Simultaneous and Converse Approximation Theorems in Weighted Orlicz Spaces

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Abstract

In the present work, we investigate the simultaneous and converse approximation by trigonometric polynomials of the functions in the Orlicz spaces with weights satisfying so called Muckenhoupt's A_p condition.

1 Introduction

A function Φ is called *Young function* if Φ is even, continuous, nonnegative in $\mathbb{R}$, increasing on $(0, \infty)$ such that

$$\Phi(0) = 0, \lim_{x\to\infty} \Phi(x) = \infty.$$

A nonnegative function $M : [0, \infty) \to [0, \infty)$ is said to be *quasiconvex* if there exist a convex Young function Φ and a constant $c_1 \geq 1$ such that

$$\Phi(x) \leq M(x) \leq \Phi(c_1 x), \quad \forall x \geq 0.$$

A Young function Φ is said to be satisfy Δ_2 *condition* ($\Phi \in \Delta_2$) if there is a constant $c_2 > 0$ such that

$$\Phi(2x) \leq c_2 \Phi(x)$$

for all $x \in \mathbb{R}$.

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Two Young functions Φ and Φ_1 are said to be *equivalent* (we shall write $\Phi \sim \Phi_1$) if there are $c_3, c_4 > 0$ such that

$$\Phi_1(c_3 x) \leq \Phi(x) \leq \Phi_1(c_4 x), \quad \forall x > 0.$$

Let $\mathbb{T} := [-\pi, \pi]$. A function $\omega : \mathbb{T} \to [0, \infty]$ will be called *weight* if ω is measurable and almost everywhere (a.e.) positive.

A 2π-periodic weight function ω belongs to the *Muckenhoupt class* A_p, $p > 1$, if

$$\sup_J \left(\frac{1}{|J|} \int_J \omega(x)\, dx \right) \left(\frac{1}{|J|} \int_J \omega^{-1/(p-1)}(x)\, dx \right)^{p-1} \leq c_5$$

with a finite constant c_5 independent of J, where J is any subinterval of $\mathbb{T}$.

Let M be a quasiconvex Young function. We denote by $\tilde{L}_{M,\omega}(\mathbb{T})$ the class of Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ satisfying the condition

$$\int_{\mathbb{T}} M(|f(x)|)\, \omega(x)\, dx < \infty.$$

The linear span of the *weighted Orlicz class* $\tilde{L}_{M,\omega}(\mathbb{T})$, denoted by $L_{M,\omega}(\mathbb{T})$, becomes a normed space with the *Orlicz* norm

$$\|f\|_{M,\omega} := \sup \left\{ \int_{\mathbb{T}} |f(x) g(x)|\, \omega(x)\, dx : \int_{\mathbb{T}} \tilde{M}(|g|)\, \omega(x)\, dx \leq 1 \right\},$$

where $\tilde{M}(y) := \sup_{x \geq 0} (xy - M(x))$, $y \geq 0$, is the *complementary function* of M.

For a quasiconvex function M we define the *indice* $p(M)$ of M as

$$\frac{1}{p(M)} := \inf \{ p : p > 0,\ M^p \text{ is quasiconvex} \}.$$

If $\omega \in A_{p(M)}$, then it can be easily seen that $L_{M,\omega}(\mathbb{T}) \subset L^1(\mathbb{T})$ and $L_{M,\omega}(\mathbb{T})$ becomes a Banach space with the Orlicz norm. The Banach space $L_{M,\omega}(\mathbb{T})$ is called *weighted Orlicz space*.

Detailed information about the classical Orlicz spaces, defined with respect to the convex Young function M, can be found in [24]. Since every convex function is quasiconvex, Orlicz spaces, considered in this work, are more general than the classical one and are investigated in the books [13] and [22].

For formulation of the new results we will begin with some required informations.

Let

$$f(x) \backsim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1.1}$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

be the *Fourier* and the *conjugate Fourier series* of $f \in L^1(\mathbb{T})$, respectively. In addition, we put

$$S_n(x,f) := \sum_{k=-n}^{n} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 1, 2, \ldots.$$

By $L_0^1(\mathbb{T})$ we denote the class of $L^1(\mathbb{T})$ functions f for which the constant term c_0 in (1.1) equals zero. If $\alpha > 0$, then *α-th integral* of $f \in L_0^1(\mathbb{T})$ is defined as

$$I_\alpha(x,f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k} \text{ and } \mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}.$$

For $\alpha \in (0,1)$ let

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x,f),$$

$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x,f)$$

if the right hand sides exist, where $r \in \mathbb{Z}^+ := \{1, 2, 3, \ldots\}$.

Throughout this work by $C(r)$, c, c_1, c_2, ..., $c_i(\alpha, \ldots)$, $c_j(\beta, \ldots)$, ... we denote the constants, which can be different in different places, such that they are absolute or depend only on the parameters given in their brackets.

