On some quasilinear elliptic equations with critical Sobolev exponents and non-standard growth conditions

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Abstract

In this paper we study a class of quasilinear elliptic problems involving critical exponents and non-standard growth condition. We establish the existence of at least one nontrivial solution using as main tool Ekeland's variational principle.

1 Introduction and preliminary results

In this paper we study the following p-Laplacian elliptic equation

$$\begin{cases} -\Delta_p u = \lambda |u|^{q(x)-2}u + |u|^{p^\star-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an bounded, open domain with smooth boundary $\partial\Omega$, $p^\star = Np/(N-p)$ is the critical Sobolev exponent, $q(x) \in C(\overline{\Omega})$ with $q(x) > 1$ in $\overline{\Omega}$ and $\lambda > 0$ is a constant. In this paper we consider the case $1 < p < N$.

Problems of type (1) were intensively studied since 1980's. In the particular case when $p = 2$ and $q(x) \equiv 2$ in $\overline{\Omega}$ such equations were studied in the celebrated paper by Brézis and Nirenberg [3]. In [3] is also pointed out the fact that such equations with critical Sobolev exponent appear naturally in some problems in geometry and

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physics. For further results on problem (1) in the case $p = 2$ we refer to [2] and [11]. In the case when a Hardy potential is also involved we refer to [12] and [13].

In the case when $1 < p < N$ and $q(x) \equiv q$ in $\overline{\Omega}$ with q a constant such that $1 < q < p$ Garcia Azorero and Peral Alonso proved in [6] that problem (1) has infinitely many solutions for $\lambda > 0$ small enough. They also established the existence of a nontrivial solution when $p < q < p^\star$ and $\lambda > 0$ is sufficiently large.

In the case when $1 < p^2 \leq N$ and $q(x) \equiv p$ in $\overline{\Omega}$ Arioli and Gazzola proved in [1] that equation (1) has a positive nontrivial solution for all $\lambda \in (0, \lambda_1)$, where λ_1 is the principal eigenvalue of the p-Laplacian, i.e.

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}.$$

Moreover, denoting by $\mathcal{S}$ the best constant of the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^{p^\star}(\Omega)$, i.e.

$$\mathcal{S} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\left(\displaystyle\int_\Omega |u|^{p^\star} \, dx\right)^{p/p^\star}},$$

Arioli and Gazzola proved that if $q(x) \equiv p$ in $\overline{\Omega}$, $1 < p < N < p^2$ and $\lambda \in (\lambda_1 - \Lambda, \lambda_1)$, where $\Lambda = \mathcal{S} \cdot |\Omega|^{-p/N}$ then equation (1) admits a positive nontrivial solution.

This time, in order to study problem (1), we will appeal to the variable exponent Lebesgue spaces $L^{q(x)}(\Omega)$. We point out certain properties of that spaces according to the papers of Kováčik and Rákosník [8] and Mihăilescu and Rădulescu [9, 10].

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $q(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{q(x)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{q(x)} \, dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{q(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{q(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [8, Theorem 2.5], the Hölder inequality holds [8, Theorem 2.1], they are reflexive if and only if $1 < q^- \leq q^+ < \infty$ [8, Corollary 2.7] and continuous functions are dense if $q^+ < \infty$ [8, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [8, Theorem 2.8]: if $0 < |\Omega| < \infty$ and q_1, q_2 are variable exponent so that $q_1(x) \leq q_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{q'(x)}(\Omega)$ the conjugate space of $L^{q(x)}(\Omega)$, where $1/q(x) + 1/q'(x) = 1$. For any $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$ the Hölder type inequality

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{q^-} + \frac{1}{q'^-} \right) |u|_{q(x)} |v|_{q'(x)} \tag{2}$$

holds true.

If (u_n), $u \in L^{q(x)}(\Omega)$ and $q^+ < \infty$ then the following relations holds true

$$|u|_{q(x)} < 1 \quad \Rightarrow \quad |u|_{q(x)}^{q^+} \le \int_\Omega |u|^{q(x)}\, dx \le |u|_{q(x)}^{q^-} \tag{3}$$

$$|u_n - u|_{q(x)} \to 0 \quad \Leftrightarrow \quad \int_\Omega |u_n - u|^{q(x)}\, dx \to 0. \tag{4}$$

Finally, we remember that considering the Sobolev space $W_0^{1,p}(\Omega)$, defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_p \,,$$

we can state that if $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p^\star$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous. We refer to [8] for further properties of variable exponent Lebesgue-Sobolev spaces.

2 The main result

In this paper we study the existence of nontrivial weak solutions for problem (1) in the case when $q(x) \in C_+(\overline{\Omega})$ and assuming that there exists $x_0 \in \overline{\Omega}$ such that

$$1 < q(x_0) < p - 1 \,. \tag{5}$$

We say that $u \in W_0^{1,p}(\Omega)$ is a *weak solution* of problem (1) if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv\, dx - \int_\Omega |u|^{p^\star - 2} uv\, dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$.

