Monogenic Calculus as an Intertwining Operator

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Abstract

We revise a monogenic calculus for several non-commuting operators, which is defined through group representations. Instead of an algebraic homomorphism we use group covariance. The related notion of joint spectrum and spectral mapping theorem are discussed. The construction is illustrated by a simple example of calculus and joint spectrum of two non-commuting selfadjoint $n \times n$ matrices.

1 Introduction

Central objects of operator theory are functional calculus (usually defined as an algebra homomorphism), spectrum (defined as set of singular points of the resolvent), and spectral mapping theorem (describing transformations of the spectrum under the functional calculus). Following the discussion in [26] we arrange these objects as follows:

1. Functional calculus is an *original* notion defined in some independent terms;
2. Spectrum is derived from the previously defined functional calculus as its *support* in some appropriate sense;
3. Then Spectral mapping theorem should drop out naturally.

The full potential of such a construction depends from its source—the definition of a functional calculus. It is known that homomorphic calculi are successful

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only in few simplest cases, e.g. for a single normal operator. To increase its potential functional calculus through group covariance was defined in [17]. In that paper such a definition was applied to a complicated case of monogenic calculus for several non-commuting operators. Monogenic calculus of commuting operators was considered earlier in [29]. Monogenic calculus of non-commuting operators was carefully developed through plane wave decomposition in many subsequent papers, see [13, 14, 12, 15]. These papers contain many important results, e.g. connection between monogenic and Weyl [1] calculi, but they do not consider covariant properties of the calculus. Papers [27, 28] utilized an algebraic approach and do not develop group representations either.

Meanwhile the covariant approach to functional calculus required development of wavelets technique and its applications to analytic function theory, it was performed in [19, 20, 21, 23, 24]. It emerged from these studies that the new definition is a useful replacement for classical one across all range of problems, even in case of a single non-normal operator with finite range [26]. The key ingredient in this approach is the development of all principal objects of analytical function theory (Cauchy integral, Hardy and Bergman spaces, Cauchy-Riemann equations, Taylor series, etc.) from the group of Möbius transformations and wavelets technique [19, 23]. This allows to give a template definition of functional calculus as follows, cf. [17, Defn. 1.1]:

Definition 1.1. *Let A be a normed algebra, and M be a left $\mathfrak{A}$-module. Let G be a group, X be a left G-homogeneous space, and $A(X)$ be an associated space of analytic functions. An* analytic functional calculus *for an element $a \in \mathfrak{A}$ is a continuous linear mapping $\Phi : A(X) \to A(X, M)$ such that*

1. *Φ is an* intertwining operator

$$\Phi\rho_X = \rho_M\Phi$$

between two representations of the group G: ρ_X acts in the analytic space $A(X)$ of scalar valued functions on X and ρ_M acts in a space $A(X, M)$ of M-valued functions in a way depending from $a \in \mathfrak{A}$.

2. *There is an initialisation condition: $\Phi[f_0] = f_M$, i.e. the vacuum vector of $A(X)$ is mapped into the vacuum vector of $A(X, M)$.*

Note that our functional calculus released from the homomorphism condition can take value in any left $\mathfrak{A}$-module M, which however could be $\mathfrak{A}$ itself if suitable. This improves spectral localisation technique in our construction.

In the paper [1] joint spectrum was defined as the support of the Weyl calculus, i.e. as the set of points where the operator valued distribution does not vanish. We also define the spectrum as a support of functional calculus, but due to our Definition 1.1 it has a different meaning.

Definition 1.2. [26] *A corresponding* spectrum *of $a \in \mathfrak{A}$ is the support of the functional calculus Φ, i.e. the collection of non-vanishing intertwining operators between ρ_M and* prime representations *[16, § 8.3].*

More variations of functional calculi (Weyl, Wick, Berezin, etc.) are obtained from other groups and their representations [17, 21]. There are also recent papers of other researchers devoted to covariant calculus [2, 3].

2 Preliminaries on Clifford Algebras and Möbius Transformations

Let $\mathbb{R}^n$ be a real n-dimensional vector space with a fixed frame e_1, e_2, …, e_n. Let $\mathcal{C}\ell(n)$ be the *real Clifford algebra* generated by 1, $\mathbf{e}_j$, $1 \le j \le n$ and the relations

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}.$$

Then there is the natural embedding of $\mathbb{R}^n$ into $\mathcal{C}\ell(n)$ and a bilinear form $B(\mathbf{x},\mathbf{x})$ on $\mathbb{R}^n$ [7]. We identify $\mathbb{R}^n$ with its image in $\mathcal{C}\ell(n)$ and call its elements *vectors*. There are two linear anti-automorphisms $*$ (reversion) and $^-$ (main anti-automorphisms) and automorphism $'$ of $\mathcal{C}\ell(n)$ defined on its basis $A_\nu = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_r}$, $1 \le j_1 < \cdots < j_r \le n$ by the rule:

$$(A_\nu)^* = (-1)^{\frac{r(r-1)}{2}} A_\nu, \qquad \bar{A}_\nu = (-1)^{\frac{r(r+1)}{2}} A_\nu, \qquad A'_\nu = (-1)^r A_\nu.$$

In particular, for vectors, $\bar{\mathbf{x}} = \mathbf{x}' = -\mathbf{x}$ and $\mathbf{x}^* = \mathbf{x}$.

It is easy to see that $\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x} = 1$ for any $\mathbf{x} \in \mathbb{R}^n$ such that $B(\mathbf{x},\mathbf{x}) \ne 0$ and $\mathbf{y} = \bar{\mathbf{x}}\,\|\mathbf{x}\|^{-2}$, which is the *Kelvin inverse* of $\mathbf{x}$. Finite products of invertible vectors are invertible in $\mathcal{C}\ell(n)$ and form the *Clifford group* $\Gamma(n)$ [7, (1.39)]. Elements $a \in \Gamma(n)$ such that $a\bar{a} = \pm 1$ form the $\mathrm{Pin}(n)$ group—the double cover of the group of orthogonal rotations $\mathrm{O}(n)$. We also consider [6, § 5.2] $T(n)$ to be the set of all products of vectors in $\mathbb{R}^n$.

Let (a, b, c, d) be a quadruple from $T(n)$ with the properties:

1. $(ad^* - bc^*) \in \mathbb{R} \setminus 0$;
2. a^*b, c^*d, ac^*, bd^* are vectors.

Then [6, Thm. 5.2.3], [7, (4.10)] 2×2-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form the group $\Gamma(1, n+1)$ under the usual matrix multiplication. It has a representation $\rho_{\mathbb{R}^n}$ by transformations of $\dot{\mathbb{R}}^n$ given by:

$$\rho_{\mathbb{R}^n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{x} \mapsto (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}, \tag{1}$$

which form the *Möbius* (or the *conformal*) group of $\dot{\mathbb{R}}^n$. Here $\dot{\mathbb{R}}^n$ the compactification of $\mathbb{R}^n$ by the point at infinity (see [6, § 5.1]). The analogy with fractional-linear transformations of the complex line $\mathbb{C}$ is useful, as well as representations of shifts $\mathbf{x} \mapsto \mathbf{x} + y$, orthogonal rotations $\mathbf{x} \mapsto k(a)\mathbf{x}$, dilations $\mathbf{x} \mapsto \lambda\mathbf{x}$, and the Kelvin inverse $\mathbf{x} \mapsto \mathbf{x}^{-1}$ by the matrices $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$, $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively.

Following e.g. [7, (1.8)] we adopt the next agreement.

