left|f_{j,m}(tz)\right|^2 dt + \theta G_m(z) \\ &\leq v_m(z) + 2\varepsilon_m + ((\delta^{-2} - \delta^2)\delta^2 + \theta) G_m(z) \\ &\leq v_m(z) + \frac{1-\theta}{4}(H_{m-1}(z) - v_{m-1}(z)) + \left(\frac{1-\theta}{4} + \theta\right) G_m(z). \end{aligned}$$

In particular we obtain property (b):

$$\begin{aligned} H_m(z) - v_m(z) &< \frac{1+\theta}{2}(H_{m-1}(z) - v_{m-1}(z)) + \frac{1+3\theta}{4}(H_m(z) - H_{m-1}(z)) \\ &\leq (m-1)\left(\frac{1+\theta}{2}\right)^{m-1} u(z) + \frac{1+\theta}{2}\frac{u(z)}{2^m} \leq m\left(\frac{1+\theta}{2}\right)^{m-1} u(z). \end{aligned}$$

Moreover for $z \in \overline{\Omega}$ we have

$$|v_{m+1}(z) - v_m(z)| \leq h_m(z) := \varepsilon_m + \delta^{-p} \int_0^1 \left|f_{j,m}(tz)\right|^2 dt.$$

Due to (b) we may estimate, for $z \in \partial\Omega$,

$$\begin{aligned} h_m(z) &\leq \varepsilon_m + G_m(z) \leq H_m(z) - v_{m-1}(z) \leq (H_m - H_{m-1} + H_{m-1} - v_{m-1})(z) \\ &\leq 2^{-m} u(z) + (m-1)\left(\frac{1+\theta}{2}\right)^{m-2} u(z) \leq m\left(\frac{1+\theta}{2}\right)^{m-2} u(z). \end{aligned}$$

As h_m is a continuous and plurisubharmonic function, for $z \in \overline{\Omega}$ we obtain property (c):

$$|v_{m+1}(z) - v_m(z)| \leq h_m(z) \leq m\left(\frac{1+\theta}{2}\right)^{m-2} \sup_{w \in \partial\Omega} u(w).$$

Let us now define holomorphic functions $f_j = \sum_{k=1}^{\infty} f_{j,k}$ on Ω. Observe that $v_m \to v := \sum_{j=1}^{K} \int_0^1 \left|\sum_{k=1}^{\infty} f_{j,k}(tz)\right|^2 dt$ uniformly on $\overline{\Omega}$. In particular v is a continuous function on $\overline{\Omega}$, plurisubharmonic and real analytic on Ω. Moreover $u(z) = v(z)$ for $z \in \partial\Omega$. ∎

Before we give the construction of a holomorphic function with given integrals on almost all real directions, we need some additional results.

Lemma 4.5. *Let $\varepsilon \in (0,1)$, η be a probability measure on X. Let U be an open subset of X such that $\eta(U) > 0$. Moreover let T be a compact subset of $\overline{\Omega} \setminus X$, g be a complex continuous function on $\overline{\Omega}$ and H be a continuous, strictly positive function on X. Then there exists a holomorphic function f on W and an open subset V of U such that*

1. $\|f\|_T \leq \varepsilon$;
2. $-\varepsilon < \int_0^1 |(f+g)(\gamma(z,t))|^p\, dt - \int_0^1 |g(\gamma(z,t))|^p\, dt < H(z)$ *for* $z \in X$;
3. $\frac{H(z)}{5} < \int_0^1 |(f+g)(\gamma(z,t))|^p\, dt - \int_0^1 |g(\gamma(z,t))|^p\, dt$ *for* $z \in V$;
4. $\overline{V} \subset U$ *and* $\eta(\overline{V}) = \eta(V) > \frac{\eta(U)}{K+1}$.

