On the moment map on symplectic manifolds

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Abstract

We consider a connected symplectic manifold M acted on properly and in a Hamiltonian fashion by a connected Lie group G. If G is compact, then we characterize the symplectic manifolds whose squared moment map is constant. We also give a sufficient condition for G to admit a symplectic orbit. Then we study the case when G is a non-compact Lie group proving splitting results for symplectic manifolds.

1 introduction

We shall consider symplectic manifolds (M, ω) acted on by a connected Lie group G of symplectomorphism. We shall also assume that the G-action on M is proper and Hamiltonian, i.e. there exists a moment map $\mu : M \longrightarrow \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. In [1] it was proved the existence of a G-invariant almost complex structure J such that $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot) = g$ is a Riemannian metric. Therefore, throughout the following we will denote by J and g the G-invariant almost complex structure and the corresponding Riemannian metric on M respectively.

In general, the matter of existence/uniqueness of μ is delicate. However, whenever $\mathfrak{g}$ is semisimple the moment map exists and is unique ([7]). If (M, ω) is a compact Kähler manifold and G is a connected compact Lie group of holomorphic isometries, then the existence problem is solved ([9]): a moment map exists if and only if G acts trivially on the Albanese torus Alb(M).

If G is compact, it is standard to fix an Ad(G)-invariant scalar product $\langle \cdot, \cdot \rangle$ and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of $\langle \cdot, \cdot \rangle$, regarding μ as a $\mathfrak{g}$-valued map. It is

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also natural to study the squared moment map $\parallel \mu \parallel^2$ and its critical set. This function has been intensively studied in [10], obtaining strong information on the topology of M.

Our first main result characterizes completely the symplectic manifolds whose squared moment map is constant.

Theorem 1.1. *Let (M, ω) be a connected symplectic manifold and let G be a compact connected Lie group acting effectively and in a Hamiltonian fashion on M with moment map $\mu : M \longrightarrow \mathfrak{g}$. The following are equivalent:*

1. *G is semisimple and M is symplectomorphically and G-equivariantly isometric with respect to g, to a product of a flag manifold and an almost complex manifold which is acted on trivially by G.*
2. *the squared moment map $f = \parallel \mu \parallel^2$ is constant;*
3. *M is mapped by the moment map μ to a single coadjoint orbit;*
4. *all principal G-orbits are almost complex submanifolds of (M, J);*
5. *all G-orbits are almost complex submanifolds of (M, J).*

In order to prove the above theorem, we need the following result, which might have an independent interest.

Proposition 1.2. *Let (M, ω) be a symplectic manifold and let G be a compact connected Lie group acting in a Hamiltonian fashion on M with moment map μ. Assume that $x \in M$ realizes a local maximum of the squared moment map $f = \parallel \mu \parallel^2$. Then the orbit $G \cdot x$ is symplectic. Moreover, there exists a neighborhood Y_o of x such that $G \cdot \left(Y_o \cap \mu^{-1}(\mu(x))\right)$ is a symplectic submanifold which is G-equivariantly symplectomorphic to the product of a flag manifold and a symplectic manifold which is acted on trivially by G. If we assume that $x \in M$ realizes the maximum of $f = \parallel \mu \parallel^2$ or any $z \in \mu^{-1}(\mu(x))$ realizes a local maximum of $f = \parallel \mu \parallel^2$, then the following statements hold true:*

1. *$\mu^{-1}(\mu(x))$ is a symplectic submanifold of M;*
2. *$G \cdot \mu^{-1}(\mu(x))$ is a symplectic submanifold of M which is $G-$equivariantly symplectomorphic to $(Gx \times \mu^{-1}(\mu(x)), \omega_{|_{G \cdot x}} + \omega_{|_{\mu^{-1}(\mu(x))}})$.*

These results generalize ones given in [6] and [2].

