On a Higher Order Cauchy-Pompeiu Formula for Functions with Values in a Universal Clifford Algebra

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Abstract

By constructing suitable kernel functions, a higher order Cauchy-Pompeiu formula for functions with values in a universal Clifford algebra is obtained, leading to a higher order Cauchy integral formula.

1 Introduction

The theory of functions with values in a Clifford algebra has been thoroughly studied by many authors (see e.g. [1–18], [20–23]). In 1977 Delanghe and Brackx firstly introduced the concept of a k-regular function with values in a Clifford algebra and obtained a.o. the Cauchy integral formula and Taylor expansions (see [10]). Also Begehr obtained different integral representation formulae in the Clifford analysis setting (see.g. [1-3]). However all these results only hold for functions taking values in the Clifford algebra $C(V_{n,0})$, and the question arises if similar results may be obtained for functions with values in $C(V_{n,s}),\ 0 < s \leq n$.

The generalized form of the Cauchy integral formula for functions of one complex variable is known as the Cauchy-Pompeiu formula (see [19]). In [5, 12, 22] the Cauchy integral formula and the Cauchy-Pompeiu formula for functions with values in a universal Clifford algebra $C(V_{n,s})$ were obtained and some applications were given. In [4] we proved the higher order Cauchy-Pompeiu formula for functions with values in $C(V_{n,n})$, but the result is not that satisfactory since it only holds for $k < n$ and $s = n$. Similar results can be found in [2, 3, 10, 15–18].

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In this paper the higher order Cauchy-Pompeiu formula is established for functions with values in the Clifford algebra $C(V_{n,s})$, $0 < s < n$, without the condition $k < n$. As an application the higher order Cauchy integral formula is obtained. These results generalize the results in [4-5, 12].

In the following we will always assume that $s \geq 2$ and $n - s \geq 2$.

2 Preliminaries and notations

Let $V_{n,s}$ $(0 \leq s \leq n)$ be an n–dimensional $(n \geq 1)$ real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, let $C(V_{n,s})$ be the 2^n–dimensional real linear space with basis

$$\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\},$$

where N stands for the set $\{1, \cdots, n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N. We denote $e_\emptyset$ as e_0 and e_A as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. It follows at once from the multiplication rule that

$$\begin{cases} e_i^2 = 1, & i = 1, \cdots, s, \\ e_j^2 = -1, & j = s+1, \cdots, n, \\ e_i e_j = -e_j e_i, & 1 \leq i < j \leq n, \\ e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & 1 \leq h_1 < h_2 \cdots, < h_r \leq n. \end{cases} \tag{2.1}$$

Hence $C(V_{n,s})$ is a real linear, associative, but non-commutative algebra, called the universal Clifford algebra over $V_{n,s}$.

The involution in this Clifford algebra is defined by

$$\begin{cases} \overline{e_A} = (-1)^{\sigma(A) + \#(A \cap S)} e_A, & A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A \in \mathcal{P}N} \lambda_A \overline{e_A}, & \lambda = \sum\limits_{A \in \mathcal{P}N} \lambda_A e_A, \end{cases} \tag{2.2}$$

where $\sigma(A) = \#(A)(\#(A)+1)/2$. It follows that, in particular,

$$\begin{cases} \overline{e_i} = e_i, & i = 0, 1, \cdots, s, \\ \overline{e_j} = -e_j, & j = s+1, \cdots, n, \\ \overline{\lambda\mu} = \overline{\mu}\,\overline{\lambda}, & \lambda, \mu \in C(V_{n,s}). \end{cases} \tag{2.3}$$

Frequent use will be made of the notation $\mathbb{R}^n_z$, with $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, to denote $\mathbb{R}^n \setminus \{z\}$. In particular $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{(0, \cdots, 0)\}$. The meaning of the notations $\mathbb{R}^s_0$ and $\mathbb{R}^{n-s}_0$ is obvious.