Our main result is given by the following theorem.

Theorem 1. *Assume* $1 < p < N$, $q(x) \in C_+(\overline{\Omega})$ *satisfies (5) and* $q(x) < p^\star$ *in* $\overline{\Omega}$. *Then, there exists* $\lambda^\star > 0$ *such that problem (1) has a nontrivial weak solution for any* $\lambda \in (0, \lambda^\star)$.

3 Proof of the main result

In order to prove Theorem 1 we define the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\, dx - \frac{1}{p^\star}\int_\Omega |u|^{p^\star}\, dx\,.$$

Standard arguments show that $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx - \lambda\int_\Omega |u|^{q(x)-2}uv\, dx - \int_\Omega |u|^{p^\star-2}uv\, dx\,,$$

for all $u,\ v \in W_0^{1,p}(\Omega)$. Thus, we remark that in order to find weak solutions of equation (1) it is enough to find critical points for the functional J.

Lemma 1. *There exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there exist $\xi > 0$ and $r > 0$ such that*

$$J(u) \geq r, \quad \forall\, u \in W_0^{1,p}(\Omega) \text{ with } \|u\| = \xi\,.$$

Proof. Since $q(x) < p^\star$ for all $x \in \overline{\Omega}$ it follows that $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{q(x)}(\Omega)$. Thus, there exists a positive constant c_1 such that

$$|u|_{q(x)} \leq c_1\|u\|, \quad \forall\, u \in W_0^{1,p}(\Omega)\,. \tag{6}$$

Consider $\xi \in (0,1)$ with $\xi < 1/c_1$. Then the above relation implies

$$|u|_{q(x)} < 1, \quad \forall\, u \in W_0^{1,p}(\Omega), \text{ with } \|u\| = \xi\,. \tag{7}$$

By relations (3) and (7) we deduce that

$$\int_\Omega |u|^{q(x)}\, dx \leq |u|_{q(x)}^{q^-}, \quad \forall\, u \in W_0^{1,p}(\Omega), \text{ with } \|u\| = \xi\,. \tag{8}$$

Relations (8) and (6) imply

$$\int_\Omega |u|^{q(x)}\, dx \leq c_1^{q^-}\|u\|^{q^-}, \quad \forall\, u \in W_0^{1,p}(\Omega), \text{ with } \|u\| = \xi\,. \tag{9}$$

On the other hand, since $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^\star}(\Omega)$ we obtain that there exists $c_2 > 0$ such that

$$|u|_{p^\star} \leq c_2\|u\|, \quad \forall\, u \in W_0^{1,p}(\Omega)\,. \tag{10}$$

Relations (9) and (10) yield that for any $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \xi$ the following inequalities hold true

$$\begin{array}{rcl} J(u) & = & \frac{1}{p}\|u\|^p - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\, dx - \frac{1}{p^\star}|u|_{p^\star}^{p^\star} \\ & \geq & \frac{1}{p}\|u\|^p - \frac{\lambda}{q^-}c_1^{q^-}\|u\|^{q^-} - \frac{c_2^{p^\star}}{p^\star}\|u\|^{p^\star}\,. \end{array} \tag{11}$$

Thus, there exists two positive constants $a_1,\ a_2 > 0$ such that

$$J(u) \geq \|u\|^{q^-}\left[\frac{1}{p}\|u\|^{p-q^-} - \frac{\lambda \cdot a_1}{q^-} - \frac{a_2}{p^\star}\|u\|^{p^\star-q^-}\right]\,.$$

Define $Q : [0, \infty) \to \mathbb{R}$ by

$$Q(t) = \frac{1}{p} t^{p-q^-} - \frac{a_2}{p^\star} t^{p^\star - q^-} .$$

Since relation (5) holds true we deduce that $q^- < p < p^\star$ and thus, it is clear that there exists $\xi > 0$ such that $\max_{t \geq 0} Q(t) = Q(\xi) > 0$. We take $\lambda^\star = \frac{q^-}{a_1} Q(\xi)$ and we remark that there exists $r > 0$ such that for any $\lambda \in (0, \lambda^\star)$ we have

$$J(u) \geq r, \quad \forall\, u \in W_0^{1,p}(\Omega) \text{ with } \|u\| = \xi .$$

Lemma 1 is verified. ■

Lemma 2. *There exists $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $J(t\varphi) < 0$, for $t > 0$ small enough.*