Proof. Let $M := \sup_{z\in\partial\Omega} H(z)$. There exists $a, \widetilde{\epsilon} \in (0,1)$ such that for $z \in X$ we have $H(z) > aH(z) + 2\widetilde{\epsilon} > \frac{aH(z)}{4} - 2\widetilde{\epsilon} > \frac{H(z)}{5}$ and $-\varepsilon \leq -2\widetilde{\epsilon}$. Let $\delta \in (0,1)$ be such that $(1-\delta^p)M < \widetilde{\epsilon}$ and $(\delta^{-p}-1)M < \widetilde{\epsilon}$.

Due to Theorem 4.1 and Lemma 3.2 there exist $f_1, ..., f_K$, holomorphic functions on W, such that

1. $\|f_j\|_T \leq \varepsilon$;
2. $\frac{aH(z)}{4} < \max_{j=1,...,K} \int_0^1 |f_j(\gamma(z,t))|^p\, dt < aH(z)$;
3. $\int_0^1 |(f_j+g)(\gamma(z,t))|^p\, dt \geq -\widetilde{\epsilon} + \int_0^1 |g(\gamma(z,t))|^p\, dt + \delta^p \int_0^1 |f_j(\gamma(z,t))|^p\, dt$;
4. $\int_0^1 |(f_j+g)(\gamma(z,t))|^p\, dt \leq \widetilde{\epsilon} + \int_0^1 |g(\gamma(z,t))|^p\, dt + \delta^{-p} \int_0^1 |f_j(\gamma(z,t))|^p\, dt$.

There exists $j_0 \in \{1, ..., K\}$ and an open subset V_0 of U such that $\int_0^1 |f_{j_0}(\gamma(z,t))|^p\, dt = \max_{j=1,...,K} \int_0^1 |f_j(\gamma(z,t))|^p\, dt$ for $z \in V_0$ and $\eta(V_0) \geq \frac{1}{K}$. Let $f = f_{j_0}$. Now for $z \in V_0$ we obtain

$$\begin{aligned}\frac{aH(z)}{4} &< \int_0^1 |f(\gamma(z,t))|^p\, dt \leq \int_0^1 |(f+g)(\gamma(z,t))|^p\, dt \\ &\quad + \widetilde{\epsilon} - \int_0^1 |g(\gamma(z,t))|^p\, dt + (1-\delta^p)M.\end{aligned}$$

In particular

$$\frac{H(z)}{5} < \frac{aH(z)}{4} - 2\widetilde{\epsilon} \leq \int_0^1 |(f+g)(\gamma(z,t))|^p\, dt - \int_0^1 |g(\gamma(z,t))|^p\, dt.$$

In a similar way we obtain for $z \in X$

$$-\varepsilon \leq -2\widetilde{\epsilon} \leq \int_0^1 |(f+g)(\gamma(z,t))|^p\, dt - \int_0^1 |g(\gamma(z,t))|^p\, dt.$$

Moreover for $z \in X$ we have

$$\begin{aligned}aH(z) &> \int_0^1 |f(\gamma(z,t))|^p\, dt \geq \int_0^1 |(f+g)(\gamma(z,t))|^p\, dt \\ &\quad - \widetilde{\epsilon} - \int_0^1 |g(\gamma(z,t))|^p\, dt - (\delta^{-p}-1)M.\end{aligned}$$

In particular

$$H(z) > aH(z) + 2\widetilde{\varepsilon} \geq \int_0^1 |(f+g)(\gamma(z,t))|^p\,dt - \int_0^1 |g(\gamma(z,t))|^p\,dt.$$

There exists a set S closed in X and such that $S \subset V_0$, $\eta(S) > \frac{\eta(U)}{K+1}$. Let us denote $S^r := \{z \in X : \inf_{w\in U} \|z-w\| < r\}$. Now there exists $r_0 > 0$ such that $\overline{S^r} \subset V_0$ for $0 < r < r_0$. As $(0, r_0)$ is an uncountable set there exists $r_1 \in (0, r_0)$ such that $\mu(\partial S^{r_1}) = 0$. Now it is sufficient to choose $V = S^{r_1}$. In particular $\mu(\overline{V}) = \mu(\overline{V}) > \frac{\eta(U)}{K+1}$. ∎

