An approach to Gelfand theory for arbitrary Banach algebras*

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Abstract

Let A be a Banach algebra. We say that a pair $(\mathcal{G}, \mathcal{U})$ is a (topologically Gelfand theory) Gelfand theory for A if the following hold: (G1) $\mathcal{U}$ is a C*-algebra and $\mathcal{G} : A \to \mathcal{U}$ is a homomorphism which induces the (homeomorphism) bijection $\pi \mapsto \pi \circ \mathcal{G}$ from $\widehat{\mathcal{U}}$ onto $\widehat{\mathcal{A}}$; (G2) for every maximal modular left ideal L, $\mathcal{G}(A) \nsubseteq L$. We show that this definition is equivalent to the usual definition of gelfand theory in the commutative case. We prove that many properties of Gelfand theory of commutative Banach algebras remain true for Gelfand theories of arbitrary Banach algebras. We show that unital homogeneous Banach algebras and postliminal C*-algebras have unique Gelfand theories (up to an appropriate notion of uniqueness).

1 Introduction

Let A be a commutative Banach algebra with the character space Φ_A (the set of characters) and the maximal ideal space $\mathcal{M}_A$ (the set of maximal modular ideals). The Gelfand transform $\mathcal{G}_A$ maps A into the commutative C*-algebra $C_0(\Phi_A)$ as follows:

$$\mathcal{G}_A : A \longrightarrow C_0(\Phi_A),$$
$$\mathcal{G}_A(a)(h) = h(a).$$

The triple $(A, \mathcal{G}_A, C_0(\Phi_A))$ shows that the Gelfand transform establishes a connection between the abstract commutative Banach algebra A and the concrete

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commutative C*-algebra $C_0(\Phi_A)$. Moreover, the Gelfand transform induces a bijection between $\mathcal{M}_A$ and $\mathcal{M}_{C_\circ(A)}$ as follows:

$$\mathcal{M}_{C_\circ(\Phi_A)} \longrightarrow \mathcal{M}_A,$$

$$L \mapsto \mathcal{G}_A^{-1}(L).$$

Let A be an arbitrary Banach algebra. The maximal ideal space is replaced by the set of maximal modular left ideals, which is again denoted by $\mathcal{M}_A$. Based on these facts, the following notion of a Gelfand theory was introduced in [3]:

A pair $(\mathcal{G},\mathcal{U})$ is called a Gelfand theory for a Banach algebra A if it satisfies the following set of axioms

(G1) $\mathcal{U}$ is a C*-algebra and $\mathcal{G} : A \longrightarrow \mathcal{U}$ is a homomorphism.

(G2) The assignment $L \mapsto \mathcal{G}^{-1}(L)$ is a bijection from $\mathcal{M}_\mathcal{U}$ onto $\mathcal{M}_A$.

(G3) For each $L \in \mathcal{M}_\mathcal{U}$, the linear map $\mathcal{G}_L : A/\mathcal{G}^{-1}(L) \longrightarrow \mathcal{U}/L$ induced by $\mathcal{G}$ has dense range.

It is well known that in the commutative case, characters are exactly the irreducible representations. We rise this fact to give a new definition of the Gelfand theory for arbitrary Banach algebras. We show that most results of [3] are also proved with our axioms.

2 preliminaries

We denote by $B(X)$ ($K(X)$), the algebra of all bounded linear operators (compact operators) on the Banach space X. Let A be Banach algebra. A representation of A on X is a homomorphism form A into $B(X)$. The representation $\pi : A \longrightarrow B(X)$ is called irreducible if $\{0\}$ and X are the only invariant subspaces for π. A representation $\pi : A \longrightarrow B(X)$ is called topologically representation if $\{0\}$ and X are the only closed invariant subspaces of. We recall that representations $\pi_1 : A \longrightarrow B(X_1)$ and $\pi_2 : A \longrightarrow B(X_2)$ are equivalent ($\pi_1 \cong \pi_2$) if there exists an isomorphism $T : X_1 \longrightarrow X_2$ such that for all $a \in A$, $T\pi_1(a) = \pi_2(a)T$. The relation $\cong$ defines an equivalent relation on the set of representations of A. The set of all classes of irreducible representations of A is denoted by $\widehat{A}$. For given $L \in \mathcal{M}_A$, we denote by π_L, its corresponding regular representation. That is,