Notation 2.1. *In a non-commutative algebra setting the ambiguous notation* $\frac{a}{b}$ *always means* ab^{-1}*. Consequently* $\frac{ac}{bc} = \frac{a}{b}$ *but* $\frac{ca}{cb} \ne \frac{a}{b}$ *in general.*

Study of Möbius transformation is facilitated by introduction of projective coordinates in the space $P\mathbb{R}^{1,n+1}$ of spheres in $\mathbb{R}^n$ [7, (4.12)]. The sphere with the centre $m \in \mathbb{R}^n$ and the radius r defined by the equation $(\mathbf{y} - \mathbf{m})2 = r2$ is associated with the ray of matrices by the map

$$T : \quad \{\mathbf{y} \mid B(\mathbf{y}-\mathbf{m}, \mathbf{y}-\mathbf{m}) = r2\} \quad \mapsto \quad \lambda \begin{pmatrix} \mathbf{m} & -\mathbf{m}^2 - r2 \\ 1 & -\mathbf{m} \end{pmatrix}. \tag{2}$$

A point $\mathbf{x} \in \mathbb{R}^n$ is associated with a zero radius sphere with the centre $\mathbf{x}$ and thus is represented by $\begin{pmatrix} \mathbf{x} & -\mathbf{x}^2 \\ 1 & \mathbf{x} \end{pmatrix}$. Then Möbius transformations (1) corresponds to the orthogonal rotations in the projective space $P\mathbb{R}^{1,n+1}$ as follows [7, (4.13)]:

$$\rho_P \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \begin{pmatrix} \mathbf{m} & -\mathbf{m}^2 - r2 \\ 1 & -\mathbf{m} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{m} & -\mathbf{m}^2 - r2 \\ 1 & -\mathbf{m} \end{pmatrix} \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}. \tag{3}$$

One usually says that the conformal group in $\mathbb{R}^n$, $n > 2$ is not so rich as the conformal group in $\mathbb{R}^2$. Nevertheless, the conformal covariance has many applications in Clifford analysis [6, 33]. Notably, groups of conformal mappings of unit spheres $\mathbb{S}^{n-1} = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n, B(\mathbf{x}, \mathbf{x}) = 1\}$ onto itself are similar for all n and as sets can be parametrised by the product of the unit ball $\mathbb{B}^n$ and the group of isometries of $\mathbb{S}^{n-1}$. We specialise the main result of [18] for the positive definite case as follows:

Proposition 2.2. *The group $\mathcal{M}(\mathbb{B}^n)$ of conformal mappings of the open unit ball $\mathbb{B}^n$ and the unit sphere $\mathbb{S}^{n-1}$ onto themselves is represented by matrices*

$$\begin{pmatrix} a & b' \\ b & a' \end{pmatrix}, \qquad a, b \in T(n), \quad ab^* \in \mathbb{R}^n, \quad |a|^2 - |b|^2 = 1. \tag{4}$$

Its inverse is $\begin{pmatrix} a & b' \\ b & a' \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b^* & a^* \end{pmatrix}$.

Proof. The proof is easy in the projective coordinates (2). Indeed the unit spere corresponds to the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Straightforwardly transformations ρ_P (3) with matrices of the form (4) preserve this ray:

$$\begin{pmatrix} a & b' \\ b & a' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} b' & -a \\ a' & -b \end{pmatrix} \begin{pmatrix} a^* & b^* \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus corresponding Möbius transformations preserve the unit sphere. ∎

The presentation of $\mathcal{M}(\mathbb{B}^n)$ by (4) is difficult to use due to ineffective definition through the constrain $|a|^2 - |b|^2 = 1$. Thus we will prefer a direct parametrisation, cf. [35, § VI.1.3], as follows. We can identify the unit ball $\mathbb{B}^n$ with the left coset $O(\mathbb{S}^{n-1})\backslash\mathcal{M}(\mathbb{B}^n)$, where the decomposition $\mathcal{M}(\mathbb{B}^n) \sim O(\mathbb{S}^{n-1}) \times \mathbb{B}^n$ follows from (7). Note that $K = O(\mathbb{S}^{n-1})$ is the maximal compact subgroup of $\mathcal{M}(\mathbb{B}^n)$.

$$\begin{aligned} \begin{pmatrix} a & b' \\ b & a' \end{pmatrix} &= |a| \begin{pmatrix} \frac{a}{|a|} & 0 \\ 0 & \frac{a'}{|a|} \end{pmatrix} \begin{pmatrix} 1 & \frac{\bar{a}}{|a|^2} b' \\ \frac{a^*}{|a|^2} b & 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{1+\mathbf{u}^2}} \begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix} \begin{pmatrix} 1 & \mathbf{u}' \\ \mathbf{u} & 1 \end{pmatrix}, \end{aligned} \tag{5}$$

where

$$w = \frac{a}{|a|}, \qquad \mathbf{u} = \frac{a^*}{|a|^2}b, \qquad \sqrt{1+\mathbf{u}^2} = |a|^{-1}, \qquad |\mathbf{u}| < 1. \tag{6}$$

Consequently for $\mathbf{u} \in \mathbb{B}^n$, $w \in \Gamma(n)$ the Möbius transformations $\phi_{(\mathbf{u},w)}$ with matrix

$$\frac{1}{\sqrt{1+\mathbf{u}^2}}\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}\begin{pmatrix} 1 & \mathbf{u}' \\ \mathbf{u} & 1 \end{pmatrix} = \frac{1}{\sqrt{1+\mathbf{u}^2}}\begin{pmatrix} w & w\mathbf{u}' \\ w'\mathbf{u} & w' \end{pmatrix}, \tag{7}$$

constitute $\mathcal{M}(\mathbb{B}^n)$. Sometime in Möbius transformations we will omit the normalising factor $(1+\mathbf{u}^2)^{-1/2}$ in (7) for the sake of brevity. Although one should not forget that this factor is important for the invariant integration on $\mathcal{M}(\mathbb{B}^n)$. The following properties follows from such a realisation:

Lemma 2.3. *$\mathcal{M}(\mathbb{B}^n)$ acts on $\mathbb{B}^n$ transitively. Transformations of the form $\phi_{(0,w)}$ constitute a subgroup isomorphic to $\mathrm{O}(n)$. The homogeneous space $\mathcal{M}(\mathbb{B}^n)/\mathrm{O}(n)$ is isomorphic as a set to $\mathbb{B}^n$. Moreover:*

1. $\phi^2_{(\mathbf{u},1)} = -1$ *on* $\mathbb{B}^n$, *thus* $\phi^{-1}_{(\mathbf{u},1)} = \phi_{(\mathbf{u}',1)} = \phi_{(-\mathbf{u},1)}$.
2. $\phi^{-1}_{(\mathbf{u},1)}(0) = -\mathbf{u}$ *and* $\phi^{-1}_{(\mathbf{u},1)}(\mathbf{u}) = 0$.
3. $\phi^{-1}_{(\mathbf{u}_1,1)}\phi^{-1}_{(\mathbf{u}_2,1)} = \phi^{-1}_{(\mathbf{u},w)}$ *where*

$$\mathbf{u} = \phi^{-1}_{(\mathbf{u}_1,1)}(\mathbf{u}_2) = \phi^{-1}_{(\mathbf{u}_2,1)}(\mathbf{u}_1) \quad \textit{and} \quad w = \frac{1-\mathbf{u}_1\mathbf{u}_2}{|1-\mathbf{u}_1\mathbf{u}_2|}.$$

We use the same notation for the Möbius transformation $\phi_{(\mathbf{u},w)}$ and the matrix (7) which produces it. It is a direct check to see that

$$\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}\begin{pmatrix} 1 & \mathbf{u}' \\ \mathbf{u} & 1 \end{pmatrix} = \begin{pmatrix} 1 & w\mathbf{u}'w^* \\ w'\mathbf{u}\bar{w} & 1 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix},$$

which implies that $\phi^{-1}_{(\mathbf{u},w)} = \phi_{(w^*\mathbf{u}'w,\bar{w})}$.

Lemma 2.4. *The left invariant* Haar measure *dg on $\mathcal{M}(\mathbb{B}^n) \sim \mathbb{B}^n \times O(n)$ in coordinates $(\mathbf{u}, w)$ is*

$$dg(\mathbf{u}, w) = \frac{d\mathbf{u}\, dw}{|1+\mathbf{u}^2|^n}, \tag{8}$$

where dw is a Haar measure on $O(n)$ and $d\mathbf{u}$ is the Lebesgue measure on $\mathbb{B}^n$.

Proof. It follows from 3 that left shifts on $\mathcal{M}(\mathbb{B}^n)$ in coordinates $(\mathbf{u}, w)$ acts by Möbius transformations on $\mathbf{u} \in \mathbb{B}^n$ which fixes the unit sphere. According to [6, Cor. 6.1.2] the invariant metric on $\mathbb{B}^n$ is defined through the distance of the zero radius sphere defined by $\mathbf{u}$ to the unit sphere $\mathbb{S}^{n-1}$. This distance is $|1+\mathbf{u}^2|^{-1}$, thus the invariant measure is obtained from its n-th power. ∎

The importance of the Haar measure is justified by the *invariant integration* (or *invariant functional*) it produces:

$$\int_{\mathcal{M}(\mathbb{B}^n)} f(g)\, dg = \int_{\mathcal{M}(\mathbb{B}^n)} f(g_1 g)\, dg, \qquad \text{for all } f(g) \in \mathbb{L}^1(\mathcal{M}(\mathbb{B}^n)) \text{ and } g \in \mathcal{M}(\mathbb{B}^n).$$

It is rarely realised that the Haar invariant functional is not the only possible and that other invariant functionals are useful as well. The classic example is described below.

