Composition methods and homotopy types of the gauge groups of $Sp(2)$ and $SU(3)$

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Abstract

We estimate the number of homotopy types of the gauge groups of $Sp(2)$ and $SU(3)$.

1 Introduction

Let G be a compact, connected, simple Lie group. The fact that $\pi_3(G) = \pi_4(BG) = \mathbb{Z}$ leads to the classification of principal G bundles P_k over S^4 by the integer k in $\mathbb{Z}$. The gauge group $\mathcal{G}_k(G)$ acts freely on the space $Map(P_k, EG)$ of all G equivariant maps from P_k to EG and its orbit space is given by the k–component of the space $Map_k(S^4, BG)$ of maps from S^4 to BG. Since $Map(P_k, EG)$ is contractible, we get $B\mathcal{G}_k(G) \simeq Map_k(S^4, BG)$. Similarly, if $\mathcal{G}_k^b(G)$ is the based gauge group which consists of base point preserving automorphisms on P_k, then $B\mathcal{G}_k^b(G) \simeq \Omega_k^3 G$ [1]. Then we have the following fibrations:

$$\Omega_k^3 G \to B\mathcal{G}_k(G) \to BG\,, \quad \mathcal{G}_k(G) \to G \xrightarrow{\alpha_k} \Omega_k^3 G\,.$$

In this paper we study the homotopy types of of gauge groups associated with principal $Sp(2)$ and $SU(3)$ bundles over S^4.

This paper is organized as follows. In Section 2, we collect some known facts concerning some homotopy groups of spheres, $Sp(2)$ and $SU(3)$, which will be used

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in the next section. In Section 3, we calculate the homotopy group $[G, \Omega^3 G]$ for $G = Sp(2)$ and $SU(3)$. In Section 4, we apply the results to estimate the number of homotopy types of gauge groups of $Sp(2)$ and $SU(3)$.

2 Some known results

In this section we collect some known results which will be used in Section 3.

Notation. $\pi_i(X : p)$ denotes the p-primary component of the homotopy group $\pi_i(X)$; in particular $\pi^n_m = \pi_m(S^n : 2)$.

Firstly, we recall from [8] some results on homotopy groups of spheres:

$$\begin{aligned} &\pi_4(S^3) \cong \mathbb{Z}_2\{\eta_3\}, \quad \pi_6(S^3) \cong \mathbb{Z}_4\{\nu'\} \oplus \mathbb{Z}_3\{\alpha_1(3)\}, \\ &\pi_9(S^3) \cong \mathbb{Z}_3\{\alpha_1(3) \circ \alpha_1(6)\}, \quad \pi_{10}(S^3) \cong \mathbb{Z}_3\{\alpha_2(3)\} \oplus \mathbb{Z}_5, \\ &\pi_7(S^7) \cong \mathbb{Z}\{\iota_7\}, \quad \pi_{11}(S^7) \cong 0, \quad \pi_{10}(S^7) \cong \mathbb{Z}_8\{\nu_7\} \oplus \mathbb{Z}_3\{\alpha_1(7)\}\,, \end{aligned} \tag{2.1}$$

where $\{-\}$ indicates a generator of the group;

$$\begin{aligned} &\pi^3_5 \cong \mathbb{Z}_2\{\eta_3^2\}, \quad \pi^3_6 \cong \mathbb{Z}_4\{\nu'\}, \quad \pi^3_7 \cong \mathbb{Z}_2\{\nu'\eta_6\}, \\ &\pi^5_6 \cong \mathbb{Z}_2\{\eta_5\}, \quad \pi^5_7 \cong \mathbb{Z}_2\{\eta_5^2\}, \quad \pi^5_8 \cong \mathbb{Z}_8\{\nu_5\}. \end{aligned} \tag{2.2}$$