Let $x, t \in \mathbb{R}$, $r \in \mathbb{R}^+ := (0, \infty)$ and let

$$\Delta_t^r f(x) := \sum_{k=0}^{\infty} (-1)^k [C_k^r] f(x + (r-k)t), \quad f \in L^1(\mathbb{T}), \tag{1.2}$$

where $[C_k^r] := \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k > 1$, $[C_k^r] := r$ for $k = 1$ and $[C_k^r] := 1$ for $k = 0$.

Since [34, p. 14]

$$|[C_k^r]| = \left|\frac{r(r-1)\ldots(r-k+1)}{k!}\right| \le \frac{c_6(r)}{k^{r+1}}, \quad k \in \mathbb{Z}^+$$

we have that

$$C(r) := \sum_{k=0}^{\infty} |[C_k^r]| < \infty,$$

and therefore $\Delta_t^r f(x)$ is defined a.e. on $\mathbb{R}$. Furthermore, the series in (1.2) converges absolutely a.e. and $\Delta_t^r f(x)$ is measurable [37].

If $r \in \mathbb{Z}^+$, then the fractional difference $\Delta_t^r f(x)$ coincides with usual forward difference. Now we define

$$\sigma_\delta^r f(x) := \frac{1}{\delta} \int_0^\delta |\Delta_t^r f(x)|\, dt, \quad f \in L_{M,\omega}(\mathbb{T}), \omega \in A_{p(M)}.$$

Let $M \in \triangle_2$, M^{θ} is quasiconvex for some $\theta \in (0,1)$ and $\omega \in A_{p(M)}$. Since the series in (1.2) converges absolutely a.e., we have $\sigma_{\delta}^{r} f(x) < \infty$ a.e. and using the boundedness of the Hardy-Littlewood Maximal function [13, Th. 6.4.4, p.250] in $L_{M,\omega}(\mathbb{T})$, $\omega \in A_{p(M)}$, we get

$$\|\sigma_{\delta}^{r} f(x)\|_{M,\omega} \leq c_7(M,r)\, \|f\|_{M,\omega} < \infty. \tag{1.3}$$

Hence, if $r \in \mathbb{R}^{+}$ and $\omega \in A_{p(M)}$ we can define the *r-th mean modulus of smoothness* of a function $f \in L_{M,\omega}(\mathbb{T})$ as

$$\Omega_r(f,h)_{M,\omega} := \sup_{|\delta| \leq h} \|\sigma_{\delta}^{r} f(x)\|_{M,\omega}. \tag{1.4}$$

If $r \in \mathbb{Z}^{+}$, $M(x) := x^p/p$, $1 < p < \infty$ and $\omega \in A_p$ then $\Omega_r(f,h)_{M,\omega}$ coincides with Ky's mean modulus of smoothness, defined in [26].

Remark 1. *Let $L_{M,\omega}(\mathbb{T})$ be a weighted Orlicz space with $M \in \triangle_2$ and $\omega \in A_{p(M)}$. If M^{θ} is quasiconvex for some $\theta \in (0,1)$, then r-th mean modulus of smoothness $\Omega_r(f,h)_{M,\omega}$, $r \in \mathbb{R}^{+}$, has the following properties:*
(i) $\Omega_r(f,h)_{M,\omega}$ is non-negative and non-decreasing function of $h \geq 0$.
(ii) $\Omega_r(f_1+f_2,\cdot)_{M,\omega} \leq \Omega_r(f_1,\cdot)_{M,\omega} + \Omega_r(f_2,\cdot)_{M,\omega}$.
(iii) $\lim_{h\to 0} \Omega_r(f,h)_{M,\omega} = 0$.

Let

$$E_n(f)_{M,\omega} := \inf_{T \in \mathcal{T}_n} \|f - T\|_{M,\omega}, \quad f \in L_{M,\omega}(\mathbb{T}), \quad n = 0,1,2,\ldots,$$

where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than n.

A polynomial $T_n(x,f) := T_n(x)$ of degree n is said to be a *near best approximant* of f if

$$\|f - T_n\|_{M,\omega} \leq c_8(M)\, E_n(f)_{M,\omega}, \quad n = 0,1,2,\ldots.$$

Let $W_{M,\omega}^{\alpha}(\mathbb{T})$, $\alpha > 0$, be the class of functions $f \in L_{M,\omega}(\mathbb{T})$ such that $f^{(\alpha)} \in L_{M,\omega}(\mathbb{T})$. $W_{M,\omega}^{\alpha}(\mathbb{T})$, $\alpha > 0$, becomes a Banach space with the norm

$$\|f\|_{W_{M,\omega}^{\alpha}(\mathbb{T})} := \|f\|_{M,\omega} + \left\|f^{(\alpha)}\right\|_{M,\omega}.$$

In this work we investigate the simultaneous and inverse theorems of approximation theory in the weighted Orlicz spaces $L_{M,\omega}(\mathbb{T})$.