Let Ω be an open non-empty subset of $\mathbb{R}^n$. We introduce the following operators:

$$\begin{aligned} D_1 &= \sum_{k=1}^{s} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{n,s})) \to C^{(r-1)}(\Omega, C(V_{n,s})), \\ D_2 &= \sum_{k=s+1}^{n} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{n,s})) \to C^{(r-1)}(\Omega, C(V_{n,s})), \end{aligned}$$

Definition 2.1. *(i) A function $f \in C^{(r)}(\Omega, C(V_{n,s}))$ $(r \geq 1)$ is called (D_α) left (right) regular in Ω if $D_\alpha[f] = 0$ ($[f]D_\alpha = 0$) in Ω, ($\alpha = 1, 2$).*
(ii) A function $f \in C^{(r)}(\Omega, C(V_{n,s}))$ $(r \geq k)$ is called (D_α) left (right) k-regular in Ω if $D^k_\alpha[f] = 0$ ($[f]D^k_\alpha = 0$) in Ω, ($\alpha = 1, 2$).
(iii) A function f is said to be (D_α) biregular if and only if it is both (D_α) left and right regular in Ω, ($\alpha = 1, 2$).
(iv) A function f is said to be (D_α) k-biregular if and only if it is both (D_α) left and right k-regular in Ω.
(v) A function $f \in C^{(r)}(\Omega, C(V_{n,s}))$ $(r \geq 1)$ is said to be LR regular in Ω if and only if it is both (D_1) left regular and (D_2) right regular, i.e., $D_1[f] = 0$ and $[f]D_2 = 0$ in Ω.

We will often need to consider the special case $\Omega = \Omega_1 \times \Omega_2$ where Ω_1 is an open non-empty set in $\mathbb{R}^s$ and Ω_2 is an open non-empty set in $\mathbb{R}^{n-s}$. In this case, the points in $\Omega_1 \times \Omega_2$ are denoted by $x = (x_1, x_2, \cdots, x_n) = (x^S, x^{N\setminus S})$, where $x^S = (x_1, x_2, \cdots, x_s) \in \Omega_1$ and $x^{N\setminus S} = (x_{s+1}, x_{s+2}, \cdots, x_n) \in \Omega_2$. Correspondingly, the functions defined in Ω are denoted by

$$f(x) = f(x^S, x^{N\setminus S}).$$

In the sequel we will use the following $C(V_{n,s})$–valued $(s-1)$–differential forms and $(n-s-1)$–differential forms:

$$\mathrm{d}\sigma_1 = \sum_{k=1}^{s} (-1)^{k-1} e_k \, \mathrm{d}\widehat{x}^S_k, \qquad \mathrm{d}\sigma_2 = \sum_{k=s+1}^{n} (-1)^{k-s-1} e_k \, \mathrm{d}\widehat{x}^{N\setminus S}_k,$$

where $\mathrm{d}\widehat{x}^S_k = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{k-1} \wedge \mathrm{d}x^{k+1} \cdots \wedge \mathrm{d}x^s$, $\mathrm{d}\widehat{x}^{N\setminus S}_k = \mathrm{d}x^{s+1} \wedge \cdots \wedge \mathrm{d}x^{k-1} \wedge \mathrm{d}x^{k+1} \cdots \wedge \mathrm{d}x^n$.

3 Kernel functions

In this section we will introduce the kernel functions which play the most important role in constructing the higher order Cauchy-Pompeiu formula. Similar results can be found in [2,3, 10,15–18].

Suppose $x^S = (x_1, x_2, \cdots, x_s) \in \mathbb{R}^s_0$ and $x^{N\setminus S} = (x_{s+1}, x_{s+2}, \cdots, x_n) \in \mathbb{R}^{n-s}_0$. Define for all $j \geq 1$ the functions $H^S_j(x^S)$ and $H^{N\setminus S}_j(x^{N\setminus S})$ as follows:

$$H^S_j(x^S) = \begin{cases} \dfrac{A_{j,s}}{\omega_s} \dfrac{(\mathbf{x}^S)^j}{\rho^s(x^S)}, & s \text{ odd}; \\ \dfrac{A_{j,s}}{\omega_s} \dfrac{(\mathbf{x}^S)^j}{\rho^s(x^S)}, & 1 \leq j < s, \ s \text{ even}; \\ \dfrac{A_{j-1,s}}{2\omega_s} \log((\mathbf{x}^S)^2), & j = s, \ s \text{ even}; \\ \dfrac{A_{s-1,s}}{2\omega_s} C_{l,0,s} (\mathbf{x}^S)^l \left(\log((\mathbf{x}^S)^2) - 2 \displaystyle\sum_{i=0}^{l-1} \dfrac{C_{i+1,0,s}}{C_{i,0,s}} \right), & j = s+l, l > 0, s \text{ even}; \end{cases} \tag{3.4}$$