$$\pi_L : A \longrightarrow B(\frac{A}{L}),$$

$$\pi_L(a)(b + L) = ab + L.$$

Also, for each irreducible representation π, there exists $L \in \mathcal{M}_A$ such that $\pi \cong \pi_L$ [5, Corollary 5.14].

We denote by $\mathrm{Prim}(A)$, the set of all primitive ideals of A. A primitive ideal is the kernel of an irreducible representation of A. In fact, a primitive ideal I has the form

$$(L : A) = \{a \in A : aA \subseteq L\}$$

where L is a maximal modular left ideal of A. We recall that an ideal I is modular if there exists $e \in A$ such that $ea - a$, and $ae - a \in I$, for each $a \in A$.

The topology of $\mathrm{Prim}(A)$ is given by means of the closure operation. Given a subset W of $\mathrm{Prim}(A)$, the closure of W, $\overline{W}$, is defined as the set of all elements of $\mathrm{Prim}(A)$ containing $\bigcap_{I\in W} I$. It follows that the closure operation defines a topology on $\mathrm{Prim}(A)$, which is called the Jacobson topology or the hull-kernel topology.

The spectrum of A is the set $\widehat{A}$ endowed with the topology induced by the inverse image of the Jacobson topology, under the map

$$\theta : \widehat{A} \longrightarrow \mathrm{Prim}(A),$$

$$\theta(\pi) = \ker(\pi).$$

By this definition, θ is an open and closed map. Therefore, each closed subset of $\widehat{A}$ has the form

$$\{\pi \in \widehat{A} : \ker(\pi) \supseteq I\},$$

where I is a closed ideal of A [8, Theorems 5.4.7].

3 definition of the Gelfand theory and its properties

Let A be a commutative Banach algebra. The Gelfand transform induces a bijection between the character spaces Φ_A and $\Phi_{C_0(\Phi_A)}$ given by

$$\Phi_{C_\circ(\Phi_A)} \longrightarrow \Phi_A,$$

$$h \mapsto h \circ \mathcal{G}_A.$$

According to this fact and the arguments mentioned in the pervious section, we arrange our new axioms.

Definition 3.1. Let A be a Banach algebra. A (topological) Gelfand theory for A is a pair $(\mathcal{G},\mathcal{U})$ which satisfies the following set of axioms:

(G1) $\mathcal{U}$ is a C*-algebra and $\mathcal{G} : A \to \mathcal{U}$ is a homomorphism which induces a (homeomorphism) bijection as follows:

$$\widehat{\mathcal{U}} \longrightarrow \widehat{A},$$

$$\pi \mapsto \pi \circ \mathcal{G}.$$

(G2) For every $L \in \mathcal{M}_{\mathcal{U}}$, $\mathcal{G}(A) \nsubseteq L$.

For simplicity, we abbreviate 'topological Gelfand theory' (resp. 'Gelfand theory') with TGT (resp. GT).

Proposition 3.2. *Let A be a commutative Banach algebra and $(\mathcal{G},\mathcal{U})$ be a GT for A. There is an isomorphism $\gamma : C_\circ(\Phi_A) \to \mathcal{U}$ such that $\mathcal{G} = \gamma \circ \mathcal{G}_A$.*

