Second order non-linear strong differential subordinations

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Abstract

The concept of differential subordination was introduced in [4] by S.S. Miller and P.T. Mocanu and the concept of strong differential subordination was introduced in [1], [2], [3] by J.A. Antonino and S. Romaguera. In [7] we have studied the strong differential subordinations in the general case and in [8] we have studied the first order linear strong differential subordinations. In [6] we have studied the second order linear strong differential subordinations. In this paper we study the second order non-linear strong differential subordinations. Our results may be applied to deduce sufficient conditions for univalence in the unit disc, such as starlikeness, convexity, alpha-convexity, close-to-convexity respectively.

1 Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in U. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H};\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U\}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \quad z \in U,$$

which are analytic in the unit disk.

Received by the editors January 2008.
Communicated by Y. Félix.
2000 *Mathematics Subject Classification* : 30C45, 34A30.
Key words and phrases : analytic function, differential subordination, strong differential subordination.

In addition, we need the classes of convex, alpha-convex, close-to-convex and starlike (univalent) functions given respectively by

$$K = \left\{ f \in A;\ \text{Re}\ \left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0,\ z \in U \right\},$$

$$M_\alpha = \left\{ f \in A,\ \frac{f(z)f'(z)}{z} \neq 0, \right.$$

$$\left. \text{Re}\ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0,\ z \in U \right\}$$

$$C = \{ f \in A,\ \text{Re}\, f'(z) > 0,\ z \in U \},$$

and

$$S^* = \{ f \in A,\ \text{Re}\, zf'(z)/f(z) > 0 \}.$$

In order to prove our main results we use the following definitions and lemmas.

Definition 1. [1], [2], [3] Let $\mathcal{H}(z,\xi)$ be analytic in $U \times \overline{U}$ and let $f(z)$ analytic and univalent in U. The function $H(z,\xi)$ is strongly subordinate to $f(z)$, written $H(z,\xi) \prec\prec f(z)$, if for each $\xi \in \overline{U}$, the function of z, $H(z,\xi)$ is subordinate to $f(z)$.

Remark 1. (i) Since $f(z)$ is analytic and univalent, Definition 1 is equivalent to:

$$H(0,\xi) = f(0) \text{ and } H(U \times \overline{U}) \subset f(U).$$

(ii) If $H(z,\xi) \equiv H(z)$ then the strong subordination becomes the usual subordination.

Definition 2. [4], [5, p.21] We denote by Q the set of functions q that are analytic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U;\ \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Lemma A. [5, Lemma 2.2.d, p.24] *Let $q \in Q(a)$, with $q(0) = a$ and $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ be analytic in U, with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, and

(iii) $\text{Re}\ \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\text{Re}\ \left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right].$

Definition 3. [7, Definition 4] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$(A) \qquad \psi(r,s,t;z,\xi) \notin \Omega$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\mathrm{Re}\,\frac{t}{s} + 1 \geq m\mathrm{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right], \quad z \in U,\ \zeta \in \partial U \setminus E(q)$$

and $m \geq n$.

Remark 2. For the function $q(z) = Mz$, $M > 0$, $z \in U$, the condition of admissibility (A) becomes

$$(A') \qquad \psi(Me^{i\theta}, Ke^{i\theta}, L; z, \xi) \notin \Omega$$

whenever $K \geq nM$, $\mathrm{Re}\,[Le^{-i\theta}] \geq (n-1)K$, $z \in U$, $\xi \in \overline{U}$ and $\theta \in \mathbb{R}$.

For the function $q(z) = \dfrac{1+z}{1-z}$, $z \in U$, the condition of admissibility (A) becomes

$$(A'') \qquad \psi(\rho i, \sigma, \mu + \nu i; z, \xi) \notin \Omega$$

whenever $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $\sigma \leq -\dfrac{n}{2}[1+\rho^2]$, $\sigma + \mu \leq 0$, $z \in U$, $\xi \in \overline{U}$, and $n \geq 1$.

2 Main results

Definition 4. A strong differential subordination of the form

$$A(z,\xi)z^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)p^2(z) + E(z,\xi) \prec\prec h(z)$$

where $A, B, C, D, E : U \times \overline{U} \to \mathbb{C}$, $A(z,\xi)z^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)p^2(z) + E(z,\xi)$ is a function of z, analytic for all $\xi \in \overline{U}$ and function h is analytic and univalent in U, is called second order non-linear strong differential subordination.

Remark 3. If $D(z,\xi) \equiv 0$ then we obtain a second order linear strong differential subordination studied in [6].

Remark 4. For $A(z,\xi) = D(z,\xi) = 0$ the second order non-linear strong differential subordination reduces to the first order linear differential subordination studied in [8].