$$
\begin{aligned}
&H_j^{N\backslash S}(x^{N\backslash S})\\
=&\begin{cases}
\dfrac{A_{j,n-s}}{\omega_{n-s}}\dfrac{(\overline{\mathbf{x}}^{N\backslash S})^j}{\rho^{n-s}(x^{N\backslash S})}, \quad n-s \text{ odd};\\
\dfrac{A_{j,n-s}}{\omega_{n-s}}\dfrac{(\overline{\mathbf{x}}^{N\backslash S})^j}{\rho^{n-s}(x^{N\backslash S})}, \quad 1\le j<n-s, \quad n-s \text{ even};\\
\dfrac{A_{j-1,n-s}}{2\omega_{n-s}}(-1)^{\frac{n-s}{2}}\log(\mathbf{x}^{N\backslash S}\overline{\mathbf{x}}^{N\backslash S}), \quad j=n-s, \quad n-s \text{ even};\\
\dfrac{A_{n-s-1,n-s}}{2\omega_{n-s}}(-1)^{\frac{n-s}{2}}C_{l,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^l\left(\log(\mathbf{x}^{N\backslash S}\overline{\mathbf{x}}^{N\backslash S})-2\sum\limits_{i=0}^{l-1}\dfrac{C_{i+1,0,n-s}}{C_{i,0,n-s}}\right),\\
\qquad\qquad j=n-s+l, l>0, n-s \text{ even},
\end{cases}
\end{aligned}
\tag{3.5}
$$

where $\mathbf{x}^S=\sum\limits_{k=1}^{s}x_ke_k$, $\mathbf{x}^{N\backslash S}=\sum\limits_{k=s+1}^{n}x_ke_k$, $\rho(x^S)=\left(\sum\limits_{k=1}^{s}x_k^2\right)^{\frac12}$, $\rho(x^{N\backslash S})=\left(\sum\limits_{k=s+1}^{n}x_k^2\right)^{\frac12}$, ω_m denotes the area of the unit sphere in $\mathbb{R}^m$, $(m=s,n-s)$, i.e. $\omega_m=\dfrac{2\pi^{m/2}}{\Gamma(m/2)}$, $(m=s,n-s)$,

$$
A_{j,m}=\frac{1}{2^{[\frac{j-1}{2}]}[\frac{j-1}{2}]!\prod\limits_{r=1}^{[\frac{j}{2}]}(2r-m)}, \qquad j\ge 1, m \,\text{odd or}\, 1\le j<m, m\,\text{even} \tag{3.6}
$$

and

$$
C_{j,0,m}=\begin{cases}
1, & j=0,\\
\dfrac{1}{2^{[\frac{j}{2}]}\left([\frac{j}{2}]\right)!\prod\limits_{\mu=0}^{[\frac{j-1}{2}]}(m+2\mu)}, & j\in\mathbf{N}^*=\mathbf{N}\backslash\{0\}.
\end{cases}
\tag{3.7}
$$

Lemma 3.1. *Let $C_{j,0,m}$ $(m=s,n-s)$ be given by (3.7), and $\mathbf{x}^S=x_1e_1+\cdots+x_se_s$, $\mathbf{x}^{N\backslash S}=x_{s+1}e_{s+1}+\cdots+x_ne_n$, then for $j\in\mathbf{N}^*$,*

$$
\begin{cases}
D_1[C_{j,0,s}(\mathbf{x}^S)^j]=[C_{j,0,s}(\mathbf{x}^S)^j]D_1=C_{j-1,0,s}(\mathbf{x}^S)^{j-1};\\
D_2[C_{j,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^j]=[C_{j,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^j]D_2=C_{j-1,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^{j-1}.
\end{cases}
\tag{3.8}
$$

and

$$
\begin{cases}
\begin{aligned}
&D_1\left[C_{l,0,s}(\mathbf{x}^S)^l\log((\mathbf{x}^S)^2)\right]=\left[C_{l,0,s}(\mathbf{x}^S)^l\log((\mathbf{x}^S)^2)\right]D_1\\
=\;&C_{l-1,0,s}(\mathbf{x}^S)^{l-1}\log((\mathbf{x}^S)^2)+2C_{l,0,s}(\mathbf{x}^S)^{l-1};\\
&D_2\left[C_{l,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^l\log(\mathbf{x}^{N\backslash S}\overline{\mathbf{x}}^{N\backslash S})\right]=\left[C_{l,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^l\log(\mathbf{x}^{N\backslash S}\overline{\mathbf{x}}^{N\backslash S})\right]D_2\\
=\;&C_{l-1,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^{l-1}\log(\mathbf{x}^{N\backslash S}\overline{\mathbf{x}}^{N\backslash S})+2C_{l,0,n-s}(\overline{\mathbf{x}}^{N\backslash S})^{l-1}.
\end{aligned}
\end{cases}
\tag{3.9}
$$