Proof. Since A is commutative, every element of $\widehat{\mathcal{U}}$ is one dimensional. Then semisimplicity of $\mathcal{U}$ implies that $\mathcal{U}$ is commutative. Therefore, $\mathcal{U}$ can be identified with $C_\circ(\Phi_{\mathcal{U}})$. Since A and $\mathcal{U}$ are commutative, $\mathcal{G}$ is automatically continuous and so the bijection

$$\Phi_{\mathcal{U}} \longrightarrow \Phi_A,$$
$$h \mapsto h \circ \mathcal{G}$$

will be continuous, for it is the restriction of $\mathcal{G}^*$ to $\Phi_{\mathcal{U}}$. Moreover, if $\mathcal{U}$ is unital, then $\mathcal{G}^*$ is homeomorphism

In the non-unital case, one may consider the unitization of A and $\mathcal{U}$ to deduce that $\mathcal{G}^*$ is a homeomorphism. Therefore, the map

$$\gamma : C_\circ(\Phi_A) \longrightarrow C_\circ(\Phi_{\mathcal{U}}),$$
$$\gamma(f) = f \circ \mathcal{G}^*$$

is a *-isomorphism which satisfies $\mathcal{G} = \gamma \circ \mathcal{G}_A$. ∎

Definition 3.3. The Gelfand theories $(\mathcal{G}_1,\mathcal{U}_1)$ and $(\mathcal{G}_2,\mathcal{U}_2)$ for A are called equivalent if there exists a *-isomorphism $\gamma : \mathcal{U}_2 \to \mathcal{U}_1$ such that $\mathcal{G}_1 = \gamma \circ \mathcal{G}_2$. If any two Gelfand theories of A are equivalent, we say that A has a unique Gelfand theory.

Remark 3.4.
(i) The proposition 3.2 shows that any commutative Banach algebra has a unique Gelfand theory which is also topological.

(ii) One can see that if A has a GT, then any irreducible representation of A can be considered on a Hilbert space.

(iii) The continuity of the Gelfand transform $\mathcal{G}_A$ of a commutative Banach algebra A is well-known. The closed graph theorem and the continuity of irreducible representations guarantees that $\mathcal{G}$ is continuous.

Theorem 3.5. *Let $(\mathcal{G},\mathcal{U})$ be a GT for A. Then*

(i) the inverse image of each element of $\mathcal{M}_{\mathcal{U}}$ (Prim $(\mathcal{U})$) under $\mathcal{G}$ is an element of $\mathcal{M}_A$ (Prim(A)),

(ii) if A is unital then so is $\mathcal{U}$,

(iii) if A is unital, then $\sigma(a) = \sigma(\mathcal{G}(a))$, for each $a \in A$.

Proof. (i) Let $L \in \mathcal{M}_{\mathcal{U}}$ and J be a left ideal of A containing strictly $\mathcal{G}^{-1}(L)$. Since $\{\mathcal{G}(a) + L : a \in J\}$ is a non-zero invariant subspace of $\pi_L \circ \mathcal{G}$, $\{\mathcal{G}(a) + L : a \in J\} = \frac{\mathcal{U}}{L}$. So, there exists $\widetilde{e} \in J$ such that $\mathcal{G}(\widetilde{e}) = e + L$, where e is a unit modular for L. It is easy to see that $\widetilde{e}$ is a unit modular for J and hence $J = A$.

Suppose that $I \in \text{Prim}(\mathcal{U})$ and let $L \in \mathcal{M}_{\mathcal{U}}$ such that $I = (L : \mathcal{U})$. Let x be an element of $\mathcal{G}^{-1}((L : \mathcal{U}))$. Then $\mathcal{G}(x)\mathcal{U} \subseteq L$ and hence $xA \subseteq \mathcal{G}^{-1}(L)$. Conversely, suppose that xA is a subset of $\mathcal{G}^{-1}(L)$. Then $\pi_L \circ \mathcal{G}(x)$ vanishes on $\{\mathcal{G}(a) + L : a \in A\} = \frac{\mathcal{U}}{L}$. It means that $\mathcal{G}(x)$ is in $\ker \pi_L = (L : \mathcal{U})$.