By (5.3) and (5.5) of [8], we have

$$2\nu' = \eta_3^3, \quad \eta_5^3 = 4\nu_5. \tag{2.3}$$

Secondly we consider the symplectic group $Sp(2)$, which is a S^3-bundle over S^7:

$$S^3 \xrightarrow{i} Sp(2) \xrightarrow{p} S^7 \tag{2.4}$$

so that we have a cellular decomposition

$$Sp(2) \simeq S^3 \cup e^7 \cup e^{10}\,,$$

where e^7 is attached to S^3 by the Massey element $\omega = \langle \iota_3, \iota_3 \rangle$, the Samelson product. Associated with (2.4), we have a homotopy exact sequence

$$(\mathrm{A})_n \qquad \cdots \to \pi_n(S^3) \xrightarrow{i_*} \pi_n(Sp(2)) \xrightarrow{p_*} \pi_n(S^7) \xrightarrow{\Delta} \pi_{n-1}(S^3) \to \cdots.$$

We recall from [6] some results on homotopy groups of $Sp(2)$:

$$\begin{aligned} \pi_6(Sp(2)) &\cong 0, \\ \pi_7(Sp(2)) &\cong \mathbb{Z}\{[12\iota_7]\}, \\ \pi_{13}(Sp(2)) &\cong \mathbb{Z}_4 \oplus \mathbb{Z}_2, \end{aligned} \tag{2.5}$$

where $[x]$ denotes an element of $\pi_n(Sp(2))$ such that $p_*([x]) = x$. We also have

$$\begin{aligned} \Delta(\iota_7) &= \omega = \nu' + \alpha_1(3), \\ \Delta(\alpha_1(7)) &= \alpha_1(3) \circ \alpha_1(6). \end{aligned} \tag{2.6}$$

Thirdly we consider the special unitary group $SU(3)$, which is a S^3-bundle over S^5:

$$S^3 \xrightarrow{i} SU(3) \xrightarrow{p} S^5 \tag{2.7}$$

so that we have a cellular decomposition

$$SU(3) \simeq S^3 \cup e^5 \cup e^8 ,$$

where e^5 is attached to S^3 by η_3, a suspension of the Hopf element η_2. Associated with (2.7), we have a homotopy exact sequence

$$(\mathrm{B})_n \qquad \cdots \to \pi_n(S^3) \xrightarrow{i_*} \pi_n(SU(3)) \xrightarrow{p_*} \pi_n(S^5) \xrightarrow{\Delta} \pi_{n-1}(S^3) \to \cdots .$$

We recall from [6] some results on homotopy groups of $SU(3)$:

$$\begin{aligned} &\pi_6(SU(3):2) = \mathbb{Z}_2\{i_*(\nu')\}, \quad \pi_7(SU(3):2) = 0, \\ &\pi_8(SU(3):2) = \mathbb{Z}_4\{[2\iota_5]\nu_5\}, \quad \pi_{11}(SU(3):2) = \mathbb{Z}_4\{[\nu_5^2]\}, \end{aligned} \tag{2.8}$$

where we denote by $[x]$ an element of $\pi_n(SU(3):2)$ such that $p_*([x]) = x$.

3 The homotopy group $[G, \Omega^3 G]$ for $G = Sp(2)$ and $SU(3)$

Firstly we calculate $[Sp(2), \Omega^3 Sp(2)] \cong [S^3 \wedge Sp(2), Sp(2)]$.

Theorem 3.1. $[Sp(2), \Omega^3 Sp(2)] \cong \mathbb{Z}_{40} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Proof. Recall from [5, Lemma 2.1] that

$$S^3 Sp(2) \simeq S^6 \cup_{S^3\omega} e^{10} \vee S^{13} ,$$

where $\omega = \nu' + \alpha_1(3)$. Hence we have

$$[S^3 \wedge Sp(2), Sp(2)] \cong [S^6 \cup_{S^3\omega} e^{10}, Sp(2)] \oplus \pi_{13}(Sp(2)) ,$$

where we have $\pi_{13}(Sp(2)) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ by (2.5). Hence it is sufficient to calculate $[S^6 \cup_{S^3\omega} e^{10}, Sp(2)]$. For simplicity we put $\gamma = S^3\omega$. Then we have

$$\gamma = 2\nu_6 + \alpha_1(6) \in \pi_9(S^6) \cong \mathbb{Z}_8\{\nu_6\} \oplus \mathbb{Z}_3\{\alpha_1(6)\}.$$