Theorem 3.1. *Let for all $j\ge 1$, $H_j^S(x^S)$ and $H_j^{N\backslash S}(x^{N\backslash S})$ be given by (3.4) and (3.5) , let $x^S\in\mathbb{R}_0^s$ and $x^{N\backslash S}\in\mathbb{R}_0^{n-s}$, then*

$$
\begin{cases}
D_1\left[H_1^S(x^S)\right]=\left[H_1^S(x^S)\right]D_1=0,\\
D_1\left[H_{j+1}^S(x^S)\right]=\left[H_{j+1}^S(x^S)\right]D_1=H_j^S(x^S), \;\; \textit{for all } j\ge 1.
\end{cases}
\tag{3.10}
$$

$$\begin{cases} D_2\left[H_1^{N\setminus S}(x^{N\setminus S})\right] = \left[H_1^{N\setminus S}(x^{N\setminus S})\right]D_2 = 0, \\ D_2\left[H_{j+1}^{N\setminus S}(x^{N\setminus S})\right] = \left[H_{j+1}^{N\setminus S}(x^{N\setminus S})\right]D_2 = H_j^{N\setminus S}(x^{N\setminus S}),\ \textit{for all } j \geq 1. \end{cases} \tag{3.11}$$

Note that similar formulae may be found in [17, 18].

Corollary 3.1. *Let for all $j \geq 1$, $H_j^S(x^S)$ and $H_j^{N\setminus S}(x^{N\setminus S})$ be given by (3.4) and (3.5) , let $x^S \in \mathbb{R}_0^s$ and $x^{N\setminus S} \in \mathbb{R}_0^{n-s}$, then*

$$\begin{cases} D_1^k\left[H_k^S(x^S)\right] = \left[H_k^S(x^S)\right]D_1^k = 0, \\ D_1^j\left[H_k^S(x^S)\right] = \left[H_k^S(x^S)\right]D_1^j = H_{k-j}^S(x^S),\ \textit{for all } 1 \leq j < k. \end{cases} \tag{3.12}$$

$$\begin{cases} D_2^k\left[H_k^{N\setminus S}(x^{N\setminus S})\right] = \left[H_k^{N\setminus S}(x^{N\setminus S})\right]D_2^k = 0, \\ D_2^j\left[H_k^{N\setminus S}(x^{N\setminus S})\right] = \left[H_k^{N\setminus S}(x^{N\setminus S})\right]D_2^j = H_{k-j}^{N\setminus S}(x^{N\setminus S}),\ \textit{for all } 1 \leq j < k. \end{cases} \tag{3.13}$$

Corollary 3.2. *Let for all $j \geq 1$, $H_j^S(x^S)$ and $H_j^{N\setminus S}(x^{N\setminus S})$ be given by (3.4) and (3.5) , let $x^S \in \mathbb{R}_{z^S}^s$ and $x^{N\setminus S} \in \mathbb{R}_{z^{N\setminus S}}^{n-s}$, then*

$$\begin{cases} D_1^k\left[H_k^S(x^S - z^S)\right] = \left[H_k^S(x^S - z^S)\right]D_1^k = 0, \\ D_1^j\left[H_k^S(x^S - z^S)\right] = \left[H_k^S(x^S - z^S)\right]D_1^j = H_{k-j}^S(x^S - z^S),\ \textit{for all } 1 \leq j < k. \end{cases} \tag{3.14}$$

$$\begin{cases} D_2^k\left[H_k^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})\right] = \left[H_k^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})\right]D_2^k = 0, \\ D_2^j\left[H_k^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})\right] = \left[H_k^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})\right]D_2^j = H_{k-j}^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S}), \\ \qquad \textit{for all } 1 \leq j < k. \end{cases} \tag{3.15}$$

Remark 3.1. It follows from Corollary 3.1 and Corollary 3.2 that, for $k \geq 1$, the functions $H_k^S(x^S)$ and $H_k^{N\setminus S}(x^{N\setminus S})$ are both D_1 k-biregular and D_2 k-biregular in $\mathbb{R}_0^n$.
It also follows that $H_k^S(x^S - z^S)$ and $H_k^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})$ are both D_1 k-biregular and D_2 k-biregular in $\mathbb{R}_z^n$.