(ii) Let e be the unit of A. Let $L \in \mathcal{M}_{\mathcal{U}}$ and $x \in \mathcal{U}$. Since $\{\mathcal{G}(a) + L : a \in A\} = \frac{\mathcal{U}}{L}$, $\mathcal{G}(e)x - x$ and $x\mathcal{G}(e) - x$ are in $\ker \pi_L$. Then $\mathcal{G}(e)$ is the unit of $\mathcal{U}$ by semi-simplicity.

(iii)By [1, Theorem 4.2] and (G1),

$$\sigma(a) = \bigcup\{\sigma(\widetilde{\pi}(a)) : \widetilde{\pi} \in \widehat{A}\} = \bigcup\{\sigma(\pi \circ \mathcal{G}(a)) : \pi \in \mathcal{U}\} = \sigma(\mathcal{G}(a)). \qquad \blacksquare$$

Remark 3.6. The converse of theorem 3.5(ii) is not true in general, even in the commutative case. There exist some non-unital commutative Banach algebras with compact nonempty maximal ideal space. For example, suppose that B is a commutative Banach algebra with $\mathcal{M}_B = \emptyset$. Let $A = B \oplus C(X)$, where X is a compact Hausdorff space. Then A is non-unital with $\mathcal{M}_A = X$.

4 Existence of Gelfand theory

As mentioned in remark 3.4, if A has a GT, then each irreducible representation of A can be considered on a Hilbert space. It is proved in [2] that there is no one-to-one bounded linear map from X onto ℓ^2 when X is $c_\circ$ or ℓ^p with $p \in (1, +\infty) \setminus \{2\}$. Since the identity map id: $B(X) \longrightarrow B(X)$ is an irreducible representation, then $B(X)$ has no GT.

We obtain a condition under which $(\tau_A, C^*(A))$ is a GT for A. Here τ_A is the canonical map from A into the enveloping C*-algebra $C^*(A)$. Let A be a Banach *-algebra. Recall that A is hermitian if each self-joint element of A has real spectrum in A.

Lemma 4.1. *Let A be a unital Banach *-algebra. Then A is hermitian if and only if every irreducible representation of A is equivalent to a *-representation on some Hilbert space.*

Proof. Assume A is hermitian. Each irreducible representation π of A is equivalent to π_L for some maximal modular left ideal L. By [6, IV.6.12], there exists a pure state f on A such that $L = \{a \in A : f(a^*a) = 0\}$. So $\frac{A}{L}$ is a Hilbert space and π_L is a *-representation for some Hilbert space. Conversely, suppose $a = a^*$. Since

$$\sigma(a) = \cup\{\sigma(\pi(a)) : \pi \in \widehat{A}\}$$

and every element in $\widehat{A}$ is equivalent to a *-representation on some Hilbert space, we have $\sigma(a) \subseteq \mathbb{R}$. $\blacksquare$

Theorem 4.2. *Suppose that A is a unital Banach *-algebra. The pair $(\tau_A, C^*(A))$ is a GT (TGT) for A if and only if A is hermitian and any topologically irreducible *-representation of A is a (algebraically) irreducible representation.*

Proof. Let $(\tau_A, C^*(A))$ be a GT for A. Let π' be a topologically *-representation of A. By [4, 2.7.4], there exist a irreducible representation π of $C^*(A)$ such that $\pi' = \pi \circ \tau_A$. By assumption, π' is irreducible representation. Let A be a self-adjoint element in A. Since $\sigma(a) = \sigma(\tau_A(a))$ (3.5), $\sigma(a) \subseteq \mathbb{R}$. Conversely, By lemma 4.1 and [4, 2.7.4] the map defined by

$$\eta : \widehat{C^*(A)} \longrightarrow \widehat{A},$$

$$\eta(\pi) = \pi \circ \tau_A$$

is a bijection. Let $\widetilde{I}$ be a closed ideal of A. Since $\tau_A(A)$ is dense in $C^*(A)$, $\overline{\tau_A(\widetilde{I})}$ is a closed ideal of $C^*(A)$. Therefore, the continuity of η is obtained from the following equality:

$$\eta^{-1}(\{\widetilde{\pi} \in \widehat{A} : \ker \widetilde{\pi} \supseteq \widetilde{I}\}) = \{\pi \in \widehat{C^*(A)} : \ker \pi \supseteq \overline{\tau_A(\widetilde{I})}\}.$$