Lemma 2.5. *The invariant functional H on $\mathcal{M}(\mathbb{B}^n)$ of* Hardy type *is given by:*

$$H(f) = \lim_{r\to 1} \int_{O(n)} \int_{\mathbb{S}^{n-1}} f(r\mathbf{u}, w) \frac{dw\, d\mathbf{u}}{|1+\mathbf{u}^2|^{n-1}}, \qquad \text{where } w \in O(n), \mathbf{u} \in \mathbb{S}^{n-1}. \tag{9}$$

Proof. This result follows from the discussion in the proof of Lemma 2.4 and observation that the limit in (9) is Möbius invariant. ∎

Definition 2.6. *The* Hardy inner product *in a space of Clifford valued functions on $\mathcal{M}(\mathbb{B}^n)$ is derived from the Hardy functional* (9)*:*

$$\langle f_1, f_2 \rangle = H(\bar{f}_1 f_2). \tag{10}$$

Note that the invariance of the Hardy functional (9) implies that left shifts are isometries with respect to norm defined through (10).

3 Construction of Clifford Analysis from $\mathcal{M}(\mathbb{B}^n)$ Group

3.1 Wavelet Transform and Cauchy Kernel

To understand the functional calculus from Definition 1.1 we need first to realise the function theory of monogenic functions from the representation theory of $\mathcal{M}(\mathbb{B}^n)$, see [19, 20, 23, 24] for more details.

Each element $g \in \mathcal{M}(\mathbb{B}^n)$ acts by the linear-fractional transformation (the Möbius map) on $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ *from the left* as follows:

$$g^{-1} : \mathbf{x} \mapsto \frac{\bar{a}\mathbf{x} - \bar{b}}{a^* - b^*\mathbf{x}}, \qquad \text{where} \quad g^{-1} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b^* & a^* \end{pmatrix}. \tag{11}$$

In the decomposition (5) the first matrix on the right hand side acts by transformation (11) as an orthogonal rotation of $\mathbb{S}^{n-1}$ and $\mathbb{B}^n$; and the second one—by transitive family of maps of the unit ball onto itself.

Möbius transformations (11) could be linearised to the representation ρ_1 on functions, cf. [7, (4.56)] and [10, Thm. 5.4.1], by the induced representation technique [16, § 13]:

$$\rho_1(g) : f(z) \mapsto \frac{a' - \bar{\mathbf{x}}b'}{|a' - \bar{\mathbf{x}}b'|^n} f\left(\frac{\bar{a}\mathbf{x} - \bar{b}}{a^* - b^*\mathbf{x}}\right), \qquad \text{where} \quad g^{-1} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b^* & a^* \end{pmatrix}. \tag{12}$$

Indeed one can directly verify:

$$\begin{aligned}
\rho_1(g_1)(\rho_1(g_2)f(z)) &= \rho_1(g_1)\left(\frac{a_2' - \bar{\mathbf{x}}b_2'}{|a_2' - \bar{\mathbf{x}}b_2'|^n}\, f\left(\frac{\bar{a}_2\mathbf{x} - \bar{b}_2}{a_2^* - b_2^*\mathbf{x}}\right)\right)\\
&= \frac{a_1' - \bar{\mathbf{x}}b_1'}{|a_1' - \bar{\mathbf{x}}b_1'|^n}\,\frac{a_2' - \overline{(\bar{a}_1\mathbf{x} - \bar{b}_1)(a_1^* - b_1^*\mathbf{x})^{-1}}b_2'}{\left|a_2' - \overline{(\bar{a}_1\mathbf{x} - \bar{b}_1)(a_1^* - b_1^*\mathbf{x})^{-1}}b_2'\right|^n}\\
&\quad \times f\left(\frac{\bar{a}_2(\bar{a}_1\mathbf{x} - \bar{b}_1)(a_1^* - b_1^*\mathbf{x})^{-1} - \bar{b}_2}{(a_2^*(a_1^* - b_1^*\mathbf{x})(a_1^* - b_1^*\mathbf{x})^{-1} - b_2^*}\right)\\
&= \frac{(a_1' - \bar{\mathbf{x}}b_1')a_2' - (\bar{\mathbf{x}}a_1 - b_1)b_2'}{|(a_1' - \bar{\mathbf{x}}b_1')a_2' - (\bar{\mathbf{x}}a_1 - b_1)b_2'|^n}\, f\left(\frac{\bar{a}_2(\bar{a}_1\mathbf{x} - \bar{b}_1) - \bar{b}_2(a_1^* - b_1^*\mathbf{x})}{a_2^*(a_1^* - b_1^*\mathbf{x}) - b_2^*(\bar{a}_1\mathbf{x} - \bar{b}_1)}\right)\\
&= \frac{(a_1'a_2' + b_1b_2') - \bar{\mathbf{x}}(b_1'a_2' + a_1b_2')}{|(a_1'a_2' + b_1b_2') - \bar{\mathbf{x}}(b_1'a_2' + a_1b_2')|^n}\, f\left(\frac{(\bar{a}_2\bar{a}_1 + \bar{b}_2b_1^*)\mathbf{x} - (\bar{a}_2\bar{b}_1 + \bar{b}_2a_1^*)}{(a_2^*a_1^* + b_2^*\bar{b}_1) - (a_2^*b_1^* + b_2^*\bar{a}_1)\mathbf{x}}\right)\\
&= \frac{a' - \bar{\mathbf{x}}b'}{|a' - \bar{\mathbf{x}}b'|^n}\, f\left(\frac{\bar{a}\mathbf{x} - \bar{b}}{a^* - b^*\mathbf{x}}\right) \qquad [\text{where } a = a_1a_2 + b_1'b_2, b = b_1a_2 + a_1'b_2]\\
&= \rho_1(g_1g_2)f(\mathbf{x}).
\end{aligned}$$

Let $L_2(\mathbb{S}^{n-1})$ be equipped with a Clifford valued inner product, cf. [7, (1.29)]:

$$\langle f_1, f_2\rangle = \int_{\mathbb{S}^{n-1}} \bar{f}_1(\mathbf{x})f_2(\mathbf{x})\,d\mathbf{x} \tag{13}$$

normalised such that $\int_{\mathbb{S}^{n-1}} d\mathbf{x} = 1$. Then [7, (4.56)] the representation (12) became unitary in $L_2(\mathbb{S}^{n-1})$.

We choose [19, 21, 23] K-invariant function $f_0(\mathbf{x}) = (\mathbf{x}) \equiv 1$ be *vacuum vector* or *mother wavelet* [21]. Then *coherent states* or *wavelets* are all transformations of the vacuum vector by ρ_1:

$$f_g(\mathbf{x}) = \rho_1(g)f_0(\mathbf{x}) = \frac{a' - \bar{\mathbf{x}}b'}{|a' - \bar{\mathbf{x}}b'|^n}, \qquad g^{-1} = \begin{pmatrix} \bar{a} & -\bar{b}\\ -b^* & a^*\end{pmatrix}. \tag{14}$$

They are mainly determined by the point on the unit disk $\mathbf{u} = a^*b/|a|^2$. The linear span of all wavelets is called the *Hardy space* $H_2(\mathbb{S}^{n-1})$, and f_0 is *cyclic* in $H_2(\mathbb{S}^{n-1})$. Möbius transformations provide a natural family of intertwining operators for ρ_1 coming from inner automorphisms of $\mathcal{M}(\mathbb{B}^n)$ (will be used later).

The *wavelet transform* [19, 21] $\mathcal{W} : L_2(\mathbb{S}^{n-1}) \to H_2(\mathcal{M}(\mathbb{B}^n))$ is defined by:

$$\begin{aligned}
\mathcal{W}f(g) &= \langle f_g, f\rangle \qquad (15)\\
&= \int_{\mathbb{S}^{n-1}} \frac{a^* - b^*\mathbf{x}}{|a^* - b^*\mathbf{x}|^n} f(\mathbf{x})\,d\mathbf{x} \qquad (16)\\
&= \int_{\mathbb{S}^{n-1}} \frac{a^*\bar{\mathbf{x}} - b^*}{|a^*\bar{\mathbf{x}} - b^*|^n}\,\mathbf{x}d\mathbf{x}\,f(\mathbf{x}).\\
&= \frac{a^*}{|a|^n}\int_{\mathbb{S}^{n-1}} \frac{\bar{\mathbf{x}} - \bar{\mathbf{u}}}{|\mathbf{x} - \mathbf{u}|^n}\,d\sigma(\mathbf{x})\,f(\mathbf{x}), \quad \text{where } \mathbf{u} = \frac{b'a^*}{|a|^2},\ d\sigma(\mathbf{x}) = \mathbf{x}d\mathbf{x}. \qquad (17)
\end{aligned}$$

If we consider the *reduced wavelet transform* [19, 21] $\mathcal{W} : L_2(\mathbb{S}^{n-1}) \to H_2(\mathbb{B}^n)$ then the last formula is the Cauchy integral formula in Clifford analysis up to the factor $\frac{a^*}{|a|^n}$. This factor is similar to the factor $\sqrt{1-|u|^2}$ in the Cauchy formula in complex analysis derived in [19, (3.20)]. Their appeared due to the invariant measures on $SL(2,\mathbb{R})$ and $\mathcal{M}(\mathbb{B}^n)$. Note the appearance of the important Clifford valued differential form $d\sigma(\mathbf{x}) = \mathbf{x}d\mathbf{x}$ in (17), cf. [4, § 9.1], [9, § II.0.2.1]. A standard derivation of the Cauchy formula in Clifford analysis are based on Stokes's Theorem.