4 Higher order Cauchy-Pompeiu formula

Let M_1 and M_2 be an s–dimensional, respectively an $(n-s)$–dimensional, differentiable oriented manifold with boundary contained in Ω_1 and in Ω_2, where Ω_1 and Ω_2 are open non-empty sets in $\mathbb{R}^s$ and $\mathbb{R}^{n-s}$ respectively. In the following, we shall only consider the higher order Cauchy-Pompeiu formula on the distinguished boundary $\partial M_1 \times \partial M_2$ of $M_1 \times M_2$. We will also use the following lemma (see [5]).

Lemma 4.1. *Let M_1 be an s–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega_1 \subset \mathbb{R}^s$ and let M_2 be an $(n-s)$–dimensional compact differentiable oriented manifold contained in some open non-empty subset $\Omega_2 \subset \mathbb{R}^{n-s}$. Let $f \in C^{(r)}(\Omega_1 \times \Omega_2, C(V_{n,s}))$, $g \in C^{(r)}(\Omega_1, C(V_{n,s}))$,*

$h \in C^{(r)}(\Omega_2, C(V_{n,s}))$, $r \geq 2$, and let ∂M_1 and ∂M_2 be given the induced orientations. Then

$$\int\limits_{\partial M_1 \times \partial M_2} g(x^S)\mathrm{d}\sigma_1 f(x^S, x^{N\setminus S})\mathrm{d}\sigma_2 h(x^{N\setminus S})$$

$$= \int\limits_{M_1 \times M_2} \Big\{ \Big[([g]D_1)(x^S)([f]D_2)(x^S, x^{N\setminus S}) + g(x^S)([D_1[f]]D_2)(x^S, x^{N\setminus S}) \Big] h(x^{N\setminus S})$$

$$+ \Big[([g]D_1)(x^S) f(x^S, x^{N\setminus S}) + g(x^S)(D_1[f])(x^S, x^{N\setminus S}) \Big] (D_2[h])(x^{N\setminus S}) \Big\} \mathrm{d}x.$$

Remark 4.1. It follows from Lemma 3.1 in [12] that the above integral over the distinguished boundary $\partial M_1 \times \partial M_2$ may be regarded as a repeated integral independent of the order of integration.

Theorem 4.1. (**Higher order Cauchy-Pompeiu formula**) *Let M_1 be an $s-$dimensional differentiable compact oriented manifold contained in some open non-empty subset $\Omega_1 \subset \mathbb{R}^s$, let M_2 be an $(n-s)-$dimensional compact differentiable oriented manifold contained in some open non-empty subset $\Omega_2 \subset \mathbb{R}^{n-s}$, let $f \in C^{(r)}(\Omega_1 \times \Omega_2, C(V_{n,s}))$, $r \geq k_1 + k_2$, $k_1, k_2 \in \mathbf{N}^*$, let ∂M_1 and ∂M_2 be given the induced orientations, and let for all $j \geq 1$, $H_j^S(x^S)$ and $H_j^{N\setminus S}(x^{N\setminus S})$ be given by (3.4) and (3.5). Then, for $z \in \overset{\circ}{M}_1 \times \overset{\circ}{M}_2$,*

$$\begin{aligned} & f(z) \\ = & \sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1}(-1)^{j_1+j_2}\int\limits_{\partial M_1\times\partial M_2} H_{j_1+1}^S(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H_{j_2+1}^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S}) \\ & +(-1)^{k_1}\int\limits_{M_1} H_{k_1}^S(x^S - z^S)\left(D_1^{k_1}[f]\right)(x^S, z^{N\setminus S})\,\mathrm{d}x^S \\ & +(-1)^{k_2}\int\limits_{M_2}\left([f]D_2^{k_2}\right)(z^S, x^{N\setminus S})\,H_{k_2}^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})\,\mathrm{d}x^{N\setminus S} \\ & +(-1)^{k_1+k_2+1}\int\limits_{M_1\times M_2} H_{k_1}^S(x^S - z^S)\left(D_1^{k_1}[f]D_2^{k_2}\right)(x^S, x^{N\setminus S})\,H_{k_2}^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S})\,\mathrm{d}x. \end{aligned}$$

Remark 4.2. It is easily verified that $\left[D_1^{j_1}[f]\right]D_2^{j_2} = D_1^{j_1}\left[[f]D_2^{j_2}\right]$, whence $D_1^{j_1}[f]D_2^{j_2}$ is well defined.