Simultaneous approximation problems in nonweighted Orlicz spaces, defined with respect to the convex Young function M, was studied in [12]. In the weighted case, where the weighted Orlicz spaces are defined as the subclass of the measurable functions on $\mathbb{T}$ satisfying the condition

$$\int_{\mathbb{T}} M(|f(x)|\, \omega(x))\, dx < \infty,$$

some direct and inverse theorems of approximation theory were obtained in [17].

Some generalizations of these results to the weighted Lebesgue and Orlicz spaces defined on the curves of complex plane, were proved in [19], [21], [14], [15], [18], [16], [2] and [1].

Since Orlicz spaces considered by us in this work are more general than the Orlicz space studied in the above mentioned works, the results obtained in this paper are new also in the nonweighted cases.

The similar problems in the weighted Lebesgue spaces $L_p(\mathbb{T},\omega)$, under different conditions on the weight function ω, were investigated in the works [11], [25], [6], [30], [29], [31], [8], [10] and also in the books [39], [7], [9], [32].

Our new results are the following.

Theorem 1. *Let M^θ be quasiconvex for some $\theta \in (0,1)$, $M \in \Delta_2$, $\omega \in A_{p(M)}$ and $f \in W^\alpha_{M,\omega}(\mathbb{T})$, $\alpha \in \mathbb{R}^+_0 := [0,\infty)$. If $T_n \in \mathcal{T}_n$ is a near best approximant of f, then*

$$\left\| f^{(\alpha)} - T_n^{(\alpha)} \right\|_{M,\omega} \le cE_n\left(f^{(\alpha)}\right)_{M,\omega}, \quad n = 0,1,2,\ldots \tag{1.5}$$

with a constant $c = c(M,\alpha) > 0$.

This simultaneous approximation theorem in case of $\alpha \in \mathbb{Z}^+$ for Lebesgue spaces $L^p(\mathbb{T})$, $1 \le p \le \infty$, was proved in [5]. In the classical Orlicz spaces $L_M(\mathbb{T})$ some results about simultaneous trigonometric and algebraic approximation of type (1.5), where $E_n\left(f^{(\alpha)}\right)_{M,1}$ is replaced by the modulus of smoothness of $f^{(\alpha)}$, $\alpha \in \mathbb{Z}^+$, were obtained in [33] and [12].

Theorem 2. *If M^θ is quasiconvex for some $\theta \in (0,1)$, $M \in \Delta_2$, $\omega \in A_{p(M)}$ and $f \in W^r_{M,\omega}(\mathbb{T})$, $r \in \mathbb{R}^+$, then*

$$\Omega_r(f,h)_{M,\omega} \le ch^r \left\| f^{(r)} \right\|_{M,\omega}, \quad 0 < h \le \pi$$

with a constant $c = c(M,r) > 0$.

In the case of $r \in \mathbb{Z}^+$, for the usual non weighted modulus of smoothness defined in the Lebesgue spaces $L^p(\mathbb{T})$, $1 \le p \le \infty$, this inequality was proved in [28] and for the general case $r \in \mathbb{R}^+$ was obtained by Butzer, Dyckhoff, Görlich and Stens in [4] (See also Taberski [37]). In case of $r \in \mathbb{Z}^+$, $\omega \in A_p$, $1 < p < \infty$, this inequality in the weighted Lebesgue spaces $L^p(\mathbb{T},\omega)$ was proved in [26]. For the classical Orlicz spaces similar result in nonweighted and weighted cases were obtained in [33] and [17] (see also [3]).

The following converse theorem holds:

Theorem 3. *Let $L_{M,\omega}(\mathbb{T})$ be a weighted Orlicz space with $M \in \triangle_2$ and $\omega \in A_{p(M)}$. If M^θ is quasiconvex for some $\theta \in (0,1)$ and $f \in L_{M,\omega}(\mathbb{T})$, then for a given $r \in \mathbb{R}^+$*

$$\Omega_r(f,\pi/(n+1))_{M,\omega} \le \frac{c}{(n+1)^r} \sum_{\nu=0}^{n} (\nu+1)^{r-1} E_\nu(f)_{M,\omega}, \quad n = 0,1,2,\ldots$$

with a constant $c = c(M,r) > 0$.

In the space $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, this inequality was proved in [37]. In case of $r \in \mathbb{Z}^+$ this theorem in the spaces $L^p(\mathbb{T},\omega)$, $1 < p < \infty$, $\omega \in A_p$, was proved by Ky in [26]. For the positive and even integer r this theorem in the spaces $L^p(\mathbb{T},\omega)$, $1 < p < \infty$, $\omega \in A_p$, by using Butzer-Wehrens's type modulus of smoothness was obtained in [11]. In case of $r \in \mathbb{Z}^+$ for weighted Orlicz spaces $L_M(\mathbb{T},\omega)$, $\omega \in A_p$, similar results were obtained in [17] and [3].

