Additional structure on algebraic groups in Hasse fields

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Model theory, a branch of mathematical logic which involves the study of sets definable in a mathematical structure using an appropriate first order language, has recently shown its capacity to produce applications to other domains of mathematics, principally to algebra, algebraic geometry and number theory. The main example is the proof by Ehud Hrushovski of a conjecture of algebraic geometry, known as the Mordell-Lang conjecture for function fields (see [Hru96]). One point of this proof was to augment the usual language of fields, suitable for algebraic geometry, via new operators, essentially by adding a Hasse derivation.

Our aim here is to describe, in a geometric fashion, the new objects which are definable after adding a Hasse derivation. We develop the relevant context of D-algebraic geometry, and we concentrate on the description of "small" (*rationally thin*) D-algebraic subgroups of algebraic groups (i.e. objects of "finite dimension" in a strong sense, in a universe where each point comes with the infinite sequence of its derivatives). As Alexandru Buium did for the characteristic zero case in [Bui92], we relate these small subgroups to the notion of D-structure on an algebraic group (Theorem 3). Then we obtain some consequences about the existence of D-algebraic subgroups which are both rationally thin and Zariski-dense in a given algebraic group. By a precise study of the conditions under which an abelian variety admits a D-structure, we obtain, in the positive characteristic case, a link between the field of definition of an abelian variety and rational thinness of the subgroups of its divisible points in a given Hasse field (Corollary 4).

This first account gives only basic ideas of the objects described, and sketches of the proofs. Further publications on this topic are in preparation, and will provide additional details concerning D-algebraic geometry, including a schematic treatment, and its applications to the study of algebraic groups in positive characteristic.

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1 Preliminaries on Hasse fields

We give here some basic facts about fields with a single Hasse derivation. In characteristic zero a Hasse derivation is nothing more than a usual derivation, but in positive characteristic the Hasse derivation introduces additional structure. Our main references for the following facts are [Mar96] and Chapter 6 of [Poi87], for the characteristic zero case. In positive characteristic, the following can be found in [MW95] and [Zie03]. The book [Poi87] is also our main reference for all the various model-theoretic notions. For further description of Hasse derivations, see also [Oku63].

Definition 1. *A Hasse field is a field K equipped with a Hasse derivation D, which is a sequence $D = (D_i)_{i\in\mathbb{N}}$ satisfying:*

- *for each $i \in \mathbb{N}$, $D_i : K \longrightarrow K$ is an additive map, with $D_0 = id_K$*
- *the generalized Leibniz rule: for $x, y \in K$ and $i \in \mathbb{N}$,*

$$D_i(xy) = \sum_{l+m=i} D_l(x)D_m(y)$$

- *the composition rule: for $m, n \in \mathbb{N}$, $D_m \circ D_n = \binom{m+n}{n} D_{m+n}$.*

Definition 2. *For fixed p (prime or zero), we denote by HF_p the theory of Hasse fields of characteristic p in the language $\{0, 1, +, -, ., (D_i)_{i\in\mathbb{N}}\}$.*

Definition 3. *Let k be a model of HF_p.*

- *The subfield of constants of k is*

$$C_k := \{x \in k \mid D_1(x) = 0\}.$$

- *The subfield of absolute constants of k is*

$$C_k^\infty := \{x \in k \mid \forall i \geq 1, D_i(x) = 0\}.$$

- *If $p > 0$, the Hasse field k is said to be strict if $C_k = k^p$.*
- *Let X be a tuple of variables. We denote by $k\{X\}$ the k-algebra of D-polynomials:*

$$k\{X\} := k[d_iX; i \in \mathbb{N}].$$

We extend the Hasse derivation from k to $k\{X\}$ by:

$$D_i(d_jX) := \binom{i+j}{i} d_{i+j}X.$$

If P is a D-polynomial different from 0, and Y a single variable, the order of P in Y is the largest index i such that d_iY appears in P with non-zero coefficient. By convention, if none of the d_iY's appears in P, the order of P in Y is -1.