Remark 4.3. The existence of the integrals over the manifolds M_1, M_2 and $M_1 \times M_2$ follows from the weak singularity of the kernels $H_j^S(x^S)$ and $H_j^{N\setminus S}(x^{N\setminus S})$, for all $j \geq 1$.

Proof Step 1. Assume that $z \in \overset{\circ}{M}_1 \times \overset{\circ}{M}_2$. Take $\delta > 0$ such that $B_1(z^S, \delta) \subset \overset{\circ}{M}_1$, $B_2(z^{N\setminus S}, \delta) \subset \overset{\circ}{M}_2$. We introduce the following functions of δ :

$$\begin{aligned} & \Theta(\delta) \\ = & \sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1}(-1)^{j_1+j_2}\int\limits_{\partial(M_1\setminus B_1(z^S,\delta))\times\partial\left(M_2\setminus B_2\left(z^{N\setminus S},\delta\right)\right)} H_{j_1+1}^S(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H_{j_2+1}^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S}). \end{aligned}$$

and

$$\Delta(\delta) = (-1)^{k_1+k_2} \int\limits_{(M_1\setminus B_1(z^S,\delta))\times\left(M_2\setminus B_2\left(z^{N\setminus S},\delta\right)\right)} H^S_{k_1}(x^S - z^S)\left(D_1^{k_1}[f]D_2^{k_2}\right)(x^S, x^{N\setminus S})\, H^{N\setminus S}_{k_2}(x^{N\setminus S} - z^{N\setminus S})\,\mathrm{d}x.$$

By Theorem 3.1 and Stokes's formula we have that

$$\Theta(\delta) = \Delta(\delta). \tag{4.16}$$

Step 2. Obviously, by Remark 4.3, we also have that

$$\lim_{\delta\to 0}\Delta(\delta) = (-1)^{k_1+k_2} \int\limits_{M_1\times M_2} H^S_{k_1}(x^S - z^S)\left(D_1^{k_1}[f]D_2^{k_2}\right)(x^S, x^{N\setminus S})\, H^{N\setminus S}_{k_2}(x^{N\setminus S} - z^{N\setminus S})\,\mathrm{d}x. \tag{4.17}$$

For $j_1 = 0, \cdots, k_1 - 1$, $j_2 = 0, \cdots, k_2 - 1$, we introduce the following functions of δ :

$$\begin{aligned} &\Theta_{j_1,j_2}(\delta) \\ = \ &(-1)^{j_1+j_2} \int\limits_{\partial(M_1\setminus B_1(z^S,\delta))\times\partial\left(M_2\setminus B_2\left(z^{N\setminus S},\delta\right)\right)} H^S_{j_1+1}(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H^{N\setminus S}_{j_2+1}(x^{N\setminus S} - z^{N\setminus S}). \end{aligned}$$

and

$$\left\{\begin{aligned} &\Theta_{j_1,j_2,1}(\delta) \\ = \ &(-1)^{j_1+j_2} \int\limits_{\partial B_1(z^S,\delta)\times\partial M_2} H^S_{j_1+1}(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H^{N\setminus S}_{j_2+1}(x^{N\setminus S} - z^{N\setminus S}) \\ &\Theta_{j_1,j_2,2}(\delta) \\ = \ &(-1)^{j_1+j_2} \int\limits_{\partial M_1\times\partial B_2(z^{N\setminus S},\delta)} H^S_{j_1+1}(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H^{N\setminus S}_{j_2+1}(x^{N\setminus S} - z^{N\setminus S}), \\ &\Theta_{j_1,j_2,3}(\delta) \\ = \ &(-1)^{j_1+j_2} \int\limits_{\partial B_1(z^S,\delta)\times\partial B_2(z^{N\setminus S},\delta)} H^S_{j_1+1}(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H^{N\setminus S}_{j_2+1}(x^{N\setminus S} - z^{N\setminus S}). \end{aligned}\right. \tag{4.18}$$

where $\partial B_1(z^S, \delta)$ and $\partial B_2(z^{N\setminus S}, \delta)$ are given the induced orientations.