Although the Cauchy formula (i.e. reduced wavelet transform) is an established tool in analytic function theory its unreduced version (16) acting $\mathcal{W}: L_2(\mathbb{S}^{n-1}) \rightarrow H_2(\mathcal{M}(\mathbb{B}^n))$ is also valuable for the functional calculus of several non-commuting operators.

The wavelet transform of the vacuum vector f_0

$$\begin{aligned}\mathcal{W}f_0(g) = \langle f_g, f_0\rangle &= \int_{\mathbb{S}^{n-1}} \frac{a^*-b^*\mathbf{x}}{|a^*-b^*\mathbf{x}|^n}\,d\mathbf{x} = \frac{a^*}{|a|^n}, \qquad \text{where } g=\begin{pmatrix} a & b' \\ b & a'\end{pmatrix}, \text{ or}\\ &= w^*(1+\mathbf{u}^2)^{(n-1)/2}, \qquad \text{where } g=\frac{1}{\sqrt{1+\mathbf{u}^2}}\begin{pmatrix} w & w\mathbf{u}' \\ w'\mathbf{u} & w'\end{pmatrix}.\end{aligned} \tag{18}$$

Consequently $\mathcal{W}f_0$ has a finite norm with respect to (10).

Definition 3.1. *The* Hardy space $H_2(\mathcal{M}(\mathbb{B}^n))$ *of Clifford valued functions on* $\mathcal{M}(\mathbb{B}^n)$ *is a left* $\mathcal{C}\ell(n)$*-module invariant under left shifts on* $\mathcal{M}(\mathbb{B}^n)$*, which is generated by the vacuum vector* $\mathcal{W}f_0$ (18)*.*

¿From the general wavelet technique [21] we obtain the following result:

Lemma 3.2. *1.* $H_2(\mathcal{M}(\mathbb{B}^n))$ *is an inner product space with the product derived from the Hardy functional* (9)*:*

$$\langle f_1, f_2\rangle = H(\bar{f}_1 f_2), \qquad \textit{where } f_1, f_2 \in H_2(\mathcal{M}(\mathbb{B}^n)). \tag{19}$$

2. Wavelet transform (16) *is a unitary operator intertwining the representation* ρ_1 *on* $H_2(\mathbb{S}^{n-1})$ *and the left regular representation on* $H_2(\mathcal{M}(\mathbb{B}^n))$ *by shifts:*

$$\mathcal{W}\rho_1(g) = \lambda(g)\mathcal{W}, \qquad \textit{for all } g\in\mathcal{M}(\mathbb{B}^n).$$

3.2 Taylor Series

Other classical objects of Clifford analysis (the Cauchy-Riemann equation, the Bergman space, etc.) can be also obtained [19, 23] from representation ρ_1. However we need only the Taylor series in the present paper. It is known [5, § 11.2.2] that there is the orthonormal basis $V_m(\mathbf{x})$ of $H_2(\mathbb{S}^{n-1})$ labelled by a multiindex $m=(m_1,\ldots,m_n)\in\mathbb{Z}_+^n$. Elements $V_m(\mathbf{x})$ can be constructed as symmetric polynomials of hypercomplex variables $e_1x_j - e_jx_1$, $j=1,\ldots,n$. Consequently there is a decomposition of the Cauchy kernel (i.e. coherent states (14)):

$$\rho_1(g)f_0(\mathbf{x}) = f_g(\mathbf{x}) = \sum_{m\in\mathbb{Z}_+^n} W_m(g)\,V_m(\mathbf{x}), \qquad \text{where } g\in\mathcal{M}(\mathbb{B}^n) \tag{20}$$

with some functions on $\mathcal{M}(\mathbb{B}^n)$ defined by

$$W_m(g) = \langle V_m, f_g\rangle \qquad \text{where } g\in\mathcal{M}(\mathbb{B}^n) \text{ and } m\in\mathbb{Z}_+^n. \tag{21}$$

The explicit expression of $W_m(g)$ could be derived from the decomposition of the Cauchy kernel in [5, § 11.4.2], but it is important for us now that formula (21) for a fixed g is a sort of *wavelet transform* $H_2(\mathbb{S}^{n-1}) \to C(\mathbb{Z}_+^n)$, cf. (15). We also use the following properties of functions $V_m(\mathbf{x})$ related to the representation theory:

1. Functions $V_m(\mathbf{x})$ with fixed $|m| = m_1 + \cdots + m_n$ form an $O(n)$-invariant irreducible module [10, § 3.3], which is required by the general construction of Taylor series [19, § 3.4].

2. There is the set of creation a_j^+ and annihilation a_j^- operators (known from quantum mechanics):

$$\begin{array}{lll} a_j^+ : & V_m(\mathbf{x}) \mapsto V_{m'}(\mathbf{x}), & \text{where } m' = (m_1, \ldots, m_j + 1, \ldots, m_n). \\ a_j^- : & V_m(\mathbf{x}) \mapsto m_j V_{m'}(\mathbf{x}), & \text{where } m' = (m_1, \ldots, m_j - 1, \ldots, m_n). \end{array}$$

   These operators satisfied [8] to the Heisenberg commutation relations:

$$[a_j^+, a_k^-] = \delta_{j,k} I, \qquad [a_j^+, a_k^+] = 0, \qquad [a_j^-, a_k^-] = 0.$$

   Thus we have [8] a representation of the Heisenberg group $\mathbb{H}^n$ in $H_2(\mathbb{S}^{n-1})$. Note also that in [28] operators A_j^+ and A_j^-were associated with operator of "$\times$-product" with the hypercomplex variable $e_1 x_j - e_j x_1$ partial derivative ∂_j correspondingly.

3. The function $V_0(\mathbf{x}) \equiv 1$ coincides with the vacuum vector $f_0(\mathbf{x})$ and:

$$V_m(\mathbf{x}) = (a_1^+)^{m_1} (a_2^+)^{m_2} \cdots (a_n^+)^{m_n} f_0(\mathbf{x}).$$

Clearly we can decompose any shifted function $\rho_1(g)V_k(\mathbf{x})$ over the basis $V_m(\mathbf{x})$ in a way similar to (20):

$$\rho_1(g)V_k(\mathbf{x}) = \sum_{m \in \mathbb{Z}_+^n} W_{k,m}(g)\, V_m(\mathbf{x}), \qquad \text{where } W_{k,m}(g) = \langle V_m, \rho_1(g)V_k \rangle\,. \tag{22}$$

The representation property $\rho_1(g_1)\rho_1(g_2) = \rho_1(g_1 g_2)$ implies an addition formula:

$$W_{l,m}(g_1 g_2) = \sum_{l \in \mathbb{Z}_+^n} W_{l,k}(g_1)\, W_{k,m}(g_2). \tag{23}$$

Thus functions $W_{k,m}(g)$ are *tokens* [22, 25] from the cancellative semigroup $\mathbb{Z}_+^n$ to $\mathcal{M}(\mathbb{B}^n)$. This means that the formula (22) defines the representation ρ_1 of $\mathcal{M}(\mathbb{B}^n)$ through the convolution on $\mathbb{Z}_+^n$.

4 Representations of $\mathcal{M}(\mathbb{B}^n)$ in Algebras and Moduli

A simple but important observation is that the Möbius transformations (11) can be easily extended to some non-commutative C^*-algebras.

Let $\mathfrak{A}$ be a C^*-algebra with the unit I, and an n-tuple A of self-adjoint elements $A_j \in \mathfrak{A}$, $j = 1, \ldots, n$ be fixed. We consider the tensor product $\mathfrak{A} \otimes \mathcal{C}\ell(n)$, which we *denote by* $\mathfrak{A}_n$ for the brevity. Its unit element will be again denoted by I. Then the tuple A can be associated with the element $\mathbf{A} = e_1 A_1 + e_2 A_2 + \cdots + e_n A_n$ in $\mathfrak{A}_n$. Let M be a left normed $\mathfrak{A}$-module, We denote by M_n the tensor product $M \otimes \mathcal{C}\ell(n)$. M_n is a left $\mathfrak{A}_n$ module of course. All constructed functional calculi according to Definition 1.1 are M_n-valued.