One may prove Proposition 1.2 assuming that $\mathrm{Ad}(G)$ is compact. This means that G is covered by a compact Lie group and a vector group which lies in the center (see [5]). Nevertheless, if G acts properly on M, then the existence of a symplectic G-orbit implies that G must be compact. Indeed, if $G \cdot x = G/G_x$ is symplectic, then, from Proposition 2.1, $G_x^o = G_{\mu(x)}^o$. We recall that if G is a group, then G^o denotes the connected component of G containing the identity e. Since $Z(G) \subset G_{\mu(x)}^o$, $Z(G)$ must be compact. Therefore G must be compact as well.

Then we study the case when G is a non-compact Lie group acting properly and in a Hamiltonian fashion on M. Our main result is the following theorem.

Theorem 1.3. *Let (M, ω) be a symplectic manifold and let G be a connected non-compact Lie group acting effectively, properly and in a Hamiltonian fashion on M with moment map μ. Assume also that for every $\alpha \in \mathfrak{g}^*$ the coadjoint orbit $G \cdot \alpha$ is locally closed. The following are equivalent:*

1. *all G-orbits are symplectic;*

2. *all principal G-orbits are symplectic;*

3. *M is mapped by the moment map μ to a single coadjoint orbit;*

4. *let x be a regular point of M. Then $G \cdot x$ is a symplectic orbit, $\mu^{-1}(\mu(x))$ is a symplectic submanifold on which G_x acts trivially and the following G-equivariant application*
$$\phi : G \cdot x \times \mu^{-1}(\mu(x)) \longrightarrow M, \; \phi([gx, z]) = gz,$$
is surjective and satisfies
$$\phi^* \omega = \omega_{|_{G \cdot x}} + \omega_{|_{\mu^{-1}(\mu(x))}}.$$

If G is a reductive Lie group acting effectively on M, then in (4) it turns out that G has to be semisimple and ϕ is a G-equivariant symplectomorphism. Moreover, if we assume that $N(G_x)/G_x$ is a finite group whenever $x \in M$ is a regular point, then our result holds in the almost-Kähler setting. Indeed, in (4) the map ϕ turns out to be an isometry with respect to g while in (1) and (2) all G-orbits and all principal G-orbits are almost complex submanifolds of (M, J) respectively.

Observe that the condition for a coadjoint orbit to be locally closed is automatic for reductive groups and for their semidirect products with vector spaces. There exists an example of a solvable Lie group due to Mautner [17, p.512], with non-locally closed coadjoint orbits. These assumptions are needed to apply the symplectic slice theorem (see [1, 7, 14, 16]), and the symplectic stratification of the reduced space given in [1].

Finally, as an immediate corollary of Theorem 1.1 and Theorem 1.3, we give the following splitting result.

Let G be a non-compact semisimple Lie group. The Killing form B on $\mathfrak{g}$ is a non-degenerate $\mathrm{Ad}(G)$-invariant bilinear form. Therefore, we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of $-B$, regarding μ as a $\mathfrak{g}$-valued map. The squared moment map can be defined as the smooth function $f(x) = -B(\mu(x), \mu(x)) = \parallel \mu(x) \parallel^2$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ ([8]). Then f is positive on $\mathfrak{k}$ and negative on $\mathfrak{p}$.

An element $X \in \mathfrak{g}$ is called *elliptic* if $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$ is diagonalizable and all eigenvalues are purely imaginary; the orbit $\mathrm{Ad}(G) \cdot X$ is called *elliptic orbit*. See [12, 13] for more details about elliptic orbits.

Corollary 1.4. *Let M be a symplectic manifold acted on by a connected non-compact semisimple Lie group G, properly and in a Hamiltonian fashion with moment map μ. Assume that $\mu(M) \subset \{X \in \mathfrak{g} : X \text{ is elliptic}\}$ and $f = \parallel \mu \parallel^2$ is constant. Then all G-orbits are symplectic and M is G-equivariantly symplectomorphic to a product of a* *flag manifold and a symplectic manifold which is acted on trivially by G. Moreover, if $N(G_x)/G_x$ is a finite group whenever $x \in M$ is a regular point, then the symplectomorphism turns out to be an isometry with respect to g and all G-orbits are almost complex submanifolds of (M, J).*

2 Proof of the main results

Let M be a connected differentiable manifold equipped with a non-degenerate closed $2-$form ω. The pair (M, ω) is called a *symplectic manifold*. Here we consider a finite-dimensional connected Lie group acting smoothly and properly on M so that $g^*\omega = \omega$ for all $g \in G$, i.e. G acts as a group of canonical or symplectic diffeomorphisms.