It is clear that

$$\begin{aligned} &\Theta_{j_1,j_2}(\delta) \\ = \ &(-1)^{j_1+j_2} \int\limits_{\partial M_1\times\partial M_2} H^S_{j_1+1}(x^S - z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H^{N\setminus S}_{j_2+1}(x^{N\setminus S} - z^{N\setminus S}) \\ &-\Theta_{j_1,j_2,1}(\delta) - \Theta_{j_1,j_2,2}(\delta) + \Theta_{j_1,j_2,3}(\delta), \end{aligned} \tag{4.19}$$

Moreover it is easily shown that

$$\left\{\begin{aligned} \lim_{\delta\to 0}\Theta_{0,0,1}(\delta) &= \int\limits_{\partial M_2} f\left(z^S, x^{N\setminus S}\right)\mathrm{d}\sigma_2 H_1^{N\setminus S}(x^{N\setminus S} - z^{N\setminus S}) \\ \lim_{\delta\to 0}\Theta_{0,0,2}(\delta) &= \int\limits_{\partial M_1} H_1^S(x^S - z^S)\,\mathrm{d}\sigma_1 f\left(x^S, z^{N\setminus S}\right), \\ \lim_{\delta\to 0}\Theta_{0,0,3}(\delta) &= f\left(z^S, z^{N\setminus S}\right). \end{aligned}\right. \tag{4.20}$$

and

$$\begin{cases} \displaystyle\int_{\partial M_2} f\left(z^S, x^{N\setminus S}\right) \mathrm{d}\sigma_2 H_1^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S}) \\ \displaystyle = \quad f\left(z^S, z^{N\setminus S}\right) + \int_{M_2} ([f]D_2)(z^S, x^{N\setminus S})\, H_1^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S})\,\mathrm{d}x^{N\setminus S}; \\ \displaystyle\int_{\partial M_1} H_1^S(x^S-z^S)\,\mathrm{d}\sigma_1 f\left(x^S, z^{N\setminus S}\right) \\ \displaystyle = \quad f\left(z^S, z^{N\setminus S}\right) + \int_{M_1} H_1^S(x^S-z^S)\,(D_1[f])(x^S, z^{N\setminus S})\,\mathrm{d}x^S. \end{cases} \tag{4.21}$$

whence by (4.19), (4.20) and (4.21), we obtain

$$\begin{aligned} &\lim_{\delta\to 0}\Theta_{0,0}(\delta) \\ = \quad & \int_{\partial M_1\times\partial M_2} H_1^S(x^S-z^S)\,\mathrm{d}\sigma_1 f\left(x^S, x^{N\setminus S}\right) \mathrm{d}\sigma_2 H_1^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S}) \\ - \quad & \int_{M_2} ([f]D_2)(z^S, x^{N\setminus S})\, H_1^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S})\,\mathrm{d}x^{N\setminus S} \\ - \quad & \int_{M_1} H_1^S(x^S-z^S)\,(D_1[f])(x^S, z^{N\setminus S})\,\mathrm{d}x^S - f\left(z^S, z^{N\setminus S}\right). \end{aligned} \tag{4.22}$$

In view of the weaker singularity of the kernels $H_j^S(x^S)$ and $H_j^{N\setminus S}(x^{N\setminus S})$, for all $j>1$, it may be proved, by Lemma 4.1, that

$$\begin{cases} \lim\limits_{\delta\to 0}\Theta_{j_1,0,1}(\delta) = 0, \quad j_1>0. \\ \lim\limits_{\delta\to 0}\Theta_{j_1,0,3}(\delta) = 0, \quad j_1>0. \end{cases} \tag{4.23}$$

Hence, for $j_1>0$,

$$\begin{aligned} &\lim_{\delta\to 0}\Theta_{j_1,0}(\delta) \\ = \quad & (-1)^{j_1} \int_{\partial M_1\times\partial M_2} H_{j_1+1}^S(x^S-z^S)\,\mathrm{d}\sigma_1 \left(D_1^{j_1}[f]\right)(x^S, x^{N\setminus S})\,\mathrm{d}\sigma_2 H_1^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S}) \\ & -(-1)^{j_1}\int_{\partial M_1} H_{j_1+1}^S(x^S-z^S)\,\mathrm{d}\sigma_1 \left(D_1^{j_1}[f]\right)(x^S, z^{N\setminus S}). \end{aligned} \tag{4.24}$$

In a similar way as for (4.21), it may be proved that

$$\begin{aligned} &\int_{\partial M_1} H_{j_1+1}^S(x^S-z^S)\,\mathrm{d}\sigma_1 \left(D_1^{j_1}[f]\right)(x^S, z^{N\setminus S}) \\ = \quad & \int_{M_1} \left(H_{j_1}^S(x^S-z^S)\left(D_1^{j_1}[f]\right)(x^S, z^{N\setminus S}) + H_{j_1+1}^S(x^S-z^S)\left(D_1^{j_1+1}[f]\right)(x^S, z^{N\setminus S})\right)\mathrm{d}x^S. \end{aligned} \tag{4.25}$$