We consider the exact sequence

$$\cdots \to \pi_7(Sp(2)) \xrightarrow{(S\gamma)^*} \pi_{10}(Sp(2)) \xrightarrow{\pi^*} [S^6 \cup_\gamma e^{10}, Sp(2)] \xrightarrow{j^*} \pi_6(Sp(2)) \to \cdots$$

associated with the cofibration

$$S^9 \xrightarrow{\gamma} S^6 \xrightarrow{j} S^6 \cup_\gamma e^{10},$$

where we have $\pi_6(Sp(2)) = 0$ by (2.5). Hence we have

$$[S^6 \cup_\gamma e^{10}, Sp(2)] \cong \mathrm{Coker}(S\gamma)^*.$$

We consider the exact sequence $(\mathrm{A})_n$ associated with the fibration (2.4). In particular, we consider the case $n = 7$:

$$\pi_7(S^3) \xrightarrow{i_*} \pi_7(Sp(2)) \xrightarrow{p_*} \pi_7(S^7) \xrightarrow{\Delta} \pi_6(S^3),$$

where by (2.1) we have

$$\pi_7(S^7) = \mathbb{Z}\{\iota_7\}, \quad \pi_6(S^3) = \mathbb{Z}_4\{\nu'\} \oplus \mathbb{Z}_3\{\alpha_1(3)\}$$

and we have

$$\Delta(\iota_7) = \nu' + \alpha_1(3), \quad \pi_7(Sp(2)) = \mathbb{Z}\{[12\iota_7]\}$$

respectively by (2.6) and (2.5).

We consider the exact sequence $(\mathrm{A})_{10}$:

$$\cdots \to \pi_{11}(S^7) \to \pi_{10}(S^3) \xrightarrow{i_*} \pi_{10}(Sp(2)) \xrightarrow{p_*} \pi_{10}(S^7) \xrightarrow{\Delta} \pi_9(S^3) \to \cdots$$

associated with (2.4), where by (2.6) we have

$$\Delta(\alpha_1(7)) = \alpha_1(3) \circ \alpha_1(6),$$

which implies that

$$\pi_{10}(Sp(2)) = \mathbb{Z}_8\{[\nu_7]\} \oplus \mathbb{Z}_3\{i_*(\alpha_2(3))\} \oplus \mathbb{Z}_5. \tag{3.1}$$

We will calculate

$$(S\gamma)^* : \pi_7(Sp(2)) \to \pi_{10}(Sp(2)).$$

Since $\Delta(\iota_7)$ is of order 12, we can define a Toda bracket

$$\{\Delta(\iota_7), 12\iota_6, \alpha_1(6)\} \subset \pi_{10}(S^3).$$

Then we have

$$\begin{aligned}
\{\Delta(\iota_7), 12\iota_6, \alpha_1(6)\} &= \{\nu' + \alpha_1(3), 12\iota_6, \alpha_1(6)\} && \text{by (2.6)} \\
&\supset \{(\nu' + \alpha_1(3))4\iota_6, 3\iota_6, \alpha_1(6)\} && \text{by Proposition 1.2 of [8]} \\
&= \{\alpha_1(3), 3\iota_6, \alpha_1(6)\} && \text{by the fact } 4\nu' = 0 \\
&\ni \alpha_2(3) && \text{by Lemma 13.5 of [8].}
\end{aligned}$$

Hence by Theorem 2.1 of [6], there exists an element β of $\pi_7(Sp(2))$ such that

$$\begin{aligned} p_*(\beta) &= 12\iota_7, \\ i_*(\alpha_2(3)) &= \beta \circ \alpha_1(7). \end{aligned} \tag{3.2}$$

Since $p_* : \pi_7(Sp(2)) \to \pi_7(S^7)$ is an injective by (2.5), we have $\beta = [12\iota_7]$. So by (3.2) we have

$$i_*(\alpha_2(3)) = [12\iota_7] \circ \alpha_1(7).$$

Then we have

$$\begin{aligned} (S\gamma)^*([12\iota_7]) &= [12\iota_7] \circ 2\nu_7 + [12\iota_7] \circ \alpha_1(7) \\ &= [12\iota_7] \circ 2\nu_7 + i_*(\alpha_2(3)). \end{aligned} \tag{3.3}$$

By the facts that $p_*([12\iota_7]\circ 2\nu_7) = 24\nu_7 = 0$ and that $p_* : \pi_{10}(Sp(2):2) \to \pi_{10}(S^7:2)$ is an isomorphism, we have $[12\iota_7]\circ 2\nu_7 = 0$. So by (3.3) we have

$$(S\gamma)^*([12\iota_7]) = i_*(\alpha_2(3)).$$

Then by (3.1) we have $\mathrm{Coker}(S\gamma)^* = \mathbb{Z}_8\{[\nu_7]\} \oplus \mathbb{Z}_5$. ■

Next we calculate $[SU(3), \Omega^3 SU(3)] \cong [S^3 \wedge SU(3), SU(3)]$.