Theorem 1. *For fixed characteristic p, the existentially closed models of HF_p form an elementary class. Its theory CHF_p can be axiomatized by:*

- *if* $p = 0$
  - *the axioms of* HF_0
  - *an axiom scheme which states that for any non-zero D-polynomials P and Q in a single variable, such that the order of P is greater than the order of Q, there is a solution to the system* $P(x) = 0 \wedge Q(x) \neq 0$
- *if* $p > 0$
  - *the axioms of* HF_p
  - *the axiom of non triviality:* $\exists x, D_1(x) \neq 0$
  - *the axiom of strictness:* $\forall x, \exists y, (D_1(x) = 0 \rightarrow x = y^p)$
  - *an axiom scheme which states that the field is separably closed.*

Remark In characteristic zero, a Hasse field in nothing more than a differential field, since all the D_i's for $i > 1$ are given by iteration using the usual derivation D_1; and the theory CHF_0 is exactly the theory of differentially closed fields.
In positive characteristic, the axioms given here do not give which D-polynomial equation is solvable. This is not so surprising since, after adding as constants an element x such that $D_1(x) \neq 0$ and its derivatives, the Hasse derivation is definable in the pure language of fields (see [Zie03]).
However, other axiomatizations of the theories CHF_p exist, with a geometric flavor (see [PP98] for characteristic zero and [Kow05] for positive characteristic), and provide a better knowledge about solutions of D-polynomial equations.

Theorem 2. *Each theory CHF_p is complete and admits quantifier elimination.*

Corollary 1. *Let $k \models HF_p$. The n-types over k of the theory CHF_p are given by the prime D-ideals of $k\{X\}$, where X is a n-tuple of variables, and a D-ideal is an ideal stable under the Hasse derivation.*
To be specific, if K is a model of CHF_p containing k and $a \in K^n$, the type $tp(a/k)$ is given by the D-ideal

$$I_{a/k} := \{P \in k\{X\} \mid P(a) = 0\}.$$

2 D-algebraic varieties and thinness

From now on, K is an $\aleph_1$-saturated model of CHF_p and k is a Hasse subfield of K. Because CHF_p admits elimination of quantifiers, together with the fact that $K\{X\}$ is a countable union of Noetherian rings, we can obtain an analogue of the Nullstellensatz for sets of zeros of a family of D-polynomials. This allows us to define D-algebraic varieties as sets of points in K^n. We give quick definitions of the basic notions, which are just the analogues of the basic notions of naïve algebraic geometry (see [Lan58] or [Har77] for the usual notions). More precise definitions (and more detailed proofs of the following results) can be found in [Ben05].

Definition 4.
- *A D-affine variety is a subset of K^n for some n which is of the form*
$$V = \{x \in K^n \mid \forall P \in A, P(x) = 0\}$$
where A is a subset of $K\{X\}$ and X is an n-tuple of variables.
- *The D-topology on K^n is the topology whose closed sets are the D-affine varieties of K^n. Each D-affine variety is equipped with the induced D-topology.*
- *A D-regular function on a D-affine variety is a function which can be written locally (for the D-topology) as a quotient of two D-polynomials.*
- *A morphism between two D-affine varieties is a map whose whose coordinates are D-regular.*
- *A D-algebraic variety V is a topological space covered by a finite family of open sets $V = U_1 \cup \ldots \cup U_m$, such that each U_i is homeomorphic to a D-affine variety V_i via a map $f_i : U_i \longrightarrow V_i$, and such that the maps $f_j^\circ f_i^{-1} : f_i(U_i \cap U_j) \longrightarrow f_j(U_i \cap U_j)$ from an open subset of V_i to an open subset of V_j are isomorphisms of D-affine varieties.*
- *The notions of D-topology, of D-regular functions and of morphisms extend directly from D-affine varieties to D-algebraic varieties.*

Remark By a suitable cut-and-paste of the D-affine charts of a D-algebraic variety V, we can easily consider V as an infinitely definable set in some cartesian power of K. Because $K\{X\}$ is a countable union of Noetherian rings, we can always choose the D-subfield of parameters for V to be countable.