4.1 Resolvent Approach

We define an action of the Möbius group $\mathcal{M}(n)$ on the algebra $\mathfrak{A}_n$ by the natural formula (in Notation 2.1) similarly to expression (1):

$$g : \mathbf{A} \mapsto g^{-1}\mathbf{A} = \frac{\bar{a}\mathbf{A} - \bar{b}I}{a^*\mathbf{A} - b^*I}, \qquad \phi_{(u,w)} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b^* & a^* \end{pmatrix} \in \mathcal{M}(\mathbb{S}^{n-1}). \tag{24}$$

To this end we need invertibility of the operator $a^*\mathbf{A} - b^*I$ in $\mathfrak{A}_n$, which due to invertibility of a^* in $\mathcal{C}\ell(n)$ is equivalent to invertibility of $\mathbf{A} - \mathbf{u}I$, where $\mathbf{u} = (a^*)^{-1}b^* = a'b^* / |a|^2$ and thus $|\mathbf{u}| < 1$. Therefore we arrive to the following definition:

Definition 4.1. *[17, Defn. 3.1] The* Clifford (algebraic) resolvent set $R(\mathbf{A})$ *of an n-tuple $A_1, A_2, \ldots, A_n$ is the maximal open subset of $\mathbb{R}^n$ such that for $\mathbf{u} = u_1e_1 + u_2e_2 + \cdots + u_ne_n \in R(\mathbf{A})$ the element $\mathbf{A} - uI$ is invertible in $\mathfrak{A}_n$.*

The Clifford (algebraic) spectrum *is the completion of the Clifford resolvent set $\mathbb{R}^n \setminus R(A)$.*

Remark 4.2. *The Clifford (algebraic) spectrum is mainly an abbreviation for "the complement of the resolvent set" rather than an important characterisation of operator $\mathbf{A}$. Such a characterisation is provided instead by the spectrum, defined through the support of functional calculus, see below.*

Under the assumption that the Clifford algebraic spectrum of $\mathbf{A}$ belongs to the open unit ball $\mathbb{B}^n$ the orbit $\mathbb{A} = \{g^{-1}\mathbf{A} \mid g \in \mathcal{M}(\mathbb{B}^n)\}$ is a well defined subset of $\mathfrak{A}_n$. As any orbit $\mathbb{A}$ is a $\mathcal{M}(\mathbb{B}^n)$-homogeneous space.

Lemma 4.3 ([17, Lem. 3.18]). *For $g \in \mathcal{M}(\mathbb{B}^n)$ such that $(a^*)^{-1}b^* \in R(\mathbf{A})$ we have:*

$$\frac{\bar{a}\mathbf{A} - \bar{b}I}{a^*I - b^*\mathbf{A}} - \frac{\bar{a}\mathbf{x} - \bar{b}I}{a^* - b^*\mathbf{x}} = (a - \mathbf{x}^*b)^{-1}(\mathbf{A} - \mathbf{x}I)(a^*I - b^*\mathbf{A})^{-1}.$$

Consequently $\mathbf{x} \in R(\mathbf{A})$ implies $\frac{\bar{a}\mathbf{x} - \bar{b}}{a^* - b^*\mathbf{x}} \in R\left(\frac{\bar{a}\mathbf{A} - \bar{b}I}{a^*I - b^*\mathbf{A}}\right)$.

Proof. Möbius transforms of vectors are vectors, for them $\mathbf{y}^* = \mathbf{y}$, thus we have:

$$\begin{aligned}
\frac{\bar{a}\mathbf{A} - \bar{b}I}{a^*I - b^*\mathbf{A}} - \frac{\bar{a}\mathbf{x} - \bar{b}}{a^* - b^*\mathbf{x}} &= \frac{\bar{a}\mathbf{A} - \bar{b}I}{a^*I - b^*\mathbf{A}} - \left(\frac{\bar{a}\mathbf{x} - \bar{b}}{a^* - b^*\mathbf{x}}\right)^* \\
&= (\bar{a}\mathbf{A} - \bar{b}I)(a^*I - b^*\mathbf{A})^{-1} - (a - \mathbf{x}b)^{-1}(\mathbf{x}a' - b') \\
&= (a - \mathbf{x}b)^{-1}\left((a - \mathbf{x}b)(\bar{a}\mathbf{A} - \bar{b}I) - (\mathbf{x}a' - b')(a^*I - b^*\mathbf{A})\right)(a^*I - b^*\mathbf{A})^{-1} \\
&= (a - \mathbf{x}^*b)^{-1}(\mathbf{A} - \mathbf{x}I)(a^*I - b^*\mathbf{A})^{-1}.
\end{aligned}$$

The second statement follows from that result immediately. ∎

We define the *resolvent* function $R(g, \mathbf{A}) : \mathcal{M}(\mathbb{B}^n) \times \mathbb{A} \to \mathfrak{A}_n$ by the familiar expression:

$$R(g, \mathbf{A}) = (a^*I - b^*\mathbf{A})^{-1}$$

then a direct calculation shows that

$$R(g_1, \mathbf{A})R(g_2, g_1^{-1}\mathbf{A}) = R(g_1g_2, \mathbf{A}). \tag{25}$$

The last identity is well known in representation theory [16, § 13.2(10)] and is a key ingredient of *induced representations*. Thus we can again linearise (24) (cf. (12)) in a suitable space of M_n valued functions, where M_n is a left $\mathfrak{A}_n$ module as discussed at the beginning of this section. We linearise (24) in the space of continuous functions $C(\mathbb{A}, M_n)$ as follows:

$$\begin{aligned}\rho_{\mathbf{A}}(g_1) : f(g^{-1}\mathbf{A}) \ \mapsto\ & R(g_1^{-1}g^{-1}, \mathbf{A})\, f(g_1^{-1}g^{-1}\mathbf{A}) \\ &= (a^* I - b^*\mathbf{A})^{-1}\, f\left(\frac{\bar{a}\mathbf{A} - \bar{B}I}{a^* I - b^*\mathbf{A}}\right).\end{aligned} \tag{26}$$

However such a representation is not unitary in $H_2(\mathcal{M}(\mathbb{B}^n))$ for $n > 2$ as can be seen from a comparison with (12). To fix this we need an operator which is symbolically represented by $|a^* I - b^*\mathbf{A}|^{-n}$. When all operators A_j commute each other we can simply define:

$$\begin{aligned}|a^* I - b^*\mathbf{A}|^{-2} &= (a^* I - b^*\mathbf{A})^{-1}(a' I - \mathbf{A}b')^{-1} = \left(|a|^2 + |b|^2\, \mathbf{A}^2\right)^{-1} \\ &= \left(|a|^2 - |b|^2 \sum_{j=1}^{n} A_j 2\right)^{-1}.\end{aligned} \tag{27}$$

Then for an even $n \geq 4$ we can straightforwardly define $|a^* I - b^*\mathbf{A}|^{-n+2}$. For an odd n we can define a square root of the selfadjoint element $(a' I - \mathbf{A}b')(a^* I - b^*\mathbf{A})$ of $\mathfrak{A}_n$ by various means. However the Clifford algebraic spectrum may not guarantee the invertibility in (27), thus some additional assumptions of the type $\|\mathbf{A}\| < (1+\sqrt{2})^{-1}$ are required [14]. Consequently for for a commuting n-tuple ($n > 2$) of operators A_j one defines a representation $\rho_{\mathbf{A}}$ in the $C(\mathbb{A}, M_n)$ by the expression:

$$\rho_{\mathbf{A}} f(\mathbf{A}) = R(g, \mathbf{A})\, |a^* I - b^*\mathbf{A}|^{-n+2}\, f(g^{-1}\mathbf{A}). \tag{28}$$

In this way we obtain the monogenic calculus of commuting operators studied in [29].

For any $v \in M$ we can again define a K-invariant *vacuum vector* as $f_0(\mathbf{A}, v) = v \otimes f_0(\mathbf{A}) \equiv v \in C(\mathbb{A}, M_n)$. It generates the associated with $f_{\mathbf{u}}$ family of *coherent states* $f_g(\mathbf{A}, v) = R(g, \mathbf{A})\, |a^* I - b^*\mathbf{A}|^{-n+2}\, v$, where $g \in \mathcal{M}(\mathbb{B}^n)$. The *wavelet transform* defined by the same common formula based on coherent states (cf. (15)):

$$\mathcal{W}_m f(g) = \langle \rho_{\mathbf{A}}(g) f_0, f \rangle,$$

is a version of Cauchy integral, which maps $L_2(\mathbb{A}, M')$ to $C(\mathcal{M}(\mathbb{B}^n), \mathbb{C})$, where M' is the dual of the module M. The classical Riesz-Dunford functional calculus is a particular realisation of this approach [26].