Theorem 4. *Let M^θ be quasiconvex for some $\theta \in (0,1)$, $M \in \Delta_2$ and $\omega \in A_{p(M)}$. If*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{M,\omega} < \infty$$

for some $\alpha \in (0,\infty)$, then $f \in W^\alpha_{M,\omega}(\mathbb{T})$ and

$$E_n\left(f^{(\alpha)}\right)_{M,\omega} \leq c\left\{(n+1)^\alpha E_n(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{M,\omega}\right\} \tag{1.6}$$

with a constant $c = c(M,\alpha) > 0$.

In the space $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, this inequality for $\alpha \in \mathbb{Z}^+$ was proved in [35]. When $\alpha \in \mathbb{R}^+$ in the classical Orlicz spaces $L_M(\mathbb{T})$, similar inequality was proved in [20]. In case of $\alpha \in \mathbb{Z}^+$, in $L^p(\mathbb{T},\omega)$, $1 < p < \infty$, $\omega \in A_p$, an inequality of type (1.6) was recently proved in [23].

Corollary 1. *Let M^θ be quasiconvex for some $\theta \in (0,1)$, $M \in \Delta_2$, $\omega \in A_{p(M)}$ and $r > 0$. If*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{M,\omega} < \infty$$

for some $\alpha \in (0,\infty)$, then $f \in W^\alpha_{M,\omega}(\mathbb{T})$ and for $n = 0,1,2,\ldots$

$$\Omega_r\left(f^{(\alpha)}, \frac{\pi}{n+1}\right)_{M,\omega} \leq c\left\{\frac{1}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{\alpha+r-1} E_\nu(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{M,\omega}\right\}$$

with a constant $c = c(M,\alpha,r) > 0$.

In cases of $\alpha, r \in \mathbb{Z}^+$ and $\alpha, r \in \mathbb{R}^+$, this corollary in the spaces $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, was proved in [38] (See also [35]) and in [36], respectively. In the case of $\alpha \in \mathbb{R}^+$ and $r \in \mathbb{Z}^+$, in the classical Orlicz spaces $L_M(\mathbb{T})$ the similar result was obtained [20]. For the weighted Lebesgue spaces $L^p(\mathbb{T},\omega)$, $1 < p < \infty$, when $\omega \in A_p$ and $\alpha, r \in \mathbb{Z}^+$, similar type inequality was obtained in [23].

2 Auxiliary Facts

We begin with

Lemma 1. *Let $M \in \triangle_2$, $\omega \in A_{p(M)}$ and $r \in \mathbb{R}^+$. If M^θ is quasiconvex for some $\theta \in (0,1)$ and $T_n \in \mathcal{T}_n$, $n \geq 1$, then there exists a constant $c > 0$ depends only on r and M such that*

$$\Omega_r (T_n, h)_{M,\omega} \leq ch^r \left\| T_n^{(r)} \right\|_{M,\omega}, \quad 0 < h \leq \pi/n.$$

Proof. Since

$$\Delta_t^r T_n \left(x - \frac{r}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i \sin \frac{t}{2} \nu \right)^r c_\nu e^{i\nu x},$$

$$\Delta_t^{[r]} T_n^{(r-[r])} \left(x - \frac{[r]}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i \sin \frac{t}{2} \nu \right)^{[r]} (i\nu)^{r-[r]} c_\nu e^{i\nu x}$$

with $\mathbb{Z}_n^* := \{\mp 1, \mp 2, \ldots, \mp n\}$, $[r] \equiv$ integer part of r, putting

$$\varphi(z) := \left(2i \sin \frac{t}{2} z \right)^{[r]} (iz)^{r-[r]}, \quad g(z) := \left(\frac{2}{z} \sin \frac{t}{2} z \right)^{r-[r]}, \quad -n \leq z \leq n,$$

$$g(0) := t^{r-[r]},$$

we get

$$\Delta_t^{[r]} T_n^{(r-[r])} \left(x - \frac{[r]}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu x}, \quad \Delta_t^r T_n \left(x - \frac{r}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) g(\nu) c_\nu e^{i\nu x}.$$

Taking into account the fact that [37]

$$g(z) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi z/n}$$

uniformly in $[-n, n]$, with $d_0 > 0$, $(-1)^{k+1} d_k \geq 0$, $d_{-k} = d_k$ $(k = 1, 2, \ldots)$, we have

$$\Delta_t^r T_n (\cdot) = \sum_{k=-\infty}^{\infty} d_k \Delta_t^{[r]} T_n^{(r-[r])} \left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t \right).$$