Theorem 1. *Let $A, B, C, D, E : U \times \overline{U} \to \mathbb{C}$ with*

$$(1) \qquad A(z,\xi) = A > 0, \quad E(z,\xi) \equiv 0, \quad \mathrm{Re}\,B(z,\xi) > 0,$$

$$A(n-1)n + n\mathrm{Re}\,B(z,\xi) + \mathrm{Re}\,C(z,\xi) \geq 1 + M|D(z,\xi)|, \quad M > 0,$$

and $Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)p^2(z)$ a function of z analytic for all $\xi \in \overline{U}$.

If $p \in \mathcal{H}[0,n]$ and the second order non-linear strong differential subordination

$$(2) \qquad Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)p^2(z) \prec\prec Mz$$

holds, then

$$p(z) \prec Mz, \quad z \in U,\ M > 0.$$

Proof. Let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. For $r = p(z)$, $s = zp'(z)$, $t = z^2 p''(z)$, let

$$\psi(r, s, t; z, \xi) = At + B(z, \xi)s + C(z, \xi)r + D(z, \xi)r^2. \tag{3}$$

Then (2) becomes

$$\psi(r, s, t; z, \xi) \prec\prec Mz, \quad z \in U,\ \xi \in \overline{U}. \tag{4}$$

If we let $h(z) = Mz$, $z \in U$, $M > 0$ then $h(U) = U(0, M)$ and (4) is equivalent to

$$\psi(r, s, t; z, \xi) \in U(0, M), \quad z \in U,\ \xi \in \overline{U}. \tag{5}$$

Suppose that p is not subordinate to function h. Then, by Lemma A, we have that there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ such that $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$, $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) = Ke^{i\theta_0}$, $z_0^2 p''(z_0) = \zeta_0^2 h''(\zeta_0) = L$ with $K \geq nM$, $\text{Re}\,[Le^{-i\theta_0}] \geq (n-1)K$ where $z \in U$, $\theta_0 \in \mathbb{R}$.

By replacing r with $p(z_0)$, s with $z_0 p'(z_0)$, t with $z_0^2 p''(z_0)$ in (3) and using the conditions given by (1) we obtain

$$|\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0); z_0, \xi)| = \tag{6}$$

$$\begin{aligned}
&= |Az_0^2 p''(z_0) + B(z_0, \xi) z_0 p'(z_0) + C(z_0, \xi) p(z_0) + D(z_0, \xi) p^2(z_0)| \\
&= |AL + B(z_0, \xi) Ke^{i\theta_0} + C(z_0, \xi) Me^{i\theta_0} + D(z_0, \xi) M^2 e^{2i\theta_0}| \\
&= |ALe^{-i\theta_0} + B(z_0, \xi) K + C(z_0, \xi) M + D(z_0, \xi) M^2 e^{i\theta_0}| \\
&\geq |ALe^{-i\theta_0} + B(z_0, \xi) L + C(z_0, \xi) M| - M^2 |D(z_0, \xi)| \\
&\geq \text{Re}\,[ALe^{-i\theta_0} + B(z_0, \xi) K + C(z_0, \xi) M] - M^2 |D(z_0, \xi)| \\
&\geq A\text{Re}\, Le^{-i\theta_0} + K\text{Re}\, B(z_0, \xi) + M\text{Re}\, C(z_0, \xi) - M^2 |D(z_0, \xi)| \\
&\geq A(n-1)nM + nM\text{Re}\, B(z_0, \xi) + M\text{Re}\, C(z_0, \xi) - M^2 |D(z_0, \xi)| \\
&\geq M[A(n-1)n + n\text{Re}\, B(z_0, \xi) + \text{Re}\, C(z_0, \xi)] - M^2 |D(z_0, \xi)| \geq M.
\end{aligned}$$

Since (6) contradicts (5), the assumption made is false and hence, $p(z) \prec Mz$, $z \in U$, $M > 0$. ∎

Example 1. Let

$$A(z, \xi) = 2, \quad B(z, \xi) = z + \xi + 3 - 2i,$$

$$C(z, \xi) = 2z + \xi + 5 - i, \quad D(z, \xi) = z + \xi + 2,$$

$$E(z, \xi) = 0, \quad n = 1, \quad M = \frac{1}{4}, \quad z \in U, \quad \xi \in \overline{U}.$$

Since $z \in U$, $\xi \in \overline{U}$, we have

$$\text{Re}\, B(z, \xi) \geq 0, \quad \text{Re}\, D(z, \xi) \geq 0,$$

$$\text{Re}\, B(z, \xi) + \text{Re}\, C(z, \xi) \geq 1 + \frac{|D(z, \xi)|}{4}.$$