Fact If V is an irreducible D-algebraic variety defined with parameters in k, then $\aleph_1$-saturation guarantees that K contains a generic point a of V over k, which is to say that the only closed subset of V defined with parameters in k and containing a is V itself.

Definition 5. *A D-algebraic group $(G, \cdot, ^{-1}, e)$ is a D-algebraic variety G together with morphisms $\cdot : G \times G \longrightarrow G$ and $^{-1} : G \longrightarrow G$ and a point $e \in G$ such that $(G, \cdot, ^{-1}, e)$ is a group.*

It is easy to see that if G is an algebraic group defined over K, $G(K)$ is in particular a D-algebraic group. As such, it has more structure than the algebraic group G: it may have D-algebraic subgroups which do not come from algebraic subgroups of G. For example, if G is defined over C_K (or C_K^∞), then $G(C_K)$ (or $G(C_K^\infty)$) is a D-algebraic subgroup of $G(K)$ which is Zariski-dense in G. Our main task here is to understand this new structure provided by the D-algebraic geometry, and more particularly to describe the D-algebraic subgroups which are *rationally thin.*

Definition 6. *Let $a \in K$. We denote by $k(\{a\})$ the Hasse field generated by a over k:*
$$k(\{a\}) := k(D_i(a))_{i \in \mathbb{N}}.$$
We say that a is rationally thin over k if $k(\{a\})$ is finitely generated over k as a field.

Definition 7. *Let V be an irreducible D-algebraic variety, and let a be a generic point of V over a countable field of parameters k for V. We say that V is rationally thin if a is rationally thin over k.*

Remark This definition does not depend on which D-affine charts are used to find the point a, since these D-affine charts are related by D-regular bijective maps (with parameters in k) which leave the notion of Hasse field generated over k unchanged. Neither does it depend on the generic point chosen, because any two generic points generate Hasse fields which are isomorphic over k.

3 D-structures on algebraic varieties

Another way to study an algebraic group in a Hasse field, apparently different from the study of a D-algebraic subgroup, is to put additional structure on the sheaf of regular functions of this group. This approach is based on the work of Alexandru Buium in characteristic zero (see [Bui92]). The notion of *D-structure* in positive characteristic has also been studied in [KP06].
In the following, (K, D) will be an $\aleph_1$-saturated model of CHF_p, for p prime or zero.

Definition 8. *Let V be an algebraic variety defined over K. A D-structure on V is an extension of the Hasse derivation of K to the sheaf of regular functions $\mathcal{O}_V$.*

Definition 9. *Let (V, E) and (W, F) be D-structures on the algebraic varieties V and W. A morphism $f : V \longrightarrow W$ preserves D-structure if the induced map $f^\# : \mathcal{O}_W \longrightarrow \mathcal{O}_V$ satisfies $f^\# \circ F = E \circ f^\#$.*

In the special case where (L, D) is a Hasse field extension of (K, D), $Spec\, L$ is an algebraic variety over K reduced to one point, with a Hasse derivation D on its structural sheaf L. By analogy with the definition of the points of an algebraic variety V with value in a field L, we define:

Definition 10. *Let (V, E) be an algebraic variety with a D-structure, and (L, D) a Hasse field extension of (K, D). A morphism $Spec\, L \longrightarrow V$ preserving D-structure is called a D-point of V with value in L; we denote by $V^E(L)$ the set of D-points of V with value in L (it is a subset of $V(L)$, the set of morphisms $Spec\, L \longrightarrow V$).*

Examples

- Let V be an affine variety defined over (K, D), with coordinate ring $K[V]$. A D-structure on V is given by a Hasse derivation E on $K[V]$ which extends D. Let X be the system of coordinates in $K[V]$. Then the D-points of V in an extension (L, D) of (K, D) are given by:

$$V^E(L) = \{x \in V(L) \mid \forall i \in \mathbb{N}, E_i(X)(x) = D_i(x)\}.$$

- Let V be an algebraic variety defined over C_K^∞. Then there is a D-structure on V, called the trivial D-structure, defined in the following way: for each basic affine open subset U of V, the Hasse derivation E of $\mathcal{O}_V(U)$ is trivial on the coordinate functions (which means that $E_i(X) = 0$ for $i \geq 1$).