Consequently we get

$$\left\| \frac{1}{\delta} \int_0^\delta |\Delta_t^r T_n (\cdot)| \, dt \right\|_{M,\omega} = \left\| \frac{1}{\delta} \int_0^\delta \left| \sum_{k=-\infty}^{\infty} d_k \Delta_t^{[r]} T_n^{(r-[r])} \left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t \right) \right| dt \right\|_{M,\omega}$$

$$\leq \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta} \int_0^\delta \left| \Delta_t^{[r]} T_n^{(r-[r])} \left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t \right) \right| dt \right\|_{M,\omega}$$

and since [39, p.103]

$$\Delta_t^{[r]} T_n^{(r-[r])}(\cdot) = \int_0^t \cdots \int_0^t T_n^{(r)}\left(\cdot + t_1 + \ldots + t_{[r]}\right) dt_1 \ldots dt_{[r]}$$

we find

$$\Omega_r(T_n, h)_{M,\omega} \le \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta} \int_0^{\delta} \left| \Delta_t^{[r]} T_n^{(r-[r])}\left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t\right)\right| dt \right\|_{M,\omega}$$

$$= \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta} \int_0^{\delta} \left| \int_0^t \cdots \int_0^t T_n^{(r)}\left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t + t_1 + \ldots + t_{[r]}\right) dt_1 \ldots dt_{[r]} \right| dt \right\|_{M,\omega}$$

$$\le h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta} \int_0^{\delta} \frac{1}{\delta^{[r]}} \int_0^{\delta} \cdots \int_0^{\delta} \left| T_n^{(r)}\left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t + t_1 + \ldots + t_{[r]}\right)\right| \times \right.$$

$$\left. \times dt_1 \ldots dt_{[r]} dt \right\|_{M,\omega}$$

$$\le h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta^{[r]}} \int_0^{\delta} \cdots \int_0^{\delta} \left\{ \frac{1}{\delta} \int_0^{\delta} \left| T_n^{(r)}\left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t + t_1 + \ldots + t_{[r]}\right)\right| \times \right.$$

$$\left. \times dt \right\} dt_1 \ldots dt_{[r]} \Big\|_{M,\omega}$$

$$\le c_9(M, r)\, h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\delta} \int_0^{\delta} \left| T_n^{(r)}\left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t\right)\right| dt \right\|_{M,\omega}$$

$$\le c_9(M, r)\, h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\| \frac{1}{\frac{r-[r]}{2}\delta} \int_{\cdot + \frac{k\pi}{n}}^{\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2}\delta} \left| T_n^{(r)}(u)\right| du \right\|_{M,\omega}.$$

On the other hand [37]

$$\sum_{k=-\infty}^{\infty} |d_k| < 2g(0) = 2t^{r-[r]}, \quad 0 < t \le \pi/n$$

and for $0 < t < \delta < h \le \pi/n$ we have

$$\sum_{k=-\infty}^{\infty} |d_k| < 2h^{r-[r]}.$$

Therefore the boundedness of Hardy-Littlewood Maximal function in $L_{M,\omega}(\mathbb{T})$ implies that

$$\Omega_r(T_n, h)_{M,\omega} \le c_{10}(M, r)\, h^r \left\| T_n^{(r)} \right\|_{M,\omega}.$$

Further, by the similar way for $0 < -h \le \pi/n$, the same inequality also holds and the proof of Lemma 1 is completed. ∎

Lemma 2. *Let $M \in \triangle_2$, M^θ is quasiconvex for some $\theta \in (0,1)$ and $\omega \in A_{p(M)}$. If $T_n \in \mathcal{T}_n$ and $\alpha > 0$, then there exists a constant $c > 0$ depending only on α and M such that*

$$\left\| T_n^{(\alpha)} \right\|_{M,\omega} \leq c n^\alpha \left\| T_n \right\|_{M,\omega}.$$

Proof. Since $M \in \triangle_2$, M^θ is quasiconvex for some $\theta \in (0,1)$ and $\omega \in A_{p(M)}$ we have [3]

$$\left\| S_n f \right\|_{M,\omega} \leq c_{11}(M) \left\| f \right\|_{M,\omega},$$

$$\left\| \tilde{f} \right\|_{M,\omega} \leq c_{12}(M) \left\| f \right\|_{M,\omega}.$$

Following the method given in [27] we obtain the required result. ∎

Definition 1. *For $f \in L_{M,\omega}(\mathbb{T})$, $\delta > 0$ and $r = 1,2,3,\ldots$, the Peetre K-functional is defined as*

$$K\left(\delta, f; L_{M,\omega}(\mathbb{T}), W^r_{M,\omega}(\mathbb{T})\right) := \inf_{g \in W^r_{M,\omega}(\mathbb{T})} \left\{ \left\| f - g \right\|_{M,\omega} + \delta \left\| g^{(r)} \right\|_{M,\omega} \right\}. \quad (2.1)$$

Lemma 3. *Let $M \in \triangle_2$, M^θ is quasiconvex for some $\theta \in (0,1)$ and $\omega \in A_{p(M)}$. If $f \in L_{M,\omega}(\mathbb{T})$ and $r = 1,2,3,\ldots$, then*

(i) the K-functional (2.1) and the modulus (1.4) are equivalent and

(ii) there exists a constant $c > 0$ depending only on r and M such that

$$E_n(f)_{M,\omega} \leq c \Omega_r \left(f, \frac{1}{n} \right)_{M,\omega}.$$

Proof. (i) can be proved by the similar way to that of Theorem 1 in [26] and later (ii) is proved by standard way (see for example, [17]). ∎