Theorem 3.2. $[SU(3), \Omega^3 SU(3)] \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{24}$.

Proof. Recall from [5, Lemma 2.1] that

$$S^3 SU(3) \simeq S^6 \cup_{\eta_6} e^8 \vee S^{11}.$$

Since η_6 is of order 2, we have

$$S^3 SU(3) \simeq_p S^6 \vee S^8 \vee S^{11},$$

where p is an odd prime. So localized at $p > 2$, we have

$$\begin{aligned}[S^3 \wedge SU(3), SU(3)] &= [S^6 \vee S^8 \vee S^{11}, S^3 \times S^5] \\ &= \pi_6(S^3 \times S^5) \oplus \pi_8(S^3 \times S^5) \oplus \pi_{11}(S^3 \times S^5).\end{aligned}$$

So localized at primes $p > 2$, we have $[S^3 \wedge SU(3), SU(3)] \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Now we concentrate on 2-primary components of $[S^6 \cup_{\eta_6} e^8, SU(3)]$. So in the following we work in the 2-local category. We consider the following commutative diagram

$$\begin{tikzcd}
 & \pi^5_9 \arrow[d, "\Delta"'] & & \pi^5_7 \arrow[d, "\Delta"'] & \\
 & \pi^3_8 \arrow[d, "i_*"'] & & \pi^3_6 \arrow[d, "i_*"'] & \\
\pi_7(SU(3)) \arrow[r, "\eta_7^*"] \arrow[d, "p_*"'] & \pi_8(SU(3)) \arrow[r, "\pi^*"] \arrow[d, "p_*"'] & {[S^6 \cup_{\eta_6} e^8, SU(3)]} \arrow[r, "j^*"] \arrow[d, "p_*"'] & \pi_6(SU(3)) \arrow[r, "\eta_6^*"] \arrow[d, "p_*"'] & \pi_7(SU(3)) \arrow[d, "p_*"'] \\
\pi^5_7 \arrow[r, "\eta_7^*"] \arrow[d, "\Delta"'] & \pi^5_8 \arrow[r, "\pi^*"] \arrow[d, "\Delta"'] & {[S^6 \cup_{\eta_6} e^8, S^5]} \arrow[r, "j^*"] \arrow[d] & \pi^5_6 \arrow[r, "\eta_6^*"] \arrow[d, "\Delta"'] & \pi^5_7 \arrow[d, "\Delta"'] \\
\pi^3_6 \arrow[r, "\eta_6^*"] & \pi^3_7 \arrow[r, "\pi^*"] & {[S^5 \cup_{\eta_5} e^7, S^3]} \arrow[r, "j^*"] & \pi^3_5 \arrow[r, "\eta_5^*"] & \pi^3_6
\end{tikzcd}$$

where vertical and horizontal sequences are exact associated with the fibrations $(\mathrm{B})_8$ and $(\mathrm{B})_6$ and the cofibration

$(\mathrm{C})_n$ $$S^{n+1} \xrightarrow{\eta_n} S^n \xrightarrow{j} S^n \cup_{\eta_n} e^{n+2}$$

respectively.