For a non-commuting tuple $\mathbf{A}$ one can, for example, define representation $\rho_{\mathbf{A}}$ using the fruitful approach [14] based on the plain wave decomposition [34]. An alternative is the Taylor expansion construction initiated in [17].

4.2 Taylor Expansion Approach

To define a functional calculus for $\mathbf{A}$ we fix images $\Phi(V_m)$ of V_m (see Subsection 3.2), $m \in \mathbb{Z}_+^n$ in $\mathfrak{A}_n$, cf. [27, 28]. Seemingly this could be done in many different ways, but the covariance property fixes one preferred assignment. Indeed subgroup $O(n)$ of $\mathcal{M}(\mathbb{B}^n)$ contains permutations of elements of orthonormal basis e_k. Functions $V_m(\mathbf{x})$ are symmetric polynomials of x_j and are invariant under such permutations. To preserve $O(n)$ invariance we define $\Phi(V_m) = \Phi_{\mathbf{A},x}(V_m)$ associated to the tuple $\mathbf{A}$ to be

$$\Phi(V_m) = A_m := \frac{1}{|m|!} \sum_{\sigma \in S_{|m|}} e_{\sigma(1)} A_{\sigma(1)} e_{\sigma(2)} A_{\sigma(2)} \cdots e_{\sigma(n)} A_{\sigma(|m|)} \tag{29}$$

is the averaging of products of m_j copies of $e_j A_j$ over the permutation group $S_{|m|}$.

The value $r_R(\mathbf{A}) = \lim_{j\to\infty} \sup_\sigma \left\| A_{\sigma(1)} \cdots A_{\sigma(j)} \right\|^{1/j}$, $1 \le \sigma(i) \le n$ is known as the *Rota-Strang joint spectral radius* [32]. We give a similar definition which is better tailored to our circumstances:

Definition 4.4. *Let $m \in \mathbb{Z}_+^n$, $v \in M$ and A_m be defined in* (29). *We call*

$$r_S(\mathbf{A}) = \limsup_{|m|\to\infty} \|A_m\|_{\mathfrak{A}}^{1/|m|} \qquad \textit{and} \qquad r_L(\mathbf{A}, v) = \limsup_{|m|\to\infty} \|A_m v\|_M^{1/|m|}$$

the symmetric joint spectral radius *of* $\mathbf{A}$ *and* local spectral radius *of* $\mathbf{A}$ *at v correspondingly. Obviously $r_S(\mathbf{A}) \le r_R(A)$ and $r_L(\mathbf{A}, v) \le r_S(A)\,\|v\|$.*

Let $r_L(\mathbf{A}, v) < 1$, $v \in M$ and a sequence c_m, $m \in \mathbb{Z}_+^n$ be a square summable. Then the infinite series $\sum_{m\in\mathbb{Z}_+^n} c_m A_m v$ is absolutely convergent by norm in M_n. The linear space of all such sequences is denoted by $H_2(\mathbf{A}, v)$. Analogously to representation ρ_1 in (22) we define an action $\rho_{\mathbf{A},v}$ of $\mathcal{M}(\mathbb{B}^n)$ on $H_2(A, v)$ by:

$$\rho_{\mathbf{A},v}(g) : \sum_{k\in\mathbb{Z}_+^n} c_k A_k v \mapsto \sum_{k\in\mathbb{Z}_+^n} d_k A_k v, \quad \text{where } d_k = \sum_{m\in\mathbb{Z}_+^n} W_{k,m}(g)\, c_m. \tag{30}$$

Then the identity (23) implies that $\rho_{\mathbf{A},v}$ is a representation of $\mathcal{M}(\mathbb{B}^n)$.

Definition 4.5. *Let $r_L(A, v) < 1$ then the monogenic functional calculus $\Phi = \Phi_{\mathbf{A},v}$ associated to a n-tuple $\mathbf{A}$ and a vector $v \in M$ is a continuous linear map $\Phi : H_2(\mathbb{S}^{n-1}) \to H_2(A, v)$ is defined by the following two conditions:*

1. *Φ intertwines ρ_1* (12) *and $\rho_{\mathbf{A},v}$* (30)*: $\Phi\rho_1(g) = \rho_{\mathbf{A},v}(g)\Phi$ for all $g \in \mathcal{M}(\mathbb{B}^n)$.*
2. *The map of vacuum vectors is $\Phi(f_0) = f_v$, where $f_0(\mathbf{x}) \equiv 1$ and $f_v = v$.*

This defines monogenic calculus uniquely, particularly its integral formula.

Proposition 4.6. *Let $E(g, \mathbf{A})$ be the family of coherent states for $\rho_{\mathbf{A},v}$:*

$$E(g, \mathbf{A}) = \rho_{\mathbf{A},v} f_v = \sum_{k\in\mathbb{Z}_+^n} W_{m,0}(g) A_m v. \tag{31}$$

Then the functional calculus $\Phi_{\mathbf{A},v}$ is defined by the Integral formula:

$$\Phi_{\mathbf{A},v} f = \int_{\mathcal{M}(\mathbb{B}^n)} E(g, \mathbf{A}) f(g)\, dg.$$

Proof. Indeed using the Definition 4.5 we calculate for $f = \langle \rho_1(g) f_0, f \rangle$:

$$\begin{aligned}
\Phi_{\mathbf{A},v} f &= \Phi_{\mathbf{A},v} \langle \rho_1(g) f_0, f \rangle \\
&= \langle \Phi_{\mathbf{A},v} \rho_1(g) f_0, f \rangle && (32)\\
&= \langle \rho_{\mathbf{A},v}(g) \Phi_{\mathbf{A},v} f_0, f \rangle && (33)\\
&= \langle \rho_{\mathbf{A},v}(g) f_v, f \rangle && (34)\\
&= \langle E(g, \mathbf{A}), f \rangle && (35)\\
&= \int_{\mathbb{S}^{n-1}} E(g, \mathbf{A}) f(\mathbf{x})\, d\mathbf{x},
\end{aligned}$$

where (32) is obtained by linearity and continuity of functional calculus, (33) follows from the intertwining property 1, (34) is obtained from the initialisation property 2, and finally (35) uses expression (31) for $E(g, \mathbf{A})$. ■

The full consideration of the monogenic calculus and the corresponding joint spectrum requires a solid background from the representation theory of semisimple Lie groups. We will consider a simpler but still illustrative case in the next section.

5 Functional Calculus and Spectrum for a Pair of Matrices

In this section we demonstrate the previous construction by the simplest non-trivial example: functional calculus for a pair A_1, A_2 of self-adjoint non-commuting operators with finite dimensional ranges, cf. [15]. Instead of tensor product $e_1 A_1 + e_2 A_2$ with Clifford algebra $\mathcal{C}\ell(2)$ we can consider the complexification $A_1 + iA_2$ since the product $i = e_1 e_2$ has all properties of the complex imaginary unit. The group $\mathcal{M}(\mathbb{B}^n)$ is the $SL(2, \mathbb{R})$ group in this case, $O(2)$ consists from the orthogonal rotations of the plane, and $\mathcal{M}(\mathbb{B}^n)/O(n) = SL(2, \mathbb{R})/O(2)$ is the unit disk $\mathbb{D}$. In two dimensions the formula (26) defines an isometric representation of $\mathcal{M}(\mathbb{B}^n)$ without a normalising factor.

5.1 Jet Bundles and Prolongations of ρ_1

To formulate the complete description of monogenic calculus and spectrum of $\mathbf{A} = A_1 + iA_2$ we use the language of jet spaces and prolongations of representations introduced by S. Lie, see [30, 31] for a detailed exposition.

Definition 5.1. [31, Chap. 4] *Two holomorphic functions have nth* order contact *in a point if their value and their first n derivatives agree at that point, in other words their Taylor expansions are the same in first $n + 1$ terms.*

A point $(z, u^{(n)}) = (z, u, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ of the jet space $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ *is the equivalence class of holomorphic functions having nth contact at the point z with the polynomial:*

$$p_n(w) = \mathbf{u}_n \frac{(w - z)^n}{n!} + \cdots + \mathbf{u}_1 \frac{(w - z)}{1!} + u. \qquad (36)$$

For a fixed n each holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has nth *prolongation* (or *n-jet*) $\mathrm{j}_n f : \mathbb{D} \to \mathbb{C}^{n+1}$:

$$\mathrm{j}_n f(z) = (f(z), f'(z), \ldots, f^{(n)}(z)). \tag{37}$$

The graph $\Gamma_f^{(n)}$ of $\mathrm{j}_n f$ is a submanifold of $\mathbb{J}^n$ which is section of the *jet bundle* over $\mathbb{D}$ with a fibre $\mathbb{C}^{n+1}$. We also introduce a notation J_n for the map $J_n : f \mapsto \Gamma_f^{(n)}$ of a holomorphic f to the graph $\Gamma_f^{(n)}$ of its n-jet $\mathrm{j}_n f(z)$ (37).