3 Proof of the results

Proof of Theorem 1. We set

$$W_n(f) := W_n(x,f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x,f), \quad n = 0,1,2,\ldots.$$

Since

$$W_n(\cdot, f^{(\alpha)}) = W_n^{(\alpha)}(\cdot, f)$$

we have

$$\begin{aligned} \left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot,f) \right\|_{M,\omega} \leq & \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)}) \right\|_{M,\omega} + \\ & \left\| T_n^{(\alpha)}(\cdot, W_n(f)) - T_n^{(\alpha)}(\cdot,f) \right\|_{M,\omega} + \\ & \left\| W_n^{(\alpha)}(\cdot,f) - T_n^{(\alpha)}(\cdot, W_n(f)) \right\|_{M,\omega} =: I_1 + I_2 + I_3. \end{aligned}$$

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n to f in $L_{M,\omega}(\mathbb{T},\omega)$. In this case, from the boundedness of W_n in $L_{M,\omega}(\mathbb{T},\omega)$ we have

$$I_1 \le \left\| f^{(\alpha)}(\cdot) - T_n^*(\cdot, f^{(\alpha)}) \right\|_{M,\omega} + \left\| T_n^*(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)}) \right\|_{M,\omega}$$

$$\le c_{13}(M)\, E_n\left(f^{(\alpha)}\right)_{M,\omega} + \left\| W_n(\cdot, T_n^*(f^{(\alpha)}) - f^{(\alpha)}) \right\|_{M,\omega} \le c_{14}(M,\alpha)\, E_n\left(f^{(\alpha)}\right)_{M,\omega}.$$

From Lemma 2 we get

$$I_2 \le c_{15}(M,\alpha)\, n^{\alpha} \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f) \right\|_{M,\omega}$$

and

$$\begin{aligned} I_3 &\le c_{16}(M,\alpha)\,(2n)^{\alpha} \left\| W_n(\cdot, f) - T_n(\cdot, W_n(f)) \right\|_{M,\omega} \\ &\le c_{17}(M,\alpha)\,(2n)^{\alpha} E_n(W_n(f))_{M,\omega}. \end{aligned}$$

Now we have

$$\begin{aligned} \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f) \right\|_{M,\omega} &\le \left\| T_n(\cdot, W_n(f)) - W_n(\cdot, f) \right\|_{M,\omega} \\ &+ \left\| W_n(\cdot, f) - f(\cdot) \right\|_{M,\omega} + \left\| f(\cdot) - T_n(\cdot, f) \right\|_{M,\omega} \\ &\le c_{18}(M)\, E_n(W_n(f))_{M,\omega} + c_{19}(M)\, E_n(f)_{M,\omega} + c_{20}(M)\, E_n(f)_{M,\omega}. \end{aligned}$$

Since

$$E_n(W_n(f))_{M,\omega} \le c_{21}(M)\, E_n(f)_{M,\omega},$$

we get

$$\begin{aligned} \left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{M,\omega} &\le c_{14}(M,\alpha)\, E_n\left(f^{(\alpha)}\right)_{M,\omega} + c_{22}(M)\, n^{\alpha} E_n(W_n(f))_{M,\omega} \\ &+ c_{23}(M)\, n^{\alpha} E_n(f)_{M,\omega} + c_{17}(M,\alpha)\,(2n)^{\alpha} E_n(W_n(f))_{M,\omega} \\ &\le c_{24}(M,\alpha)\, E_n\left(f^{(\alpha)}\right)_{M,\omega} + c_{25}(M)\, n^{\alpha} E_n(f)_{M,\omega}. \end{aligned}$$

Since [3]

$$E_n(f)_{M,\omega} \le \frac{c_{26}(M,\alpha)}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{M,\omega}, \tag{3.1}$$

we obtain

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{M,\omega} \le c_{27}(M,\alpha)\, E_n\left(f^{(\alpha)}\right)_{M,\omega}$$

and the proof is completed. ∎

Proof of Theorem 2. Let $T_n \in \mathcal{T}_n$ be the trigonometric polynomial of best approximation of f in $L_{M,\omega}(\mathbb{T})$ metric. By Remark 1(ii), Lemma 1 and (1.3) we get

$$\Omega_r(f,h)_{M,\omega} \le \Omega_r(T_n,h)_{M,\omega} + \Omega_r(f - T_n, h)_{M,\omega}$$

$$\le c_{10}(M,r)\, h^r \left\| T_n^{(r)} \right\|_{M,\omega} + c_7(M,r)\, E_n(f)_{M,\omega}, \quad 0 < h \le \pi/n.$$