For the continuity of η^{-1}, it is enough to consider the following equality

$$\eta(\{\pi \in \widehat{C^*(A)} : \ker \pi \supseteq I\}) = \{\widetilde{\pi} \in \widehat{A} : \ker \widetilde{\pi} \supseteq \tau_A^{-1}(I)\},$$

where I is a closed ideal of $C^*(A)$. By density of $\tau_A(A)$ in $C^*(A)$, (G2) follows. ∎

Remark 4.3. It is well-known that for any element x in a C*-algebra $\mathcal{U}$, we have:

$$\| x \| = \sup\{\| \pi(x) \| : \pi \in \widehat{\mathcal{U}}\}.$$

Thus $C^*(A)$ is the smallest C^*-algebra that constructs a Gelfand theory for a hermitian semisimple Banach *-algebra A. It means that, if $(\mathcal{U}, \mathcal{G})$ is any Gelfand theory for A, then $C^*(A)$ is embedded in $\mathcal{U}$.

Let n be a natural number. A Banach algebra A satisfies the standard polynomial identity $S_n = 0$ if for all $a_1, ...a_n$ in A

$$S_n(a_1, ..., a_n) = \sum sgn(\tau) a_{\tau(1)}, ..., a_{\tau(n)} = 0$$

where the sum runs over the symmetric group on n symbols. It is proved in [7, p. 338] and [10, 1.4.5] that $M_n(\mathbb{C})$ satisfies the polynomial identity $S_{2n} = 0$. A Banach algebra A is called n-homogeneous if for every $\pi \in \widehat{A}$, $\dim(\pi) = n$.

Proposition 4.4. *Every unital n-homogeneous Banach algebra has a unique GT.*

Proof. Let $\mathcal{U}$ be the C*-direct sum of $M_n(\mathbb{C})$. That is,

$$\mathcal{U} = \ell^\infty - \oplus M_n(\mathbb{C})$$

where direct sum is taken on the set $\widehat{A}$. We define

$$\mathcal{G} : A \longrightarrow \mathcal{U},$$

$$\mathcal{G}(a) = (\widetilde{\pi}(a))_{\widetilde{\pi} \in \widehat{A}}.$$

It is enough to check that (G1) holds. Let π be an irreducible representation of $\mathcal{U}$. The simplicity of $M_n(\mathbb{C})$, implies that $\ker \pi$ has the form

$$\ell^\infty - \oplus I_{\widetilde{\pi}},$$

where $I_{\widetilde{\pi}} = M_n(\mathbb{C})$ or zero. The C*-algebra $\mathcal{U}$ satisfies the polynomial identity $S_{2n} = 0$. Therefore, $\dim(\pi) = n$ [10, 1.4.5]. Hence there is $\widetilde{\pi}_0 \in \widehat{A}$ such that

$$I_{\widetilde{\pi}} = \begin{cases} 0 & \widetilde{\pi} = \widetilde{\pi}_0 \\ M_n(\mathbb{C}) & \widetilde{\pi} \neq \widetilde{\pi}_0. \end{cases}$$

Let $\Pi_{\widetilde{\pi_0}}$ be the projection on $\widetilde{\pi_0}$-th component. Since $\Pi_{\widetilde{\pi}_0}$ and π are irreducible representations with the same kernel and finite dimension, $\Pi_{\widetilde{\pi}_0} \cong \pi$. Therefore, every irreducible representation of $\mathcal{U}$ has the form $\Pi_{\widetilde{\pi}}$ for some $\widetilde{\pi} \in \widehat{A}$. These show that (G1) holds.