One can prolong any map of functions $\psi : f(z) \mapsto [\psi f](z)$ to a map $\psi^{(n)}$ of n-jets by the formula

$$\psi^{(n)}(J_n f) = J_n(\psi f). \tag{38}$$

For example such a prolongation $\rho_1^{(n)}$ of the representation ρ_1 of the group $\mathcal{M}(\mathbb{B}^n)$ in $H_2(\mathbb{D})$ (as any other representation of a Lie group [31]) will be again a representation of $\mathcal{M}(\mathbb{B}^n)$. Equivalently we can say that J_n *intertwines* ρ_1 and $\rho_1^{(n)}$:

$$J_n \rho_1(g) = \rho_1^{(n)}(g) J_n \quad \text{for all } g \in \mathcal{M}(\mathbb{B}^n).$$

Of course, the representation $\rho_1^{(n)}$ is not irreducible: any jet subspace $\mathbb{J}^k$, $0 \le k \le n$ is $\rho_1^{(n)}$-invariant subspace of $\mathbb{J}^n$. However the representations $\rho_1^{(n)}$ are *primary* [16, § 8.3] in the sense that they are not sums of two subrepresentations.

The following statement explains why jet spaces appeared in our study.

Proposition 5.2. *Let the matrix $\mathbf{A} = A_1 + iA_2$ be a Jordan block of a length k with the eigenvalue $u = 0$, and v be its root vector of order k, i.e. $\mathbf{A}^{k-1}v \neq \mathbf{A}^k v = 0$. Then the restriction of $\rho_{\mathbf{A},v}$ on the subspace generated by v_m is equivalent to the representation ρ_1^k.*

5.2 Spectrum and the Jordan Normal Form of a Matrix

Now we are prepared to describe a spectrum of a matrix $\mathbf{A} = A_1 + iA_2$. Since the functional calculus is an intertwining operator its support is a decomposition into intertwining operators with prime representations (we could not expect generally that these prime subrepresentations are irreducible).

Recall the group of inner automorphisms $b_g : g_1 \mapsto b_g(g_1) = g^{-1} g_1 g$ of $\mathcal{M}(\mathbb{B}^n)$. The representation $\rho_g(g_1) = \rho_1(b_g(g_1))$ is equivalent to ρ_1 and they are obviously intertwined by the operator $\rho_1(g^{-1})$: $\rho_g \rho_1(g^{-1}) = \rho_1(g^{-1})\rho_1$. For a Jordan block $\mathbf{A}$ with an eigenvalue $\mathbf{u}$ its Möbius transformation with the matrix $\begin{pmatrix} 1 & \mathbf{u}' \\ \mathbf{u} & 1 \end{pmatrix}$ will be a Jordan block with eigenvalue $\mathbf{0}$ due to Lemma 4.3. Thus inner automorphisms extend Proposition 5.2 to the complete characterisation of $\rho_{\mathbf{A},v}$ for matrices.

Proposition 5.3. *Representation $\rho_{\mathbf{A},v}$ is equivalent to a direct sum of the prolongations $\rho_1^{(k)}$ of ρ_1 in the kth jet space $\mathbb{J}^k$ intertwined with inner automorphisms. Consequently the spectrum of $\mathbf{A}$ (defined via the functional calculus $\Phi_{\mathbf{A},v}$) labelled exactly by n pairs of numbers $(\mathbf{u}_i, k_i)$, where $\mathbf{u}_i \in \mathbb{D}$, $k_i \in \mathbb{Z}_+$ for $1 \le i \le n$ some of whom could coincide.*

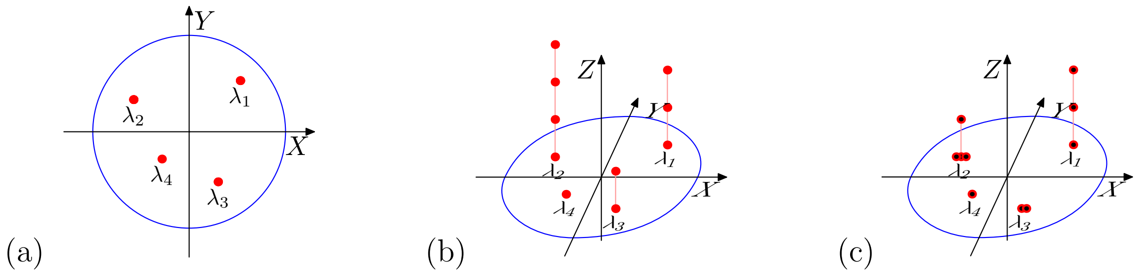

Figure 1: (a) Classical spectrum for a pair of matrices on ; (b) the new version of joint spectrum; (c) is an example mapping for the new spectrum.

Obviously this spectral theory is a fancy restatement of the *Jordan normal form* of matrices.

Example 5.4. *Let* $J_k(\mathbf{u})$ *denote the Jordan block of the length* k *for the eigenvalue* $\mathbf{u}$. *On the Fig. 1 there are two pictures of the spectrum for the matrix*

$$a = J_3(\mathbf{u}_1) \oplus J_4(\mathbf{u}_2) \oplus J_1(\mathbf{u}_3) \oplus J_2(\mathbf{u}_4),$$

where

$$\mathbf{u}_1 = \frac{3}{4}e^{i\pi/4}, \quad \mathbf{u}_2 = \frac{2}{3}e^{i5\pi/6}, \quad \mathbf{u}_3 = \frac{2}{5}e^{-i3\pi/4}, \quad \mathbf{u}_4 = \frac{3}{5}e^{-i\pi/3}.$$

Part (a) represents the conventional two-dimensional image of the spectrum, i.e. eigenvalues of $\mathbf{A}$, *and (b) describes spectrum* $\operatorname{sp} a$ *arising from the wavelet construction. The first image does not allow to distinguish* $\mathbf{A}$ *from many other essentially different matrices, e.g. the diagonal matrix*

$$diag\,(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4),$$

which even have a different dimensionality. At the same time the Fig. 1(b) completely characterise $\mathbf{A}$ *up to a similarity. Note that each point of* $\operatorname{sp}\mathbf{A}$ *on Fig. 1(b) corresponds to a particular root vector, which spans a primary subrepresentation.*

In light of the previous discussions [17, p. 29], [13, Ex. 6.3] the following simple example is still of interest.

Example 5.5. *For a pair Pauli matrices* $J_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ *and* $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *the joint Clifford algebraic spectrum (Definition 4.1) as found in [17, p. 29] is the single point* $(0,0) \in \mathbb{D}$. *The joint spectrum found in [13, Ex. 6.3] coincides with the Weyl joint spectrum and the numerical range [15]: all of them are the entire unit disk* $\mathbb{D}$. *Finally the joint spectrum from Proposition 5.3 is a pair of points* $(\mathbf{0},0)$ *and* $(\mathbf{0},1)$ *from* $\mathbb{R}^2 \times \mathbb{Z}_+$ *since* $J_1 + iJ_2$ *is similar to the Jordan block of the length* 2 *with the eigenvalue* 0.

5.3 Spectral Mapping Theorem

As was mentioned in the Introduction a reasonable spectrum should be linked to the corresponding functional calculus by an appropriate spectral mapping theorem. The new version of spectrum is based on prolongation of ρ_1 into jet spaces. Naturally a correct version of spectral mapping theorem should operate in jet spaces as well.

Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic map, let us define its action on functions $[\phi_* f](\mathbf{z}) = f(\phi(\mathbf{z}))$. According to the general formula (38) we can define the prolongation $\phi_*^{(n)}$ onto the jet space $\mathbb{J}^n$. Its associated action $\rho_1^k \phi_*^{(n)} = \phi_*^{(n)} \rho_1^n$ on the pairs $(\mathbf{u}, k)$ is given by the formula:

$$\phi_*^{(n)}(\mathbf{u}, k) = \left(\phi(\mathbf{u}), \left[\frac{k}{\deg_{\mathbf{u}} \phi}\right]\right), \tag{39}$$

where $\deg_{\mathbf{u}} \phi$ denotes the degree of zero of the function $\phi(\mathbf{z}) - \phi(\mathbf{u})$ at the point $\mathbf{z} = \mathbf{u}$ and $[x]$ denotes the integer part of x. We are ready to state

Theorem 5.6 (Spectral mapping). *Let ϕ be a holomorphic mapping $\phi : \mathbb{D} \to \mathbb{D}$ and its prolonged action $\phi_*^{(n)}$ defined by* (39), *then*

$$\operatorname{sp} \phi(\mathbf{A}) = \phi_*^{(n)}(\operatorname{sp} \mathbf{A}).$$

The explicit expression of (39) for $\phi_*^{(n)}$, which involves derivatives of ϕ up to nth order, is known, see for example [11, Thm. 6.2.25]. However it was not recognised before as a form of spectral mapping.