The G-action is called *Hamiltonian*, and we said that G acts in a Hamiltonian fashion on M or M is G-Hamiltonian, if there exists a map $\mu : M \longrightarrow \mathfrak{g}^*$, called the *moment map*, satisfying the following:

1. For each $X \in \mathfrak{g}$, let
   - $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(p) = \mu(p)(X)$, be the component of μ along X, and
   - $X^\#$ be the vector field on M generated by the one parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\} \subseteq G$.

   Then
   $$\mathrm{d}\mu^X = i_{X^\#}\omega,$$
   i.e., μ^X is a Hamiltonian function for the vector field $X^\#$.

2. μ is G-equivariant, i.e. $\mu(gp) = \mathrm{Ad}^*(g)(\mu(p))$, where Ad^* is the coadjoint representation on $\mathfrak{g}^*$.

Let $x \in M$ and $\mathrm{d}\mu_x : T_xM \longrightarrow T_{\mu(x)}\mathfrak{g}^*$ being the differential of μ at x. Then

$$Ker\mathrm{d}\mu_x = (T_xG \cdot x)^{\perp_\omega} := \{v \in T_xM : \omega(v, w) = 0,\ \forall w \in T_xG \cdot x\}$$

and the pullback by the restricted moment map $\mu : G \cdot x \longrightarrow \mathrm{Ad}^*(G) \cdot \mu(x)$ of the symplectic form on the coadjoint orbit through $\mu(x)$, it equals the restriction of the ambient symplectic form ω to the orbit $G \cdot x$:

$$\omega_{|_{G\cdot x}} = \mu^*(\omega_{Ad^*(G)\cdot\mu(x)})_{|_{G\cdot x}}, \tag{1}$$

see [1] p. 211, where $\omega_{\mathrm{Ad}^*(G)\cdot\mu(x)}$ is the Kirillov-Konstant-Souriau (KKS) symplectic form on the coadjoint orbit of $\mu(x)$ in $\mathfrak{g}^*$. This implies the following well-known fact ([7]).

Proposition 2.1. *The orbit of G through $x \in M$ is symplectic if and only if the stabilizer group of x is an open subgroup of the stabilizer of $\mu(x)$ hence if and only if the restricted moment map $\mu : G \cdot x \longrightarrow \mathrm{Ad}^*(G) \cdot \mu(x)$ is a covering map. In particular if G is compact or semisimple, then the restricted moment map is a diffeomorphism.*

Proof. The first affirmation follows immediately from (1). If G is compact or semisimple, then $G_{\mu(x)}$ is connected so $G_x = G_{\mu(x)}$. Therefore the restricted moment map is a diffeomorphism. ∎

Proof of Proposition 1.2. Let $\beta = \mu(x)$ and let G_x be the isotropy group at x. It follows from the symplectic slice theorem, see [1, 7, 14, 16], there exists a G-invariant neighborhood of $G \cdot x$ in M which is equivariantly symplectomorphic to a neighborhood Y_o of the zero section of $(Y = G \times_{G_x} (\mathfrak{q} \oplus V), \tau)$ and the moment map is given by

$$\mu([g, m, v]) = \mathrm{Ad}(g)(\beta + m + \mu_V(v)),$$

where $\mathfrak{q}$ is a G_x-module in the G_x-equivariant splitting $\mathfrak{g} = \mathfrak{g}_\beta \oplus \mathfrak{s} = \mathfrak{g}_x \oplus \mathfrak{q} \oplus \mathfrak{s}$ and μ_V is the moment map of the G_x-action on the symplectic subspace V of $((T_x G \cdot x)^{\perp\omega}, \omega_x)$. Note that V is isomorphic to the quotient $(T_x G \cdot x)^{\perp\omega} / (T_x G \cdot x)^{\perp\omega} \cap T_x G \cdot x$.