Using (3.1), Lemma 3 (ii) and

$$\Omega_l(f,h)_{M,\omega} \le ch^l \left\| f^{(l)} \right\|_{M,\omega}, \quad f \in W^l_{M,\omega}(\mathbb{T}), l = 1,2,3,\ldots,$$

which can be showed using the judgements given in [26, Theorem 1], we have

$$E_n(f)_{M,\omega} \le \frac{c_{26}(M,r)}{(n+1)^{r-[r]}} E_n\left(f^{(r-[r])}\right)_{M,\omega} \le \frac{c_{28}(M,r)}{(n+1)^{r-[r]}} \Omega_{[r]}\left(f^{(r-[r])}, \frac{2\pi}{n+1}\right)_{M,\omega}$$

$$\le \frac{c_{29}(M,r)}{(n+1)^{r-[r]}} \left(\frac{2\pi}{n+1}\right)^{[r]} \left\| f^{(r)} \right\|_{M,\omega}.$$

On the other hand, by Theorem 1 we find

$$\left\| T_n^{(r)} \right\|_{M,\omega} \le \left\| T_n^{(r)} - f^{(r)} \right\|_{M,\omega} + \left\| f^{(r)} \right\|_{M,\omega}$$

$$\le c_{27}(M,r)\, E_n\left(f^{(r)}\right)_{M,\omega} + \left\| f^{(r)} \right\|_{M,\omega} \le c_{30}(M,r) \left\| f^{(r)} \right\|_{M,\omega}.$$

Then choosing h with $\pi/(n+1) < h \le \pi/n$, $(n = 1,2,3,\ldots)$, we obtain

$$\Omega_r(f,h)_{M,\omega} \le c_{31}(M,r)\, h^r \left\| f^{(r)} \right\|_{M,\omega}$$

and we are done. ∎

Proof of Theorem 3. Let $T_n \in \mathcal{T}_n$ be the best approximating polynomial of $f \in L_{M,\omega}(\mathbb{T})$, $\omega \in A_{p(M)}$ and let $m \in \mathbb{Z}^+$. Then by Remark 1(ii) and (1.3) we have

$$\Omega_r(f,\pi/(n+1))_{M,\omega} \le \Omega_r(f - T_{2^m}, \pi/(n+1))_{M,\omega} + \Omega_r(T_{2^m}, \pi/(n+1))_{M,\omega}$$

$$\le c_7(M,r)\, E_{2^m}(f)_{M,\omega} + \Omega_r(T_{2^m}, \pi/(n+1))_{M,\omega}.$$

Since

$$\Omega_r(T_{2^m}, \pi/(n+1))_{M,\omega} \le c_{54}(M,r) \left(\frac{\pi}{n+1}\right)^r \left\| T_{2^m}^{(r)} \right\|_{M,\omega}, \quad n+1 \ge 2^m$$

and

$$T_{2^m}^{(r)}(x) = T_1^{(r)}(x) + \sum_{\nu=0}^{m-1} \left\{ T_{2^{\nu+1}}^{(r)}(x) - T_{2^\nu}^{(r)}(x) \right\},$$

we have

$$\Omega_r(T_{2^m}, \pi/(n+1))_{M,\omega} \le$$

$$c_{10}(M,r) \left(\frac{\pi}{n+1}\right)^r \left\{ \left\| T_1^{(r)} \right\|_{M,\omega} + \sum_{\nu=0}^{m-1} \left\| T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)} \right\|_{M,\omega} \right\}.$$

By Lemma 2 we find

$$\left\| T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)} \right\|_{M,\omega} \le c_{32}(M,r)\, 2^{\nu r} \left\| T_{2^{\nu+1}} - T_{2^\nu} \right\|_{M,\omega} \le c_{32}(M,r)\, 2^{\nu r+1} E_{2^\nu}(f)_{M,\omega}$$

and
$$\left\|T_1^{(r)}\right\|_{M,\omega}=\left\|T_1^{(r)}-T_0^{(r)}\right\|_{M,\omega}\leq c_{33}\left(M,r\right)E_0\left(f\right)_{M,\omega}.$$
Hence
$$\Omega_r\left(T_{2^m},\pi/\left(n+1\right)\right)_{M,\omega}\leq$$
$$c_{34}\left(M,r\right)\left(\frac{\pi}{n+1}\right)^r\left\{E_0\left(f\right)_{M,\omega}+\sum_{\nu=0}^{m-1}2^{(\nu+1)r}E_{2^\nu}\left(f\right)_{M,\omega}\right\}.$$
It is easily seen that
$$2^{(\nu+1)r}E_{2^\nu}\left(f\right)_{M,\omega}\leq c_{35}\left(r\right)\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{r-1}E_\mu\left(f\right)_{M,\omega},\quad \nu=1,2,3,\ldots. \tag{3.2}$$