Lemma 4.6. *Let $\varepsilon, a \in (0,1)$, η be a probability measure on X and T be a compact subset of $\overline{\Omega} \setminus X$. If H is a continuous strictly positive function on X and g is a complex continuous function on $\overline{\Omega}$ then there exists an open subset V of X and a holomorphic function f on W such that:*

1. *$|f(z)| \leq \varepsilon$ for $z \in T$;*
2. *$-\varepsilon < \int_0^1 |(g+f)(\gamma(z,t))|^p\,dt - \int_0^1 |g(\gamma(z,t))|^p\,dt < H(z)$ for $z \in \partial\Omega$;*
3. *$\int_0^1 |(g+f)(\gamma(z,t))|^p\,dt > aH(z) + \int_0^1 |g(\gamma(z,t))|^p\,dt$ for $z \in V$;*
4. *$\eta(\overline{V}) = \eta(V) > 1-\varepsilon$.*

Proof. First we prove that for $m \in \mathbb{N}$ and U an open subset of X, there exists an open subset V of $\partial\Omega$ and a holomorphic function f on W such that:

(a) $|f(z)| \leq \varepsilon$ for $z \in T$;

(b) $-\varepsilon < \int_0^1 |(g+f)(\gamma(z,t))|^p\,dt - \int_0^1 |g(\gamma(z,t))|^p\,dt < H(z)$ for $z \in X$;

(c) $\int_0^1 |(g+f)(\gamma(z,t))|^p\,dt > \left(1 - \frac{4^m}{5^m}\right) H(z) + \int_0^1 |g(\gamma(z,t))|^p\,dt$ for $z \in V$;

(d) $\overline{V} \subset U$ and $\eta(\overline{V}) = \eta(V) > \frac{\mu(U)}{(K+1)^m}$.

Due to Lemma 4.5 there exist $\{f_m\}_{m\in\mathbb{N}}$, a sequence of holomorphic functions on W, and a sequence $\{V_m\}_{m\in\mathbb{N}}$ of open subsets of X such that for $m \in \mathbb{N} \setminus \{0\}$

- $|f_m(z)| \leq \frac{\varepsilon}{2^m}$ for $z \in T$;
- $-\frac{\varepsilon}{2^m} < v_{m+1}(z) - v_m(z) < H_m(z)$ for $z \in X$;
- $v_{m+1}(z) - v_m(z) > \frac{1}{5} H_m(z)$ for $z \in V_m$;
- $\overline{V}_{m+1} \subset V_m \subset V_0 = U$ and $\eta(\overline{V}_m) = \eta(V_m) > \frac{\eta(V_{m-1})}{K+1}$,

where $v_m(z) = \int_0^1 \left|\left(g + \sum_{k=1}^{m-1} f_k\right)(\gamma(z,t))\right|^p dt$, $H_1 = H$ and $H_{m+1}(z) = H_m(z) - v_{m+1}(z) + v_m(z)$.

Let $f = \sum_{k=1}^{m} f_k$ and $V = V_m$. It is sufficient to prove the properties (b)-(c).

Observe that

$$H_m - H_1 = \sum_{k=1}^{m-1} (H_{k+1} - H_k) = -\sum_{k=1}^{m-1} (v_{k+1} - v_k) = -v_m + v_1$$