From now on, we denote by $\omega_V = (\omega_x)_{|V}$. Shrinking Y_o if necessary, we may also suppose that $[e, 0, 0]$ is the maximum of the smooth function $f = \parallel \mu \parallel^2$ on Y_o.

We now show that $\mathfrak{q} = \{0\}$, i.e. $G \cdot x$ is symplectic.

Let $m \in \mathfrak{q} - \{0\}$. Then for every $\lambda \in \mathbb{R}$ we have

$$f(e, \lambda m, 0) = \parallel \beta \parallel^2 + \lambda^2 \parallel m \parallel^2 + \lambda \langle m, \beta \rangle \leq \parallel \beta \parallel^2,$$

so

$$\lambda^2 \parallel m \parallel^2 + \lambda \langle m, \beta \rangle \leq 0,$$

which is a contradiction. Hence $G \cdot x$ is symplectic and by Proposition 2.1 $G_x = G_\beta$.

Let $Y_o^\beta = Y_o \cap \mu^{-1}(\beta)$. Then $G_y = G_x$ for every $y \in Y_o^\beta$, i.e. $G \cdot y$ is symplectic, and a $G-$orbit through an element of Y_o^β intersects $\mu^{-1}(\beta)$ in at most one point. Indeed, if both $x \in Y_o^\beta$ and kx lie in $\mu^{-1}(\beta)$, then, by the G-equivariance of μ, we have $\mu(kx) = \beta = k\mu(x) = k\beta$, proving $k \in G_x$. Therefore the map

$$\phi : G \cdot x \times Y_o^\beta \longrightarrow G \cdot Y_o^\beta$$

is well-defined and bijective.

By Proposition 13 in [1, p.216], shrinking Y_o if necessary, we have

$$G \cdot \mu^{-1}(\beta) \cap Y_o = \{[g, v] \in Y_o : \mu_V(v) = 0\}.$$

Let $Y^{(G_x)} = \{m \in Y : \ (G_m) = (G_x)\}$. It is easy to check that

$$Y^{(G_x)} = G \times_{G_x} V^{G_x} \cong G/G_x \times V^{G_x}, \tag{2}$$

where $V^{G_x} = \{x \in V : G_m = G_x\}$, and $\mu(Y_o^{(G_x)}) = G \cdot \beta$. Therefore

$$Y_o \cap G \cdot \mu^{-1}(\beta) = Y_o^{(G_x)} \text{ and } Y_o^\beta = Y_o \cap V^{G_x}. \tag{3}$$

This implies that both Y_o^β and $G \cdot Y_o^\beta$ are symplectic submanifolds of M. Indeed, $T_y Y_o^\beta = V^{G_x}$ which is a symplectic subspace, and the tangent space at y of $G \cdot Y_o^\beta$ splits as

$$T_y G \cdot Y_o^\beta = T_y G \cdot y \overset{\perp_\omega}{\oplus} T_y Y_o^\beta,$$

since $T_y Y_o^\beta \subset (T_y G \cdot y)^{\perp_\omega} = Ker \mathrm{d}\mu_y$ and $G \cdot y$ is symplectic.

Now,

$$\tau|_{G/G_x \times_{G_x} V^{G_x}} = \omega|_{G \cdot x} + (\omega_V)|_{V^{G_x}}, \tag{4}$$

see Corollary 14 in [1, p. 217]. Hence, from (1), (2), (3) and (4), we obtain that ϕ is a symplectomorphism.

Now assume that $x \in M$ realizes the maximum of f or any $z \in \mu^{-1}(\mu(x))$ is a local maximum of f. Let $\beta = \mu(x)$. Using the same arguments as before, we may prove that $G \cdot z$ is symplectic, $G_z = G_x = G_\beta$ for every $z \in \mu^{-1}(\beta)$ and a G-orbit intersects $\mu^{-1}(\beta)$ in at most one point. Therefore the following application

$$\phi : G \cdot x \times \mu^{-1}(\beta) \longrightarrow G \cdot \mu^{-1}(\beta), \;\; \phi(gx, z) = gz$$

is well-defined, G-equivariant and bijective.