Therefore,
$$\begin{aligned}&\Omega_r\left(T_{2^m},\pi/\left(n+1\right)\right)_{M,\omega}\\ \leq\ & c_{34}\left(M,r\right)\left(\frac{\pi}{n+1}\right)^r\left\{E_0\left(f\right)_{M,\omega}+2^rE_1\left(f\right)_{M,\omega}+\right.\\ &\left.\qquad c_{35}\left(r\right)\sum_{\nu=1}^{m}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{r-1}E_\mu\left(f\right)_{M,\omega}\right\}\\ \leq\ & c_{36}\left(M,r\right)\left(\frac{\pi}{n+1}\right)^r\left\{E_0\left(f\right)_{M,\omega}+\sum_{\mu=1}^{2^m}\mu^{r-1}E_\mu\left(f\right)_{M,\omega}\right\}\\ \leq\ & c_{36}\left(M,r\right)\left(\frac{\pi}{n+1}\right)^r\sum_{\nu=0}^{2^m-1}\left(\nu+1\right)^{r-1}E_\nu\left(f\right)_{M,\omega}.\end{aligned}$$
If we choose $2^m\leq n+1\leq 2^{m+1}$, then
$$\Omega_r\left(T_{2^m},\pi/\left(n+1\right)\right)_{M,\omega}\leq\frac{c_{36}\left(M,r\right)}{\left(n+1\right)^r}\sum_{\nu=0}^{n}\left(\nu+1\right)^{r-1}E_\nu\left(f\right)_{M,\omega},$$
$$E_{2^m}\left(f\right)_{M,\omega}\leq E_{2^{m-1}}\left(f\right)_{M,\omega}\leq\frac{c_{37}\left(M,r\right)}{\left(n+1\right)^r}\sum_{\nu=0}^{n}\left(\nu+1\right)^{r-1}E_\nu\left(f\right)_{M,\omega}$$
and Theorem 3 is proved. ∎

Proof of Theorem 4. If T_n is the best approximating trigonometric polynomial of f, then by Lemma 2
$$\left\|T_{2^{m+1}}^{(\alpha)}-T_{2^m}^{(\alpha)}\right\|_{M,\omega}\leq c_{38}\left(M,\alpha\right)2^{(m+1)\alpha}E_{2^m}\left(f\right)_{M,\omega}$$
and hence by this inequality, (3.2) and hypothesis of Theorem 4 we have
$$\sum_{m=1}^{\infty}\left\|T_{2^{m+1}}-T_{2^m}\right\|_{W^\alpha_{M,\omega}(\mathbb{T})}=\sum_{m=1}^{\infty}\left\|T_{2^{m+1}}-T_{2^m}\right\|_{M,\omega}+\sum_{m=1}^{\infty}\left\|T_{2^{m+1}}^{(\alpha)}-T_{2^m}^{(\alpha)}\right\|_{M,\omega}$$

$$= c_{39}(M,\alpha)\sum_{m=1}^{\infty}2^{(m+1)\alpha}E_{2^m}(f)_{M,\omega} \leq c_{40}(M,\alpha)\sum_{m=1}^{\infty}\sum_{j=2^{m-1}+1}^{2^m} j^{\alpha-1}E_j(f)_{M,\omega}$$

$$\leq c_{41}(M,\alpha)\sum_{j=2}^{\infty} j^{\alpha-1}E_j(f)_{M,\omega} < \infty.$$

Therefore,

$$\sum_{m=1}^{\infty}\|T_{2^{m+1}} - T_{2^m}\|_{W^{\alpha}_{M,\omega}(\mathbb{T})} < \infty,$$

which implies that $\{T_{2^m}\}$ is a Cauchy sequence in $W^{\alpha}_{M,\omega}(\mathbb{T})$. Since $T_{2^m} \to f$ in the Banach space $L_{M,\omega}(\mathbb{T})$, we have $f \in W^{\alpha}_{M,\omega}(\mathbb{T})$.

It is clear that

$$E_n\left(f^{(\alpha)}\right)_{M,\omega} \leq \left\|f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{M,\omega}$$

$$\leq \left\|S_{2^{m+2}}f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{M,\omega} + \sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}f^{(\alpha)} - S_{2^k}f^{(\alpha)}\right\|_{M,\omega}. \tag{3.3}$$

By Lemma 2

$$\left\|S_{2^{m+2}}f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{M,\omega} \leq c_{42}(M,\alpha)\,2^{(m+2)\alpha}E_n(f)_{M,\omega}$$

$$\leq c_{43}(M,\alpha)(n+1)^{\alpha}E_n(f)_{M,\omega} \tag{3.4}$$

for $2^m < n < 2^{m+1}$.

On the other hand, by Lemma 2 and by (3.2)

$$\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}f^{(\alpha)} - S_{2^k}f^{(\alpha)}\right\|_{M,\omega} \leq c_{44}(M,\alpha)\sum_{k=m+2}^{\infty}2^{(k+1)\alpha}E_{2^k}(f)_{M,\omega}$$

$$\leq c_{45}(M,\alpha)\sum_{k=m+2}^{\infty}\sum_{\mu=2^{k-1}+1}^{2^k}\mu^{\alpha-1}E_{\mu}(f)_{M,\omega} = c_{46}(M,\alpha)\sum_{\nu=2^{m+1}+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{M,\omega}$$

$$\leq c_{46}(M,\alpha)\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{M,\omega}. \tag{3.5}$$

Now using the relations (3.4) and (3.5) in (3.3) we obtain the required inequality. ■

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