In particular $-\varepsilon < v_{m+1}(z) - v_1(z) < H_1(z) = H(z)$. Now it is sufficient to prove that for $z \in V_m$ we have

$$v_{m+1}(z) - v_1(z) > \left(1 - \frac{4^m}{5^m}\right) H(z). \tag{4.1}$$

For $m = 1$ inequality (4.1) is true. Now we assume that (4.1) holds for some $m \in \mathbb{N}$. We then obtain for $z \in V_{m+1}$

$$\begin{aligned} v_{m+2}(z) - v_1(z) &= v_{m+2}(z) - v_{m+1}(z) + v_{m+1}(z) - v_1(z) \\ &> \frac{H_{m+1}(z)}{5} + v_{m+1}(z) - v_1(z) > \\ &> \frac{H(z)}{5} + \frac{4}{5}\left(1 - \frac{4^m}{5^m}\right) H(z) = \left(1 - \frac{4^{m+1}}{5^{m+1}}\right) H(z) \end{aligned}$$

which proves (4.1) and gives the construction of an open subset V of $\partial\Omega$ and a holomorphic function f on W such that (a)-(d) holds.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of strictly positive numbers and m be a natural number sufficiently large so that $\left(1 - \frac{4^m}{5^m}\right) H(z) - \sum_{k=1}^{\infty} \varepsilon_k > aH(z)$ for $z \in X$ and $\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon$. Now using (a)-(d) we can construct a sequence $\{V_k\}_{k\in\mathbb{N}}$ of open subsets of X and a sequence $\{f_k\}_{k\in\mathbb{N}}$ of holomorphic functions on W such that

(e) $|f_k(z)| \le \varepsilon_k$ for $z \in T$;

(f) $-\varepsilon_k < \omega_{k+1}(z) - \omega_k(z) < H_k(z)$ for $z \in X$;

(g) $\omega_{k+1}(z) > \left(1 - \frac{4^m}{5^m}\right) H_k(z) + \omega_k(z)$ for $z \in V_k$;

(h) $\overline{V}_k \subset E \setminus \bigcup_{j=1}^{k-1} \overline{V}_j$ and $\eta(\overline{V}_k) = \eta(V_k) > \frac{1-\sum_{j=1}^{k-1}\eta(V_j)}{(K+1)^m}$,

where $\omega_k(z) = \int_0^1 \left| \left(g + \sum_{j=1}^{k} f_j\right)(\gamma(z,t)) \right|^p dt$, $H_1 = H$ and $H_{m+1}(z) = H_m(z) - \omega_{m+1}(z) + \omega_m(z)$. Observe that $H_m - H_1 = -\omega_m + \omega_1$.

As $\sum_{j=1}^{\infty} \eta(V_j) \le 1$ it holds that $\lim_{k\to\infty} \frac{1-\sum_{j=1}^{k-1}\eta(V_j)}{(K+1)^m} = 0$. In particular there exists $n \in \mathbb{N}$ sufficiently large so that $1 - \varepsilon < \sum_{j=1}^{n} \eta(V_j)$. Let us now define $V = \bigcup_{j=1}^{n} V_j$ and $f = \sum_{j=1}^{n} f_j$.

First we prove the properties (1),(4): $\eta(V) = \sum_{j=1}^{n} \eta(V_j) > 1 - \varepsilon$ and $|f(z)| \le \sum_{j=1}^{n} \varepsilon_j < \varepsilon$ for $z \in T$.

As $\omega_1 = H_n - H + \omega_n$, property (2) is also obvious: $-\varepsilon < -\sum_{j=1}^{n} \varepsilon_j < \omega_{n+1}(z) - \omega_1(z) < H(z)$ for $z \in X$.

Now let $z \in V$. There exists $k \in \{1, ..., n\}$ such that $z \in V_k$. As $H_k = H - \omega_k + \omega_1$, we obtain property (3):

$$\begin{aligned}
\omega_{n+1}(z) - \omega_1(z) &= \sum_{j=k+1}^{n} (\omega_{j+1}(z) - \omega_j(z)) + \omega_k(z) - \omega_1(z) + \omega_{k+1}(z) - \omega_k(z) \\
&> -\sum_{j=k+1}^{\infty} \varepsilon_j + \omega_k(z) - \omega_1(z) + \left(1 - \frac{4^m}{5^m}\right) H_k(z) \\
&\geq -\sum_{j=k+1}^{\infty} \varepsilon_j + \frac{4^m}{5^m} (\omega_k(z) - \omega_1(z)) + \left(1 - \frac{4^m}{5^m}\right) H(z) \\
&\geq -\sum_{j=1}^{\infty} \varepsilon_j + \left(1 - \frac{4^m}{5^m}\right) H(z) \geq aH(z). \qquad \blacksquare
\end{aligned}$$

Now we are ready to prove the following result.