From Theorem 1, we obtain:
If

$$[2z^2p''(z) + (z+\xi+3-2i)zp'(z) + (2z+\xi+5-i)p(z) + (z+\xi+2)p^2(z)]$$

is a function of z, analytic for all $\xi \in \overline{U}$ and

$$[2z^2p''(z) + (z+\xi+3-2i)zp'(z) + (2z+\xi+5-i)p(z)$$
$$+(z+\xi+2)p^2(z)] \prec\prec z, \quad z \in U, \quad \xi \in \overline{U},$$

then

$$p(z) \prec z, \quad z \in U.$$

Theorem 2. *Let $A, B, C, D, E : U \times \overline{U} \to \mathbb{C}$ with*

(7) $$A(z,\xi) = A > 0, \quad \text{Re}\, B(z,\xi) \geq A, \quad \text{Re}\, D(z,\xi) \geq 0,$$

$$C(0,\xi) + D(0,\xi) + E(0,\xi) = 1, \quad \frac{n}{2}[\text{Re}\, B(z,\xi) - A] \geq \text{Re}\, E(z,\xi),$$
$$\text{Im}\, C(z,\xi) \leq$$
$$\leq \sqrt{[n\text{Re}\, B(z,\xi) - nA + 2\text{Re}\, D(z,\xi)][n\text{Re}\, B(z,\xi) - nA - 2\text{Re}\, E(z,\xi)]},$$

$z \in U$, $\xi \in \overline{U}$.

Let $Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)p^2(z) + E(z,\xi)$ be an analytic function of z for all $\xi \in \overline{U}$.

If $p \in \mathcal{H}[1,n]$ and the following second order strong differential subordination holds

(8) $$Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z)$$
$$+D(z,\xi)p^2(z) + E(z,\xi) \prec\prec \frac{1+z}{1-z}, \quad z \in U, \ \xi \in \overline{U},$$

then

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in U.$$

Proof. Let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and for $r = p(z)$, $s = zp'(z)$, $t = z^2p''(z)$ we have

(9) $$\psi(r,s,t;z,\xi) = At^2 + B(z,\xi)s + C(z,\xi)r + D(z,\xi)r^2 + E(z,\xi),$$

$z \in U$, $\xi \in \overline{U}$.

Then (8) becomes

(10) $$\psi(r,s,t;z,\xi) \prec\prec \frac{1+z}{1-z}, \quad z \in U, \ \xi \in \overline{U}.$$

If we let $q(z) = \dfrac{1+z}{1-z}$, $z \in U$ then $h(U) = \{w \in \mathbb{C};\ \text{Re}\, w > 0\}$, the strong differential subordination (10) implies

(11) $$\psi(r,s,t;z,\xi) \in h(U), \quad z \in U, \ \xi \in \overline{U}$$

and (11) implies

$$\text{(12)} \qquad \text{Re}\,\psi(r,s,t;z,\xi) > 0, \quad z \in U,\ \xi \in \overline{U}.$$

Suppose that p is not subordinate to the function $q(z) = \dfrac{1+z}{1-z}$, $z \in U$. Then, by Lemma A, we have that there exist points $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0) = \rho i$, $\rho \in \mathbb{R}$, $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = \sigma$, $\sigma \in \mathbb{R}$, $z_0^2 p''(z_0) = \zeta_0^2 q''(\zeta_0) = \mu + i\nu$, $\mu, \nu \in \mathbb{R}$ with $\sigma \le -\dfrac{n}{2}(1+\rho^2)$ and $\sigma + \mu \le 0$, $m \ge n \ge 1$.

By replacing $r = p(z_0)$, $s = z_0 p'(z_0)$, $t = z_0^2 p''(z_0)$ in (9) and using the conditions given by (7), we obtain

$$\text{(13)} \qquad \text{Re}\,\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0); z_0, \xi) =$$