Assume $(\mathcal{G}_1, \mathcal{V})$ is a GT for A and $\gamma : \mathcal{V} \to \mathcal{U}$ is the universal representation of $\mathcal{V}$. Then, $\gamma(\mathcal{V})$ is a rich *-subalgebra of $\mathcal{U}$. Since $\mathcal{U}$ is postliminal, $\gamma(\mathcal{V}) = \mathcal{U}$. Hence, γ is an isomorphism and $\mathcal{G} = \gamma \circ \mathcal{G}_1$ ∎

Proposition 4.5. *Let I be a closed ideal of A. If A has a TGT, then I and $\frac{A}{I}$ have TGT.*

Proof. Let $(\mathcal{G}, \mathcal{U})$ be a TGT for A and set

$$\mathcal{J} = \bigcap \{\ker \pi : \pi \in \widehat{\mathcal{U}}, \ker \pi \supseteq \mathcal{G}(I)\}.$$

Note that $\mathcal{J}$ is a closed two sided ideal of $\mathcal{U}$ and $\mathcal{G}|_I : I \to \mathcal{J}$ is homomorphism. Set $\widetilde{K} = \{\widetilde{\pi} \in \widehat{A} : \ker \widetilde{\pi} \not\supseteq I\}$ and $K = \{\pi \in \widehat{\mathcal{U}} : \ker \pi \not\supseteq \mathcal{J}\}$. By (G1) and the definition of $\mathcal{J}$, the map $\pi \to \pi \circ \mathcal{G}$ from K onto $\widetilde{K}$ is bijection. By [4, 3.2.1] and the following commutative diagram,

$$\begin{tikzcd} \widehat{\mathcal{J}} \arrow[r] \arrow[d] & \widehat{I} \arrow[d] \\ K \arrow[r] & \widetilde{K} \end{tikzcd}$$

this map is a homeomorphism.

For (G2), let $\mathcal{G}(I) \subseteq \widehat{L}$, where $\widehat{L} \in \mathcal{M}_{\mathcal{J}}$. There exists $L \in \mathcal{M}_{\mathcal{U}}$ such that $\widehat{L} = L \cap \mathcal{J}$ and $\mathcal{J} \not\subseteq L$. Since $\mathcal{G}(I) \subseteq \ker \pi_L$, L contains $\mathcal{J}$. But that is a contradiction.

Let $\widetilde{\mathcal{G}}$ be the following map

$$\widetilde{\mathcal{G}} : \frac{A}{I} \to \frac{\mathcal{U}}{\mathcal{J}},$$

$$\widetilde{\mathcal{G}}(a + I) = \mathcal{G}(a) + \mathcal{J}.$$

By [4, 3.2.1] and the following commutative diagram

$$\begin{tikzcd} \widehat{(\frac{\mathcal{U}}{\mathcal{J}})} \arrow[r] \arrow[d] & \widehat{(\frac{A}{I})} \arrow[d] \\ K_0 \arrow[r] & \widetilde{K}_0 \end{tikzcd}$$

the map $\pi \to \pi \circ \widetilde{\mathcal{G}}$ is a homeomorphism, where $K_0 = \{\pi \in \widehat{\mathcal{U}} : \ker \pi \supseteq \mathcal{J}\}$ and $\widetilde{K_0} = \{\widetilde{\pi} \in \widehat{A} : \ker \widetilde{\pi} \supseteq I\}$. It is easy to see that (G2) holds. ∎

5 Gelfand theory for C^*-algebras

Let A be a unital C*-algebra and u be an invertible element of A. Then $(\mathcal{G}_u, A)$ is a TGT for A, where

$$\mathcal{G}_u : A \longrightarrow A \quad : \quad \mathcal{G}_u(a) = uau^{-1}.$$

If u is not unitary, then $\mathcal{G}_u$ will not be *-isomorphism. We say that the Gelfand theory $(\mathcal{G}, A)$ is a *-GT (*-TGT) if $\mathcal{G}$ is *-homomorphism.