Theorem 4.7. *Let $\varepsilon > 0$, η be a probability measure on X and T be a compact subset of $\overline{\Omega} \setminus X$. If H is a lower semicontinuous, strictly positive function on X, then there exists a function f holomorphic on Ω and continuous on $\overline{\Omega} \setminus X$, such that $\|f\|_T < \varepsilon$, $\int_0^1 |(f \circ \gamma)(z,t)|^p \, dt \leq H(z)$ for $z \in X$ and*

$$\eta\left(\left\{z \in X : \int_0^1 |(f \circ \gamma)(z,t)|^p \, dt = H(z)\right\}\right) = 1.$$

Proof. There exists a sequence of continuous, strictly positive functions $\{G_k\}_{k \in \mathbb{N}}$ such that $0 < G_j(z) < G_{j+1}(z) < ... \lim_{j \to \infty} G_j(z) = H(z)$. Let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence of compact subsets of $\overline{\Omega}$ such that $T_k \subset T_{k+1}$, the interior of T_k is contained in the interior of T_{k+1} and $\bigcup_{k=1}^{\infty} T_k = \overline{\Omega} \setminus X$. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of strictly positive numbers such that $\sum_{k=1}^{\infty} \varepsilon_k < 1$. Due to Lemma 4.6 there exists a sequence $\{V_k\}_{k \in \mathbb{N}}$ of open subsets of X and a sequence $\{f_k\}_{k \in \mathbb{N}}$ of holomorphic functions on W such that

(a) $|f_k(z)| \leq \varepsilon_k \varepsilon$ for $z \in T_k \cup T$;

(b) $\omega_{k+1}(z) - \omega_k(z) < H_k(z)$ for $z \in X$;

(c) $\omega_{k+1}(z) - \omega_k(z) > (1 - \varepsilon_k) H_k(z)$ for $z \in V_k$;

(d) $\eta(\overline{V}_k) = \eta(V_k) > 1 - \varepsilon_k$,

where $\omega_1 = 0$, $\omega_m(z) = \int_0^1 \left|\left(\sum_{j=1}^{m-1} f_j\right)(\gamma(z,t))\right|^p dt$, $H_1 = G_1$ and $H_{m+1}(z) = G_{m+1}(z) - \omega_{m+1}(z) + \omega_m(z)$.

Observe that for $z \in X$ we have

$$\omega_{k+2}(z) < H_{k+1}(z) + \omega_{k+1}(z) = G_{k+1}(z) - \omega_k(z) \leq G_{k+1}(z).$$

Moreover for $z \in V_{k+1}$ we may estimate

$$\begin{aligned}
\omega_{k+2}(z) &> \omega_{k+1}(z) + (1 - \varepsilon_{k+1}) H_{k+1}(z) \geq \varepsilon_{k+1} \omega_{k+1}(z) + (1 - \varepsilon_{k+1}) G_{k+1}(z) \\
&\geq (1 - 2\varepsilon_{k+1}) G_{k+1}(z).
\end{aligned}$$

Let $U_k := \bigcap_{m=k}^{\infty} V_m$ and $U = \bigcup_{k=1}^{\infty} U_k$. Observe that $\eta(U_k) \geq 1 - \sum_{m=k}^{\infty} \varepsilon_m$ and $\eta(U) = \lim_{m\to\infty} \eta(U_m) = 1$. If $z \in U$ then there exists $k \in \mathbb{N}$ such that $z \in U_k$. In particular $z \in V_{m+1}$ for $m \geq k$ and

$$G(z) = \lim_{m\to\infty} (1 - 2\varepsilon_{m+1}) G_{m+1}(z) \leq \lim_{m\to\infty} \omega_{m+1}(z) \leq \lim_{m\to\infty} G_m(z) = G(z).$$