$$\begin{aligned}
&= \text{Re}\,[A(\mu + i\nu) + B(z_0,\xi)\sigma + C(z_0,\xi)\rho i - \rho^2 D(z_0,\xi) + E(z_0,\xi)] \\
&= A\mu + \sigma \text{Re}\,B(z_0,\xi) - \rho \text{Im}\,C(z_0,\xi) - \rho^2 \text{Re}\,D(z_0,\xi) + \text{Re}\,E(z_0,\xi) \\
&\le -A\sigma + \sigma \text{Re}\,B(z_0,\xi) - \rho \text{Im}\,C(z_0,\xi) - \rho^2 \text{Re}\,D(z_0,\xi) + \text{Re}\,E(z_0,\xi) \\
&\le \sigma[\text{Re}\,B(z_0,\xi) - A] - \rho \text{Im}\,C(z_0,\xi) - \rho^2 \text{Re}\,D(z_0,\xi) + \text{Re}\,E(z_0,\xi) \\
&\le -\frac{n}{2}(1+\rho^2)[\text{Re}\,B(z_0,\xi) - A] - \rho \text{Im}\,C(z_0,\xi) - \rho^2 \text{Re}\,D(z_0,\xi) + \text{Re}\,E(z_0,\xi) \\
&\le -\rho^2 \left[\frac{n}{2}\text{Re}\,B(z_0,\xi) - \frac{n}{2}A + \text{Re}\,D(z_0,\xi)\right] \\
&\quad -\rho \text{Im}\,C(z_0,\xi) - \frac{n}{2}[\text{Re}\,B(z_0,\xi) - A] + \text{Re}\,E(z_0,\xi) \le 0.
\end{aligned}$$

Since (13) contradicts (12), the assumption made is false and hence

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in U. \qquad \blacksquare$$

Remark 5. Theorem 2 can be rewritten as follows:

Corollary 1. *Let* $A, B, C, D, E : U \times \overline{U} \to \mathbb{C}$, $n \in \mathbb{N}$,

$$Az^2 p''(z) + B(z,\xi) z p'(z) + C(z,\xi) p(z) + D(z,\xi) p^2(z) + E(z,\xi)$$

a function of z*, analytic for all* $\xi \in \overline{U}$ *with*

$$A(z,\xi) = A > 0, \quad \text{Re}\,B(z,\xi) \ge A, \quad \text{Re}\,D(z,\xi) \ge 0,$$

$$C(0,\xi) + D(0,\xi) + E(0,\xi) = 1, \quad \frac{n}{2}[\text{Re}\,B(z,\xi) - A] \ge \text{Re}\,E(z,\xi),$$

$$\text{Im}\,C(z,\xi) \le$$

$$\le \sqrt{[n\text{Re}\,B(z,\xi) - nA + 2\text{Re}\,D(z,\xi)][n\text{Re}\,B(z,\xi) - nA - 2\text{Re}\,E(z,\xi)]},$$

$z \in U$, $\xi \in \overline{U}$.

If $p \in \mathcal{H}[1,n]$ and satisfies the inequality

$$\text{Re}\,[Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)p^2(z) + E(z,\xi)] > 0,$$

$$z \in U,\ \xi \in \overline{U}$$

then

$$\text{Re}\,p(z) > 0, \quad z \in U.$$

Remark 6. Note that the result contained in Theorem 2 can be applied to obtain sufficient conditions for univalence on the unit disc, such as starlikeness, convexity, alpha-convexity, close-to-convexity. Indeed, it suffices to consider

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(z) = 1 + \frac{zf''(z)}{f'(z)},$$

$$p(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right],$$

and $p(z) = f'(z), z \in U$ respectively.

Example 2. $A(z,\xi) = 2$, $B(z,\xi) = z + \xi + 6 - 2i$,

$$C(z,\xi) = -z - \xi + 1 - 10i, \quad D(z,\xi) = 2z + \xi + 3 + 10i,$$

$$E(z,\xi) = z - 3, \quad n = 2.$$

Since $z \in U$, $\xi \in \overline{U}$, we have

$$\text{Re}\,B(z,\xi) \geq 2, \quad \text{Re}\,D(z,\xi) \geq 0,$$

$$C(0,\xi) + D(0,\xi) + E(0,\xi) = 1, \quad \text{Re}\,B(z,\xi) \geq 2 + \text{Re}\,E(z,\xi),$$

$$\text{Im}\,C(z,\xi) \leq$$

$$\leq \sqrt{[2\text{Re}\,B(z,\xi) - 4 + 2\text{Re}\,D(z,\xi)][2\text{Re}\,B(z,\xi) - 4 - 2\text{Re}\,E(z,\xi)]}.$$

From Theorem 2, we obtain:
If

$$[2z^2p''(z) + (z+\xi+6-2i)zp'(z) + (-z-\xi+1-10i)p(z)$$
$$+(2z+\xi+3+10i)p^2(z) + z - 3]$$

is a function of z, analytic for all $\xi \in \overline{U}$ and

$$[2z^2p''(z) + (z+\xi+6-2i)zp'(z) + (-z-\xi+1-10i)p(z)$$

$$+(2z+\xi+3+10i)p^2(z) + z - 3] \prec\prec \frac{1+z}{1-z}, \quad z \in U,\ \xi \in \overline{U}$$

then

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in U.$$

■

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