We consider the exact sequence

$$\pi^3_6 \xrightarrow{\eta_6^*} \pi^3_7 \to [S^5 \cup_{\eta_5} e^7, S^3] \to \pi^3_5 \xrightarrow{\eta_5^*} \pi^3_6$$

associated with the cofibration (C)$_5$, where by (2.2) we have

$$\pi_5^3 = \mathbb{Z}_2\{\eta_3^2\}, \quad \pi_6^3 = \mathbb{Z}_4\{\nu'\}, \quad \pi_7^3 = \mathbb{Z}_2\{\nu'\eta_6\}.$$

So we see that η_6^* is an epimorphism. By (2.3) we have $2\nu' = \eta_3^3$, which implies that η_5^* is a monomorphism. So we have

$$[S^5 \cup_{\eta_5} e^7, S^3] = 0.$$

We consider the exact sequence

$$\pi_7^5 \xrightarrow{\eta_7^*} \pi_8^5 \xrightarrow{\pi^*} [S^6 \cup_{\eta_6} e^8, S^5] \to \pi_6^5 \xrightarrow{\eta_6^*} \pi_7^5$$

associated with the cofibration (C)$_6$, where by (2.2) we have

$$\pi_6^5 = \mathbb{Z}_2\{\eta_5\}, \quad \pi_7^5 = \mathbb{Z}_2\{\eta_5^2\}, \quad \pi_8^5 = \mathbb{Z}_8\{\nu_5\}.$$

So we see that η_6^* is an isomorphism. By (2.3) we have $\eta_7^*(\eta_5^2) = \eta_5^2 \circ \eta_7 = 4\nu_5$. Hence we have

$$[S^6 \cup_{\eta_6} e^8, S^5] = \mathbb{Z}_4\{\pi^*(\nu_5)\}.$$

Consider the following commutative diagram, which is deduced from the one above:

$$\begin{tikzcd}
 & 0 \arrow[d, "i_*"'] & & \pi_6^3 \arrow[d, "i_*"'] & \\
0 \arrow[r] & \pi_8(SU(3)) \arrow[r, "\pi^*"] \arrow[d, "p_*"'] & {[S^6 \cup_{\eta_6} e^8, SU(3)]} \arrow[r, "j^*"] \arrow[d, "p_*"'] & \pi_6(SU(3)) \arrow[r] \arrow[d, "p_*"'] & 0 \\
\pi_7^5 \arrow[r, "\eta_7^*"] \arrow[d, "\Delta"'] & \pi_8^5 \arrow[r, "\pi^*"] \arrow[d, "\Delta"'] & {[S^6 \cup_{\eta_6} e^8, S^5]} \arrow[r, "j^*"] \arrow[d] & 0 & \\
\pi_6^3 \arrow[r, "\eta_6^*"] & \pi_7^3 \arrow[r] & 0 & &
\end{tikzcd}$$

Since $p_* : [S^6 \cup_{\eta_6} e^8, SU(3)] \to [S^6 \cup_{\eta_6} e^8, S^5]$ is an epimorphism, there exists an element $\alpha \in [S^6 \cup_{\eta_6} e^8, SU(3)]$ such that

$$p_*(\alpha) = \pi^*(\nu_5).$$

Since $j^*(2\alpha) = 0$, there exists $\beta \in \pi_8(SU(3))$ such that

$$\pi^*(\beta) = 2\alpha.$$

By the commutativity of the above diagram we have

$$\pi^*(p_*(\beta)) = 2p_*(\alpha) = 2\pi^*(\nu_5).$$

By the exactness of the middle column we have

$$p_*(\beta) \equiv 2\nu_5 \mod \{\mathrm{Im}\, \eta_7^* = 4\nu_5\}.$$

Hence for some odd integer a we have

$$\beta = a[2\iota_5]\nu_5.$$

Thus we have

$$4\alpha = 2\pi^*(\beta) = 2\pi^*([2\iota_5]\nu_5) \neq 0,$$

which implies that α is of order 8. Hence we have

$$[S^6 \cup_{\eta_6} e^8, SU(3)] \cong \mathbb{Z}_8.$$

Since $\pi_{11}(SU(3)) \cong \mathbb{Z}_4$ by (2.8), we obtain that $[S^3 \wedge SU(3), SU(3)] \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8$ in the 2-local category. ∎

Remark 3.3. *These two theorems are known to K. Maruyama–H. Ooshima as* $\pi_3(map(G,G))$ [4].

Remark 3.4. *The result for $SU(3)$ is obtained by Hamanaka and Kono by an entirely different method, unstable K-theory* [2].

4 Homotopy types of $\mathcal{G}_k(Sp(2))$

As an application of the previous section, we estimate the number of homotopy types of the gauge groups of $Sp(2)$ and $SU(3)$. First we recall the following two propositions from [7, Example 4.4 and Proposition 4.2].