Theorem 5.1. *Let A be a C*-algebra and $(\mathcal{G},\mathcal{U})$ be a *-GT for A. Then*
(i) $\mathcal{G}$ is an isometry,
(ii) for any $L \in \mathcal{M}_{\mathcal{U}}$, $\frac{A}{\mathcal{G}^{-1}(L)} \cong \frac{\mathcal{U}}{L}$.

Proof. (i) Let $a \in A$. we have

$$\begin{aligned} ||\mathcal{G}(a)|| &= \sup\{||\pi(\mathcal{G}(a))|| : \pi \in \widehat{\mathcal{U}}\} \\ &= \sup\{||\widetilde{\pi}(a)|| : \widetilde{\pi} \in \widehat{A}\} \\ &= ||a||. \end{aligned}$$

(ii) Let $\mathcal{G}_L : \frac{A}{\mathcal{G}^{-1}(L)} \longrightarrow \frac{\mathcal{U}}{L}$ be the induced map by $\mathcal{G}$. It is easy to check that $\mathcal{G}_L$ is an algebraic isomorphism. For the continuity of $\mathcal{G}_L$, suppose that $a \in A$. Then

$$\begin{aligned} ||a + \mathcal{G}^{-1}(L)|| &= \inf_{x \in \mathcal{G}^{-1}(L)} ||a + x|| \\ &= \inf_{x \in \mathcal{G}^{-1}(L)} ||\mathcal{G}(a) + \mathcal{G}(x)|| \\ &\geq \inf_{y \in L} ||\mathcal{G}(a) + y|| = ||\mathcal{G}(a) + L|| \\ &= ||\mathcal{G}_L(a)||. \end{aligned}$$

Therefore, $\mathcal{G}_L$ is continuous and by the open mapping theorem, $\mathcal{G}_L^{-1}$ is also continuous. ∎

We recall that a C*-algebra A is called postliminal (liminal) if $K(H) \subseteq \pi(A)$ ($\pi(A) = K(H)$) for every irreducible representation $\pi : A \longrightarrow B(H)$. For more details about postliminal and liminal C*-algebras, see [4, Chapter 4]. A C*-subalgebra B of a C*-algebra A is rich, if the following conditions are satisfied:
(i) For every $\pi \in \widehat{A}$, $\pi|_B \in \widehat{B}$.
(ii) If π and π' are inequivalent representations of A then $\pi|_B$ and $\pi'|_B$ are inequivalent.

It is proved in [4, Poroposition 11.1.6] any postliminal C*-algebra has no proper rich subalgebra. For any C^*-algebra A, the pair (Id_A, A) is a trivial *-TGT. We will show that every postliminal C*-algebra has a unique *-Gelfand theory.

Let A be a C*-algebra and $\mathcal{I}$ be the set of all $x \in A$ such that $\pi(x)$ is compact for every irreducible representation $\pi : A \longrightarrow B(H)$. It is proved in [4, Proposition 4.2.6] that $\mathcal{I}$ is the largest liminal two-sided ideal of A.

Proposition 5.2. *Let A be a C*-algebra and $(\mathcal{G},\mathcal{U})$ be a GT for A. If A is postliminal (liminal), then so is $\mathcal{U}$.*

Proof. Suppose A is postliminal and $\pi : \mathcal{U} \longrightarrow B(H)$ is an irreducible representation. Since $\pi \circ \mathcal{G}$ is irreducible,

$$K(H) \subseteq \pi \circ \mathcal{G}(A) \subseteq \pi(\mathcal{U}).$$

Assume that A is liminal and $\mathcal{I}$ is the largest liminal two-sided ideal of $\mathcal{U}$. By (G1), $\pi \circ \mathcal{G}(A) = K(H)$ for each $\pi \in \widehat{\mathcal{U}}$. By definition of $\mathcal{I}$, $\mathcal{G}(A) \subseteq \mathcal{I}$. If $\mathcal{I} \neq \mathcal{U}$, there exists $L \in \mathcal{M}_{\mathcal{U}}$ such that $\mathcal{I} \subseteq L$. Therefore, $\mathcal{G}(A) \subseteq L$ which is a contradiction. ∎