Now we can define the function $f = \sum_{k=1}^{\infty} f_k$ which is holomorphic on Ω and continuous on $\overline{\Omega} \setminus X$, and observe that $\omega_\infty(z) \leq G(z)$ for $z \in X$ and $\omega_\infty(z) = G(z)$ for η-almost all $z \in X$, i.e. f has the required properties. ∎

As an application of Theorem 4.7 we prove the following description of exceptional sets (see 1.1) $E^p_\Omega(f)$.

Theorem 4.8. *Let $\varepsilon > 0$, T be a compact subset of $\overline{\Omega} \setminus X$ and η be a probability measure on X. If $E \subset X$ is a set of type G_δ then there exists a holomorphic function f such that (see 1.1) $\|f\|_T \leq \varepsilon$, $E^p_\Omega(f) \subset E$, $\eta(E \setminus E^p_\Omega(f)) = 0$ and $\int_{(X\setminus E)\times[0,1]} |f \circ \gamma|^p \, d\mathfrak{L}^{2N} < \infty$.*

Proof. Let σ be a natural measure on $\partial\Omega$. Due to [8, Theorem 2.6, Proposition 2.5] there exist sequences $\{D_i\}_{i\in\mathbb{N}}$, $\{T_i\}_{i\in\mathbb{N}}$ of compact subsets in X such that:

1. $\bigcup_{i\in\mathbb{N}} D_i = X \setminus E$ and $D_j \subset D_{j+1}$ for $j \in \mathbb{N}$;
2. $T_j \cap D_j = \emptyset$ for $j \in \mathbb{N}$;
3. $E = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} T_i$;
4. $\sigma(X \setminus (E \cup D_j)) \leq 2^{-j}$.

There exists a sequence of continuous functions $\{u_m\}_{m\in\mathbb{N}}$ such that $0 \leq u_m(z) \leq 1$, $u_m(z) = 0$ if and only if $z \in D_m$, and $u_m(z) = 1$ if and only if $z \in T_m$. Let $H(z) = 1 + \sum_{m=1}^{\infty} u_m(z)$. Observe that H is a strictly positive lower semicontinuous function on X and $\int_{X\setminus E} H d\sigma < \infty$. Now due to Theorem 4.7 there exists a function f, holomorphic on Ω and continuous on $\overline{\Omega} \setminus X$, such that $\|f\|_T \leq \varepsilon$, $\int_0^1 |(f \circ \gamma)(z,t)|^p \, dt \leq H(z)$ for $z \in X$ and

$$\eta\left(\left\{z \in X : \int_0^1 |(f \circ \gamma)(z,t)|^p \, dt = H(z)\right\}\right) = 1.$$

We may estimate

$$\int_{(X\setminus E)\times[0,1]} |f \circ \gamma|^p \, d\mathfrak{L}^{2N} = \int_{X\setminus E} \int_0^1 |(f \circ \gamma)(z,t)|^p \, dt d\sigma(z) \leq \int_{X\setminus E} H d\sigma < \infty.$$

Observe that $E^p_\Omega(f) \subset X$ since f is a continuous function on $\overline{\Omega} \setminus X$. If $z \in X \setminus E$ then there exists m_0 such that $z \in D_m$ for $m \geq m_0$ and $H(z) \leq 1 + \sum_{m=1}^{m_0} 1 < \infty$. In particular $E^p_\Omega(f) \subset E$. Moreover if $z \in E$ then $H(z) = \infty$ and therefore $\eta(E \setminus E^p_\Omega(f)) = 0$. ∎

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