Proposition 4.1. *If $\mathcal{G}_k(Sp(n))$ is homotopy equivalent to $\mathcal{G}_l(Sp(n))$, then $(n(2n+1), k) = (n(2n+1), l)$ for even n and $(4n(2n+1), k) = (4n(2n+1), l)$ for odd n.*

So, if $\mathcal{G}_k(Sp(2))$ is homotopy equivalent to $\mathcal{G}_l(Sp(2))$, then $(10, k) = (10, l)$. Therefore there are at least four homotopy types of $\mathcal{G}_k(Sp(2))$.

Proposition 4.2. *If $\mathcal{G}_k(SU(n))$ is homotopy equivalent to $\mathcal{G}_l(SU(n))$, then $(n(n^2-1)/(n,2), k) = (n(n^2-1)/(n,2), l)$.*

Observe that there is a minor numerical error in the integer for $n(n^2-1)/(n,2)$ in [7], that is, $n+1$ in Proposition 4.2 of [7] should be n. So, if $\mathcal{G}_k(SU(3))$ is homotopy equivalent to $\mathcal{G}_l(SU(3))$, then $(24, k) = (24, l)$. Therefore there are at least eight homotopy types of $\mathcal{G}_k(SU(3))$.

Now let us restate the following useful lemma due to Hamanaka-Kono [2, Lemma 3.2]. Let X be a connected loop space, with $*$ its base point, $\mu : X \times X \to X$ its loop multiplication and $\iota : X \to X$ its homotopy inverse. For an integer n we define a self map $n : X \to X$ as follows: $0 = *$, $1 = 1_X$, $n = \mu \circ ((n-1) \times 1_X) \circ \Delta$ for a positive integer n. If $n < 0$, then $n = \iota \circ (-n)$.

Lemma 4.3. *Let k, k' and d be non-zero integers satisfying $(k, d) = (k', d)$. Let $\pi_j(X)$ be finite for any j, Y a finite complex and $\alpha : Y \to X$ any continuous map. If $d\alpha = 0$, then there exists a homotopy equivalence*

$$(k'/k)_d : X \to X$$

where $k' \circ \alpha \simeq (k'/k)_d \circ k \circ \alpha$.

Now we consider the following fibration for $G = Sp(2)$ and $SU(3)$:

$$\mathcal{G}_k(G) \to G \xrightarrow{\alpha_k} \Omega_k^3 G \cong [S^3 \wedge G, G]\,,$$

where the map α_k is equal to $\gamma \circ (k\epsilon \wedge i_G)$ with γ the commutator map and ϵ a generator of $\pi_3(G)$. By Theorems 3.1 and 3.2, we have the following results.

Theorem 4.4. 1. $40(\gamma \circ \epsilon \wedge i_{Sp(2)}) = 40\alpha_k = 0$.

2. $24(\gamma \circ \epsilon \wedge i_{SU(3)}) = 24\alpha_k = 0$.

Using the localization technique stated in Lemma 4.3, we can obtain a self-homotopy equivalence h of $\Omega_0^3 G$ such that $h \circ (k \circ \alpha_1) \simeq (l \circ \alpha_1)$ holds if

$$\begin{aligned} (40, k) &= (40, l) \quad \text{for} \quad G = Sp(2), \\ (24, k) &= (24, l) \quad \text{for} \quad G = SU(3)\,. \end{aligned} \tag{4.1}$$

Hence we obtain the following result.

Theorem 4.5. *If $(40, k) = (40, l)$, then $\mathcal{G}_k(Sp(2))$ is homotopy equivalent to $\mathcal{G}_l(Sp(2))$.*

Therefore there are at most eight homotopy types of $\mathcal{G}_k(Sp(2))$. Together with Proposition 4.1, we conclude the following.

Corollary 4.6. *The number of homotopy types of $\mathcal{G}_k(Sp(2))$ is 4 or 6 or 8.*

Together with Proposition 4.2 and (4.1) we recover the result due to Hamanaka-Kono [2].

Theorem 4.7. *$\mathcal{G}_k(SU(3))$ is homotopy equivalent to $\mathcal{G}_l(SU(3)$ if and only if $(24, k) = (24, l)$.*

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