Theorem 5.3. *Every postliminal C*-algebra has a unique *-GT.*

Proof. Let A be a C*-algebra and $(\mathcal{G},\mathcal{U})$ be a *-GT for A. Since A is semi-simple, $\mathcal{G}$ is one-to-one. Since the range of $\mathcal{G}$ is closed and the map $\pi \mapsto \pi \circ \mathcal{G}$ is bijection, $\mathcal{G}(A)$ is a rich C*-subalgebra of $\mathcal{U}$. Therefore, $\mathcal{G}(A) = \mathcal{U}$, because $\mathcal{U}$ is postliminal. So $\mathcal{G}$ is a *-isomorphism and $\mathcal{G}^{-1} \circ \mathcal{G} = Id_A$. Then the two Gelfand theories (Id_A, A) and $(\mathcal{G},\mathcal{U})$ are equivalent. ∎

Remark 5.4. We claim that the two following are equivalent:
(i) Every C*-algebra has a unique *-Gelfand theory.
(ii) Let A be a C*-algebra and B be a rich C*-subalgebra of A. Then $B = A$.

Let A be a C*-algebra. Suppose that B is a rich C*-subalgebra for A. It is proved in [4, Poroposition 11.1.2] that if π is an irreducible representation of B, then there exists an irreducible representation π' of A such that $\pi = \pi'|_B$. Thus the map $\pi \mapsto \pi|_B$ from $\widehat{A}$ to $\widehat{B}$ is onto. Also, the second condition of the definition of a rich C*-subalgebra implies that the map $\pi \mapsto \pi|_B$ from $\widehat{A}$ to $\widehat{B}$ is one-to-one. Therefore, the pair (i, A) is a *-Gelfand theory for B, where i is the inclusion map. Since B has a unique *-Gelfand theory, there is an isomorphism γ such that $i = \gamma \circ id_B$ and hence, $B = A$.

Conversely, let A be a C*-algebra. If $(\mathcal{G},\mathcal{U})$ is a *-Gelfand theory for A, then $\mathcal{G}(A)$ is a rich C*-subalgebra of $\mathcal{U}$. Thus $\mathcal{G}(A) = \mathcal{U}$. It means that $\mathcal{G}$ is a *-isomorphism and $id_A = \mathcal{G}^{-1} \circ \mathcal{G}$. Therefore, A has a unique *-Gelfand theory.

The non-commutative version of the Stone-Weierstrass theorem (which is an open problem) says that: does there exist a proper rich *-subalgebra in a given C*-algebra A? As claimed above, this problem is equivalent to the existence of a C*-algebra with (at least) two different *-Gelfand theories.

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References

[1] B. Aupetit, Primer on spectral theory, Springer-Verlag, 1991.

[2] E. Berkson and H. Porta, Representation of $B(X)$, J. Functional Anal. **3**(1969), 1-34.

[3] R. Chouki, El H. Illoussamen and V. Runde, Gelfand Theory for non-commutative Banach algebras, Quart. J. Math.**53**(2002) 161-172

[4] J. Dixmier, C*-algebras, North Holland, New York, 1982.

[5] J. M. G. Fell and R. S. Doran, Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles, Pure and Applied Mathematics series, vol. 125 and 126, Academic Press, 1988

[6] A. Y. Helemskii, Banach and locally convex algebras, Claredon Press, Oxford, 1993.

[7] J. Levitzki, A theorem on polynomial identities, Proc. Amer. Math. Soc. 1 (1950), 449-463.

[8] G. J. Murphy, C^*- Algebras and Operator Theory. Academic Press, 1990.

[9] L. B. Ren, Introduction to operator algebras, World Scientific publishing (1992)

[10] L. H. Rowen. Polynomial Identities in Ring Theory. Academic Press, 1980.

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