Proof. Let $\Omega_0 = \{x \in \Omega;\ q(x) < p - 1\}$. Since relation (5) holds true it follows that $\Omega_0 \neq \emptyset$ and $|\Omega_0| > 0$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\varphi) \supset \overline{\Omega}_0$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \varphi \leq 1$ in Ω. For any $t \in (0, 1)$ we have

$$\begin{aligned} J(t\varphi) &= \frac{t^p}{p} \int_\Omega |\nabla \varphi|^p \, dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |\varphi|^{q(x)} \, dx - \frac{t^{p^\star}}{p^\star} \int_\Omega |\varphi|^{p^\star} \, dx \\ &\leq \frac{t^p}{p} \int_\Omega |\nabla \varphi|^p \, dx - \frac{\lambda}{q^+} \int_{\Omega_0} t^{q(x)} |\varphi|^{q(x)} \, dx \\ &\leq \frac{t^p}{p} \int_\Omega |\nabla \varphi|^p \, dx - \frac{\lambda \cdot t^{p-1}}{q^+} \int_{\Omega_0} |\varphi|^{q(x)} \, dx . \end{aligned}$$

It is clear that

$$J(t\varphi) < 0 ,$$

providing that

$$0 < t < \min\{1, \frac{\lambda \cdot p}{q^+} \cdot \frac{\int_{\Omega_0} |\varphi|^{q(x)} \, dx}{\int_\Omega |\nabla \varphi|^p \, dx}\} .$$

Lemma 2 is verified. ■

Proof of Theorem 1. By inequality (11) we obtain that J is bounded from below on $\overline{B_\xi(0)}$. Thus, using Ekeland's variational principle (see [5] or [14]) to the functional $J : \overline{B_\xi(0)} \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B_\xi(0)}$ such that

$$\begin{aligned} J(u_\epsilon) &< \inf_{\overline{B_\xi(0)}} J + \epsilon \\ J(u_\epsilon) &< J(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Using Lemmas 1 and 2 we find

$$\inf_{\partial B_\xi(0)} J \geq r > 0 \quad \text{and} \quad \inf_{\overline{B_\xi(0)}} J < 0 .$$

We choose $\epsilon > 0$ such that

$$0 < \epsilon \leq \inf_{\partial B_\xi(0)} J - \inf_{\overline{B_\xi(0)}} J .$$

Therefore, $J(u_\epsilon) < \inf_{\partial B_\xi(0)} J$ and thus, $u_\epsilon \in B_\xi(0)$.

We define $I : \overline{B_\xi(0)} \to \mathbb{R}$ by $I(u) = J(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + \delta \cdot v) - I(u_\epsilon)}{\delta} \geq 0$$

for small $\delta > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J(u_\epsilon + \delta \cdot v) - J(u_\epsilon)}{\delta} + \epsilon \cdot \|v\| \geq 0.$$

Letting $\delta \to 0$ it follows that $\langle J^{'}(u_\epsilon), v\rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|J^{'}(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{u_n\} \subset B_\xi(0)$ such that

$$J(u_n) \to c = \inf_{\overline{B_\xi(0)}} J < 0 \quad \text{and} \quad J^{'}(u_n) \to 0. \tag{12}$$

It is clear that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists $w \in W_0^{1,p}(\Omega$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in $W_0^{1,p}(\Omega$. Then Sobolev embeddings implies that $\{u_n\}$ converges strongly to u in $L^{q(x)}(\Omega)$ and weakly to u in $L^{p^\star}(\Omega)$. Thus, we get that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q(x)-2} u_n v \, dx = \int_\Omega |u|^{q(x)-2} uv \, dx \,,$$

and

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p^\star-2} u_n v \, dx = \int_\Omega |u|^{p^\star-2} uv \, dx \,,$$

for any $v \in W_0^{1,p}(\Omega)$.

On the other hand, relation (12) implies

$$\lim_{n\to\infty} \langle J^{'}(u_n), v\rangle = 0 \,,$$

for all $v \in W_0^{1,p}(\Omega)$.

The above information implies

$$J^{'}(u) = 0 \,,$$

and thus, u is a weak solution of equation (1).

We prove now that $u \neq 0$. Assume by contradiction that $u \equiv 0$ and

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx = l \geq 0 \,.$$

Since by relation (12) we have $\lim_{n\to\infty} \langle J^{'}(u_n), u_n\rangle = 0$ and $\{u_n\}$ converges strongly to 0 in $L^{q(x)}(\Omega)$ we obtain

$$\int_\Omega |\nabla u_n|^p \, dx - \int_\Omega |u_n|^{p^\star} \, dx = o(1)$$

or

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p^\star} \, dx = l \,.$$

Using again (12) we deduce

$$0 > c+o(1) = \frac{1}{p}\int_\Omega |\nabla u_n|^p\, dx - \lambda\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}\, dx - \frac{1}{p^\star}\int_\Omega |u_n|^{p^\star}\, dx \to \left(\frac{1}{p}-\frac{1}{p^\star}\right) l \geq 0$$

and that is a contradiction. We conclude that $u \neq 0$.

Thus, Theorem 1 is proved. ■

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