We claim that ϕ is a symplectomorphism. The set $\mu^{-1}(G\beta) \cap M^{(G_x)}$ is a manifold of constant rank and the quotient

$$(M_\beta)^{(G_x)} := (G \cdot \mu^{-1}(\beta) \cap M^{(G_x)})/G,$$

is a symplectic manifold, see Corollary 14 in [1]. Since $\mu^{-1}(\beta) \subset M^{G_x}$, we have

$$\mathrm{G} \cdot \mu^{-1}(\beta) = G \cdot \mu^{-1}(\beta) \cap M^{(G_x)},$$

i.e. $G \cdot \mu^{-1}(\beta)$ is a submanifold. Notice that β is a regular value of the restricted moment map

$$\mu : G \cdot \mu^{-1}(\beta) \longrightarrow G \cdot \beta.$$

This implies that $\mu^{-1}(\beta)$ is a submanifold of M and for every $z \in \mu^{-1}(\beta)$, the tangent space of $G \cdot \mu^{-1}(\beta)$ splits as

$$T_z G \cdot z \overset{\perp_\omega}{\oplus} T_z \mu^{-1}(\beta) = T_z G \cdot \mu^{-1}(\beta). \tag{5}$$

Since $T_z \mu^{-1}(\beta) = V^{G_x}$ and $G \cdot z$ is symplectic, one may conclude that both $G \cdot \mu^{-1}(\beta)$ and $\mu^{-1}(\beta)$ are symplectic submanifolds of M. Therefore, from (1) and (5) we obtain that ϕ is a G-equivariant symplectomorphism and the proposition is proved. ■

Proof of Theorem 1.1. ((1) $\Longleftrightarrow$ (2)). (1)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(1). Assume that the squared moment map is constant. Let $x \in M$. As in the proof of Proposition 1.2, we can show that $G \cdot x$ is symplectic and $G_x = G_{\mu(x)}$. Therefore all G-orbits are symplectic and G must be semisimple. Indeed, coadjoint orbits are of the form $G/C(T)$, where $C(T)$ is the centralizer of the torus T, so $Z(G) \subset G_x$ for every $x \in M$, i.e. $Z(G)$ acts trivially on M. Hence $Z(G)$ must be trivial since the G-action is effective.

Now, we show that the manifold M is mapped by the moment map to a single coadjoint orbit ((2)⇒(3)).

Let $G \cdot x$ be a principal orbit. Since G_x acts trivially on the slice, from the local normal form for the moment map, in a G-invariant neighborhood of $G \cdot x$ the moment map is given by

$$\mu([g,v]) = \mathrm{Ad}(g)(\beta).$$

This proves that there exists a G-invariant neighborhood of $G \cdot x$ which is mapped to a single coadjoint orbit. It is well-known that $M^{(G_x)}$ is an open dense subset of M and $M^{(G_x)}/G$ is connected ([15]). Since μ is G-equivariant, it induces a continuous application

$$\overline{\mu} : M^{(G_x)}/G \longrightarrow \mathfrak{g}/G,$$

which is locally constant. Hence $\overline{\mu}(M^{(G_x)}/G)$ is constant so $\overline{\mu}(M/G)$ is. Thus M is mapped by μ to a single coadjoint orbit; in particular $M = G \cdot \mu^{-1}(\beta)$. Note that this argument proves (4)⇒(3).

Let $x \in M$. As in the proof of Proposition 1.2, from (1), (3), (4) and (5), the following application

$$\phi : G \cdot x \times \mu^{-1}(\mu(x)) \longrightarrow M, \quad (gx, z) \longrightarrow gz,$$

is the desired G-equivariant symplectomorphism.

Let $y \in \mu^{-1}(\mu(x))$. Then $T_y\mu^{-1}(\mu(x)) = T_y M^{G_x}$. Namely, $G_y = G_x$ centralizes a torus so $N(G_x)/G_x$ is finite. On the other hand

$$M^{G_x} = N(G_x)/G_x \times \mu^{-1}(\mu(x)),$$

so $\mu^{-1}(\mu(x))$ is an almost complex totally geodesic submanifold of M. Now, we show that all G-orbits are almost complex ((2)⇒(5)).