- If (V, E) and (W, F) are two algebraic varieties with a D-structure, there is a natural D-structure F on $V \times W$:

$$\forall f \otimes g \in \mathcal{O}_V \otimes_K \mathcal{O}_W \simeq \mathcal{O}_{V \times W}, G_i(f \otimes g) = \sum_{l+m=i} E_l(f) \otimes F_m(g).$$

We will focus our attention on D-structures whose underlying varieties are algebraic groups.

Definition 11. *Let $(G, \cdot, ^{-1}, e)$ be an algebraic group defined over K. A D-structure on this algebraic group is a D-structure on G such that e is a D-point and $\cdot : G \times G \longrightarrow G$ and $^{-1} : G \longrightarrow G$ preserve D-structure.*

Algebraic groups with a D-structure form a category, where the maps are the homomorphisms of algebraic groups which preserve D-structure.

4 The equivalence of categories and its consequences

Our main theorem is a generalization of a theorem of [Bui92] from characteristic zero to arbitrary characteristic:

Theorem 3. *The category of rationally thin, connected D-algebraic groups is equivalent to the category of connected algebraic groups with a D-structure.*

Sketch of the proof This category equivalence is given by the functor which maps an algebraic group (G, E) with a D-structure, defined over K, to the D-algebraic group $G^E(K)$, where a homomorphism of algebraic groups preserving D-structure is sent to its restriction to the subgroup of D-points.
The key argument for this functor to be full and faithful is that $G^E(K)$ is Zariski-dense in G; this follows from the fact that K is separably closed and is existentially closed as a Hasse field.
Let us prove now that this functor is essentially surjective: let $(H, *, ^{-1})$ be a rationally thin, connected D-algebraic group, defined over a countable D-subfield of parameters k, and let a be a generic point of H over k. There is an integer m such that

$$k(\{a\}) = k(D_0(a), \ldots, D_m(a)).$$

Let $(H', *, ^{-1})$ be the image of $(H, *, ^{-1})$ by the injective map $(D_0, \ldots, D_m)$. For b, c two generic points of H', we have $k(\{b\}) = k(b)$ and $k(\{c\}) = k(c)$, thus $b*c \in k(b, c)$ and $b^{-1} \in k(b)$. By a theorem of André Weil (see [Wei55]) we obtain an algebraic group G defined over K and an injective map from H' to $G(K)$. This map is generically a homomorphism, with Zariski-dense image. This image is quite easily seen to be of the form $G^E(K)$ for some D-structure E on G: if x is a generic point of this image, we have for each i a rational function f_i over k such that $D_i(x) = f_i(x)$ (because the generic points of H' satisfy the same property); and then $E_i(X) = f_i(X)$ on an open subset U of G. This gives D-structure on $\mathcal{O}_G(U)$ for the coordinate functions X. We have $H \simeq G^E(K)$, as desired. $\square$

Remark With the same notation as in the proof, note that the dimension of G

is equal to the transcendence degree of $k(\{a\})$ over k.

We use the previous theorem to obtain several consequences about rationally thin subgroups of K-points of algebraic groups.