Hence, for $j_1 > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0}\Theta_{j_1,0}(\delta)\\
= \;& (-1)^{j_1}\int\limits_{\partial M_1\times\partial M_2} H^S_{j_1+1}(x^S-z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]\right)(x^S,x^{N\setminus S})\,\mathrm{d}\sigma_2 H_1^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S})\\
-\;& (-1)^{j_1}\int\limits_{M_1} H^S_{j_1}(x^S-z^S)\left(D_1^{j_1}[f]\right)(x^S,z^{N\setminus S})\,\mathrm{d}x^S\\
-\;& (-1)^{j_1}\int\limits_{M_1} H^S_{j_1+1}(x^S-z^S)\left(D_1^{j_1+1}[f]\right)(x^S,z^{N\setminus S})\,\mathrm{d}x^S.
\end{aligned}\tag{4.26}$$

Similarly, for $j_2 > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0}\Theta_{0,j_2}(\delta)\\
= \;& (-1)^{j_2}\int\limits_{\partial M_1\times\partial M_2} H^S_{1}(x^S-z^S)\,\mathrm{d}\sigma_1\left([f]D_2^{j_2}\right)(x^S,x^{N\setminus S})\,\mathrm{d}\sigma_2 H_{j_2+1}^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S})\\
-\;& (-1)^{j_2}\int\limits_{M_2}\left([f]D_2^{j_2}\right)(z^S,x^{N\setminus S})\,H^{N\setminus S}_{j_2}(x^{N\setminus S}-z^{N\setminus S})\,\mathrm{d}x^{N\setminus S}\\
-\;& (-1)^{j_2}\int\limits_{M_2}\left([f]D_2^{j_2+1}\right)(z^S,x^{N\setminus S})\,H^{N\setminus S}_{j_2+1}(x^{N\setminus S}-z^{N\setminus S})\,\mathrm{d}x^{N\setminus S}.
\end{aligned}\tag{4.27}$$

In the same way, for $j_1 > 0, j_2 > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0}\Theta_{j_1,j_2}(\delta)\\
= \;& (-1)^{j_1+j_2}\int\limits_{\partial M_1\times\partial M_2} H^S_{j_1+1}(x^S-z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S,x^{N\setminus S})\,\mathrm{d}\sigma_2 H_{j_2+1}^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S})\,.
\end{aligned}\tag{4.28}$$

Combining (4.16), (4.17) with (4.22), (4.26), (4.27) and (4.28), the result follows.

Remark 4.4. Theorem 3.1 in [5] is obtained as a special case of Theorem 4.1 for $k_1 = 1, k_2 = 1$

As a direct application of the above higher order Cauchy-Pompeiu formula , we obtain

Theorem 4.2. (**Higher order Cauchy integral formula**) *Let M_1 be an s–dimensional differentiable compact oriented manifold contained in some open non-empty subset $\Omega_1 \subset \mathbb{R}^s$, let M_2 be an $(n-s)$–dimensional compact differentiable oriented manifold contained in some open non-empty subset $\Omega_2 \subset \mathbb{R}^{n-s}$, let $f \in C^{(r)}(\Omega_1\times\Omega_2, C(V_{n,s}))$, $r \geq k_1+k_2$, $k_1, k_2 \in \mathbf{N}^*$, be both (D_1) left k_1-regular and (D_2) right k_2-regular in $\Omega = \Omega_1\times\Omega_2$ and let ∂M_1 and ∂M_2 be given the induced orientations. Let for all $j \geq 1$, $H^S_j(x^S)$ and $H^{N\setminus S}_j(x^{N\setminus S})$ be given by (3.4) and (3.5). Then, for $z \in \overset{\circ}{M}_1\times\overset{\circ}{M}_2$,*

$$\begin{aligned}
&f(z)\\
=\;&\sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1}(-1)^{j_1+j_2}\int\limits_{\partial M_1\times\partial M_2} H^S_{j_1+1}(x^S-z^S)\,\mathrm{d}\sigma_1\left(D_1^{j_1}[f]D_2^{j_2}\right)(x^S,x^{N\setminus S})\,\mathrm{d}\sigma_2 H_{j_2+1}^{N\setminus S}(x^{N\setminus S}-z^{N\setminus S})\,.
\end{aligned}$$

Remark 4.5. Theorem 3.2 in [12] is obtained as a special case of Theorem 4.2 for $k_1 = 1, k_2 = 1$.

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