Let $V \in T_y\mu^{-1}(\mu(x))$ and let $X^\#$ be a tangent vector of $T_yG \cdot y$ generated by the one parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\} \subseteq G$. Then

$$0 = \mathrm{d}\mu_y(V)(X) = \omega(X^\#, V) = g(J(X^\#), V) = g(X^\#, J(V)), \tag{6}$$

meaning that $T_yG \cdot y = T_y\mu^{-1}(\mu(x))^\perp$. Hence $G \cdot y$ is complex.

Finally we prove that ϕ is an isometry. Since ϕ is G-equivariant and G acts by isometries on M, it is enough to prove that $\mathrm{d}\phi_{(x,z)}$ is an isometry for every $z \in \mu^{-1}(\mu(x))$. Note that the tangent space of M splits as

$$T_zM = T_zG \cdot z \overset{\perp}{\oplus} T_z\mu^{-1}(\mu(x)), \tag{7}$$

for every $z \in \mu^{-1}(\mu(x))$. Hence it is enough to prove that the Killing vector fields generated by the one parameter subgroups of G have constant norm, measured along $\mu^{-1}(\mu(x))$.

Let $\xi \in \mathfrak{g}$ and let X be a vector field tangent to $\mu^{-1}(\mu(x))$. Therefore $[\xi^\#, X] = 0$ since ϕ is a G-equivariant diffeomorphism. Now, given $\eta^\#$ be such that $J(\eta^\#) = \xi^\#$, by the closeness of ω we have

$$0 = d\omega(X, \eta^\#, \xi^\#) = Xg(\xi^\#, \xi^\#),$$

proving that $\xi^{\#}$ has constant norm, measured along $\mu^{-1}(\mu(x))$.

Now, (2)⇒(3), (2)⇒(5) and (4)⇒(3) follow from the above discussion while (3)⇒(2) and (5)⇒(4) are easy to check.

((3)⇒(5)). We follow the notation introduced in the proof of the Proposition 1.2.

Let $G \cdot x$ be a G-orbit and let Y' be a neighborhood of the zero section of $(Y = G \times_{G_x} (\mathfrak{q} \oplus V), \tau)$ which is G-equivariant symplectomorphic to a neighborhood of $G \cdot x$. The moment map μ in Y' is given by

$$\mu([g, m, v]) = \mathrm{Ad}(g)(\beta + m + \mu_V(v)).$$

From Proposition 13 in [1], shrinking Y' if necessary, we have

$$\mu^{-1}(G \cdot \beta) \cap Y' = \{[g, m, v] : m = 0 \text{ and } \mu_V(v) = 0\}.$$

Since M is mapped by the moment map μ to a single coadjoint orbit $G \cdot \beta$, we conclude that $\mathfrak{q} = \{0\}$, i.e. $G \cdot x$ is symplectic. Moreover, one may check that any G-orbit is a principal orbit. Hence, as we have proved in (2) ⇒(3), $\mu^{-1}(\beta)$ is an almost complex submanifold of (M, J) from which one may deduce that all G-orbits are almost complex as well. ∎

Proof of Theorem 1.3. (1)⇒(2)⇒(3)⇒(1) follow using the same arguments in the proof of the Theorem 1.1 while (4)⇒(3) is easy to check.

((3)⇒(4)). Let $x \in M$ be a regular point and let $\beta = \mu(x)$. As in the proof of the Theorem 1.1, one may check that $M = G \cdot \mu^{-1}(\beta)$ and any G-orbit is symplectic. This b implies that β is a regular value of the application $\mu : M \longrightarrow G \cdot \beta$. Therefore $\mu^{-1}(\beta)$ is a closed submanifold whose tangent space is given by

$$T_y \mu^{-1}(\beta) = Ker \mathrm{d}\mu_y = (T_y G \cdot y)^{\perp_\omega}$$

and the tangent space of M splits as

$$T_y M = T_y G \cdot y \overset{\perp_\omega}{\oplus} T_y \mu^{-1}(\beta). \tag{8}$$

for every $y \in \mu^{-1}(\beta)$. In particular $\mu^{-1}(\beta)$ is symplectic.