Corollary 2. *Let G be a connected algebraic group defined over K. There is a connected, rationally thin D-algebraic subgroup of $G(K)$ which is Zariski-dense in G if and only if there is an algebraic group with a D-structure (H, E) with a dominant homomorphism from H to G (i.e. an homomorphism with Zariski-dense image).*

Sketch of the proof If Γ is a connected, rationally thin D-algebraic subgroup of $G(K)$, we can find by the previous theorem an algebraic group with a D-structure (H, E) such that $H^E(K)$ is isomorphic to Γ. The details of the proof show that the isomorphism $H^E(K) \longrightarrow \Gamma$ is regular, and not just D-regular, so it induces an homomorphism of algebraic groups $H \longrightarrow G$, which is dominant because Γ is Zariski-dense in G.
For the converse, it suffices to consider the image of $H^E(K)$ in $G(K)$ by the dominant homomorphism. □

If $p = 0$ and G commutative, we can construct such an algebraic group with a D-structure (H, E), viewing G as an extension of its projective part by its linear part, and using the notion of universal extension of an abelian variety (see [Bui92]). So we have:

Corollary 3. *Suppose $p = 0$ and let G be an irreducible commutative algebraic group. Then $G(K)$ has a connected D-algebraic subgroup which is rationally thin and Zariski-dense.*

From now on, we are interested in the case where $p > 0$ and G is an abelian variety. In this situation, we can obtain strong information on the D-structure:

Theorem 4 (Isotriviality)**.** *Let (G, E) be an abelian variety G equipped with a D-structure E. Then there is an abelian variety H defined over C_K^∞, with trivial D-structure F, such that*

$$(G, E) \simeq (H, F).$$

Sketch of the proof We use the Weil restriction of G from K to K^{p^n}, denoted $\Pi_{K/K^{p^n}} G$, as defined in [Spr98]. This is an algebraic group defined over K^{p^n}. We need the additional fact that this is an extension of G by an unipotent group.
In Chapter 3 of [Ben05] we show that a D-structure on G corresponds to a family of sections $s_n : G \longrightarrow \Pi_{K/K^{p^n}} G$, satisfying some extra compatibility conditions. We first note that this fact proves that a D-structure on an abelian variety must be unique, since the difference between two sections is an homomorphism from G into an unipotent group, hence must be 0. Then, we consider the Zariski-closure G_n of $s_n(G^E(K))$ in $\Pi_{K/K^{p^n}} G$, we can prove that it is defined over K^{p^n}. Since $G^E(K)$ is Zariski-dense in G, s_n induces an isomorphism from G to G_n, and G_n is an abelian variety defined over K^{p^n}.
In the suitable moduli space of abelian varieties $\mathcal{A}$ (we refer to [MF82] for this notion), G corresponds to the same point as G_n, which has value in K^{p^n}. Hence G is

in $\bigcap_n \mathcal{A}(K^{p^n}) = \mathcal{A}(C_K^\infty)$, which means that G is isomorphic to an abelian variety H defined over C_K^∞. This abelian variety H may be equipped with two D-structures: the trivial D-structure F and the D-structure coming from G by isomorphism. Because of the unicity previously noticed, we must have $(G, E) \simeq (H, F)$. □

In our setting, the D-algebraic subgroup of $G(K)$ of interest is

$$p^\infty G(K) := \bigcap_n p^n G(K).$$

This is the unique connected, Zariski-dense D-algebraic subgroup of $G(K)$ which has U-rank (see [BD01]). It is thus the unique candidate to be connected, rationally thin and Zariski-dense.
Our previous results allow us to relate the question of its rational thinness to a question of field of definition:

Corollary 4. *The subgroup $p^\infty G(K)$ is rationally thin if and only if G is isogenous to an abelian variety defined over C_K^∞.*

Sketch of the proof If $p^\infty G(K)$ is rationally thin, we find, by Corollary 2, an algebraic group with a D-structure (H, E) and a dominant morphism from H to G. Using the remark following Theorem 3, we get that the dimension of H equals the transcendence degree of $k(\{a\})$ over k for a a generic point of $p^\infty G(K)$ over k, and this transcendence degree equals the dimension of G (see [BD01]). So the dominant map from H to G is an isogeny and H is an abelian variety. Hence by Theorem 4, H is isomorphic to an abelian variety defined over C_K^∞.
The converse is straightforward. □

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