Example 5.7. *Let us continue with Example 5.4. Let ϕ map all four eigenvalues $\mathbf{u}_1$, ..., $\mathbf{u}_4$ of the matrix $\mathbf{A}$ into themselves. Then Fig. 1(a) will represent the classical spectrum of $\phi(a)$ as well as $\mathbf{A}$. In the contrast Fig. 1(c) shows mapping of the new spectrum for the case ϕ has orders of zeros at these points as follows: the order 1 at $\mathbf{u}_1$, exactly the order 3 at $\mathbf{u}_2$, an order at least 2 at $\mathbf{u}_3$, and finally any order at $\mathbf{u}_4$.*

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References

[1] Robert F. V. Anderson, *The Weyl functional calculus*, J. Functional Analysis **4** (1969), 240–267. **MR** 58 #30405

[2] Jonathan Arazy and Harald Upmeier, *Invariant symbolic calculi and eigenvalues of invariant operators on symmetric domains*, Function spaces, interpolation theory and related topics (Lund, 2000), de Gruyter, Berlin, 2002, pp. 151–211. **MR** 2003k:32031

[3] ———, *Weyl calculus for complex and real symmetric domains*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **13** (2002), no. 3-4, 165–181, Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001). **MR** 1 984 098

[4] F. Brackx, R. Delanghe, and H. Serras (eds.), *Clifford algebras and their applications in mathematical physics*, Fundamental Theories of Physics, vol. 55, Dordrecht, Kluwer Academic Publishers Group, 1993, **MR** 94j:00019.

[5] F. Brackx, Richard Delanghe, and F. Sommen, *Clifford analysis*, Research Notes in Mathematics, vol. 76, Pitman (Advanced Publishing Program), Boston, MA, 1982. **MR** 85j:30103

[6] Jan Cnops, *Hurwitz pairs and applications of Möbius transformations*, Habilitation dissertation, Universiteit Gent, Faculteit van de Wetenschappen, 1994, See also [7].

[7] Jan Cnops, *An introduction to Dirac operators on manifolds*, Progress in Mathematical Physics, vol. 24, Birkhäuser Boston Inc., Boston, MA, 2002. **MR** 1 917 405

[8] Jan Cnops and Vladimir V. Kisil, *Monogenic functions and representations of nilpotent Lie groups in quantum mechanics*, Mathematical Methods in the Applied Sciences **22** (1998), no. 4, 353–373, E-print: `arXiv:math/9806150`. **MR** 2000b:81044. **Zbl** 1005.22003.

[9] R. Delanghe, F. Sommen, and V. Souček, *Clifford algebra and spinor-valued functions*, Mathematics and its Applications, vol. 53, Kluwer Academic Publishers Group, Dordrecht, 1992, A function theory for the Dirac operator, Related REDUCE software by F. Brackx and D. Constales, With 1 IBM-PC floppy disk (3.5 inch). **MR** 94d:30084

[10] John E. Gilbert and Margaret A.M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge Studies in Advanced Mathematics, vol. 26, Cambridge University Press, Cambridge, 1991.

[11] Roger A. Horn and Charles R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994, Corrected reprint of the 1991 original. **MR** 95c:15001

[12] B. Jefferies and G. W. Johnson, *Feynman's operational calculi for noncommuting operators: the monogenic calculus*, Adv. Appl. Clifford Algebras **11** (2001), no. 2, 239–264 (2002). **MR** 2003k:47011

[13] Brian Jefferies and Alan McIntosh, *The Weyl calculus and Clifford analysis*, Bull. Austral. Math. Soc. **57** (1998), no. 2, 329–341.

[14] Brian Jefferies, Alan McIntosh, and James Picton-Warlow, *The monogenic functional calculus*, Studia Math. **136** (1999), no. 2, 99–119. **MR** 2000i:47030

[15] Brian Jefferies and Bernd Straub, *Lacunas in the support of the Weyl calculus for two Hermitian matrices*, J. Aust. Math. Soc. **75** (2003), no. 1, 85–124. **MR** 1 984 630

[16] A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin, 1976, Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220. **MR** 54 #447

[17] Vladimir V. Kisil, *Möbius transformations and monogenic functional calculus*, Electron. Res. Announc. Amer. Math. Soc. **2** (1996), no. 1, 26–33, (electronic) **MR** 98a:47018.

[18] ———, *Towards to analysis in* $\mathbb{R}^{pq}$, Proceedings of Symposium Analytical and Numerical Methods in Quaternionic and Clifford Analysis, June 5–7, 1996, Seiffen, Germany (Klaus (ed.) Sprößig, Wolfgang (ed.); Gürlebeck, ed.), TU Bergakademie Freiberg, Freiberg, 1996, **Zbl** 882.30030, pp. 95–100.

[19] ———, *Analysis in* $\mathbb{R}^{1,1}$ *or the principal function theory*, Complex Variables Theory Appl. **40** (1999), no. 2, 93–118, E-print: `arXiv:funct-an/9712003`. **MR** 2000k:30078.

[20] ———, *Two approaches to non-commutative geometry*, Complex Methods for Partial Differential Equations (H. Begehr, O. Celebi, and W. Tutschke, eds.), Kluwer Academic Publishers, Netherlands, 1999, E-print: `arXiv:funct-an/9703001`, **MR** 2001a:01002, pp. 219–248.

[21] ———, *Wavelets in Banach spaces*, Acta Appl. Math. **59** (1999), no. 1, 79–109, E-print: `arXiv:math/9807141`. **MR** 2001c:43013.

[22] ———, *Umbral calculus and cancellative semigroup algebras*, Z. Anal. Anwendungen **19** (2000), no. 2, 315–338, E-print: `arXiv:funct-an/9704001`. **MR** 2001g:05017. **Zbl** 0959.43004.

[23] ———, *Spaces of analytical functions and wavelets—Lecture notes*, E-print: `arXiv:math.CV/0204018`, 2000–2002, 92 p.

[24] ———, *Meeting Descartes and Klein somewhere in a noncommutative space*, Highlights of Mathematical Physics (A. Fokas, J. Halliwell, T. Kibble, and B. Zegarlinski, eds.), AMS, 2002, E-print: `arXiv:math-ph/0112059`, pp. 165–189.

[25] ———, *Tokens: An algebraic construction common in combinatorics, analysis, and physics*, Functional Analysis: Proc. of the Ukrainian Math. Congress-2001, Inst. of Math. of NAS of Ukraine, 2002, E-print: `arXiv:math.FA/0201012`, pp. 146–155.

[26] ———, *Spectrum as the support of functional calculus*, Functional Analysis and its Applications (Proceedings of the S. Banach Conference) (North-Holland) (V. Kadets and W. Zelazko, eds.), Math. Studies series, Elsevier Science Publishers, 2004, E-print: `arXiv:math.FA/0208249`, pp. 133–142.

[27] Vladimir V. Kisil and Enrique Ramírez de Arellano, *The Riesz-Clifford functional calculus for several non-commuting operators and quantum field theory*, Math. Methods Appl. Sci. **19** (1996), no. 8, 593–605, **MR** 97h:47009, E-print: `arXiv:funct-an/9502006`.

[28] ———, *A functional model for quantum mechanics: Unbounded operators*, Math. Methods Appl. Sci. **20** (1997), no. 9, 745–757, **MR** 98f:47028.

[29] Alan McIntosh and Alan Pryde, *A functional calculus for several commuting operators*, **36** (1987), 421–439.

[30] Peter J. Olver, *Applications of Lie groups to differential equations*, second ed., Springer-Verlag, New York, 1993. **MR** 94g:58260

[31] ______, *Equivalence, invariants, and symmetry*, Cambridge University Press, Cambridge, 1995. **MR** 96i:58005

[32] Gian-Carlo Rota and W.G. Strang, *A note on the joint spectral radius*, **22** (1960), 379–381.

[33] John Ryan, *Some application of conformal covariance in Clifford analysis*, Clifford Algebras in Analysis and Related Topics (John Ryan, ed.), CRC Press, Boca Raton, 1995, pp. 128–155.

[34] Frank Sommen, *Plane wave decompositions of monogenic functions*, Annales Pol. Math. **49** (1988), 101–114.

[35] N. Ja. Vilenkin, *Special functions and the theory of group representations*, American Mathematical Society, Providence, R. I., 1968, Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. **MR** 37 #5429

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