Now, we show that G_x acts trivially on $\mu^{-1}(\beta)$.

Note that $G_y^o = G_x^o = G_\beta^o$, for every $y \in \mu^{-1}(\beta)$, due the fact that $G \cdot y$ is symplectic, $G_y \subset G_\beta$ since μ is G-equivariant, and $\mu^{-1}(\beta)$ is connected since both G and $M = G \cdot \mu^{-1}(\beta)$ are. We now claim that G_x acts trivially on $\mu^{-1}(\beta)$. Indeed, suppose that we may find a sequence $x_n \to x$ in $\mu^{-1}(\beta)$ and a sequence $g_n \in G_{x_n} - G_x \subseteq G_\beta$ such that $g_n x_n = x_n$. Since the G-action is proper we may assume that $g_n \to g_o$ which lies in G_x. In particular the sequence g_n converges to g_o in G_β, since it is a closed Lie group. Now, G_x is an open subset of G_β, since $G_x^o = G_\beta^o$; therefore there exists n_o such that $g_n \in G_x$ for $n \geq n_o$ which is an absurd. Thus, there exists an open subset U' of x in $\mu^{-1}(\beta)$ such that $G_z \subset G_x$, $\forall z \in U'$. On the other hand, from the slice theorem, see [15], there exists a neighborhood U of the regular point x such that $(G_z) = (G_x)$ $\forall z \in U$. Shrinking U' if necessary, we may assume that $U' \subset U$. Therefore, keep in mind that $G_y^o = G_x^o$ for every

$y \in \mu^{-1}(\beta)$, we have that $G_x = G_y$ for every $y \in U'$. Since G_x is compact we conclude that G_x acts trivially on $\mu^{-1}(\beta)$. Hence the application

$$\phi : G \cdot x \times \mu^{-1}(\beta) \longrightarrow M, \quad \phi(gG_x, z) = gz$$

is well-defined, smooth and G-equivariant. Moreover, from (1) and (8) we get that

$$\phi^* \omega = \omega_{|_{G \cdot x}} + \omega_{|_{\mu^{-1}(\mu(x))}} \tag{9}$$

Since $Z(G) \subseteq G^o_{\mu(x)} = G^o_x$, G must be semisimple whenever G is a reductive Lie group and ϕ turns out to be a symplectomorphism since, from Proposition 2.1, a G-orbit intersects $\mu^{-1}(\beta)$ in at most one point.

Now assume that $N(G_x)/G_x$ is a finite group whenever x is a regular point. As in the proof of Theorem 1.1, one may show that $\mu^{-1}(\beta) \cap M^{G_x}$ is almost complex, $G \cdot y$ is almost complex for every $y \in \mu^{-1}(\beta) \cap M^{G_x}$ and finally

$$\overline{\phi} = \phi_{|_{G \cdot x \times (\mu^{-1}(\beta) \cap M^{G_x})}} : G \cdot x \times (\mu^{-1}(\beta) \cap M^{G_x}) \longrightarrow M^{(G_x)}, \ \phi([gx, z]) = gz,$$

is G-equivariant and it satisfies (9). Therefore it is a local diffeomorphism. Now, following the proof of Theorem 1.1 we have that $\overline{\phi}$ is an isometry with respect to g. Since $M^{(G_x)}$ is an open dense subset of M, we obtain that ϕ is an isometry with respect to g, the submanifold $\mu^{-1}(\beta)$ is almost complex, so all G-orbits are since the symplectic splitting (8) turns out to be g-orthogonal, concluding our proof. ■

Proof of Corollary 1.4

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G. Since any elliptic element is conjugate to an element of $\mathfrak{k}$, we have that the squared moment map $f = \parallel \mu \parallel^2$ is positive. Therefore, as in the proof of Theorem 1.2, one may check that all G-orbits are symplectic. The last statement follows now immediately from Theorem 1.3. ■

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