Curves of the Projective 3–space, Tangent Developables and Partial Spreads

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1 Introduction

A *twisted cubic* $\mathcal{C}$ of $PG(3,q)$, the 3–dimensional projective space over the Galois field $GF(q)$, is given in its canonical form by

$$\mathcal{C} = \{P(t) = (t^3, t^2, t, 1), t \in GF(q) \cup \{\infty\}\},$$

where $t = \infty$ gives the point $(1,0,0,0)$. Twisted cubics over Galois fields were introduced and studied by Segre [17], [18]. Further properties were investigated by Hirschfeld [12], [13]. The main property of a twisted cubic of $PG(3,q)$ is that it is a maximal arc [10, 21.2], namely it is a set of $q+1$ points of $PG(3,q)$, no four of which are coplanar.

However, twisted cubics are also interesting because of their connection with spreads and partial spreads of $PG(3,q)$.

In $PG(3,q)$, a *spread* $\mathcal{S}$ is a set of q^2+1 lines, no two of which intersect. A *partial spread* $\mathcal{P}$ is a set of mutually skew lines, and if $|\mathcal{P}| = s$, then $\mathcal{P}$ is also called a *s–span*. Hence, a (q^2+1)–span is a spread of $PG(3,q)$.

In [3] it was shown that in $PG(3,q)$, $(q+1,3)=1$, if $\mathcal{C}$ is a twisted cubic, then the set $\mathcal{S}$ of lines consisting of the imaginary chords of $\mathcal{C}$, the imaginary axes of the osculating developable of $\mathcal{C}$ and the tangents to $\mathcal{C}$ form a spread.

In particular, it is easily seen that the tangents to $\mathcal{C}$ form a $(q+1)$–span [10, Theorem 21.1.9] (actually the proof works for any field). For further results on twisted cubics over Galois fields see also [4].

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In this paper we are mainly interested in curves $\mathcal{X}$ of $PG(3, \mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field of characteristic $p \geq 0$, satisfying the following condition.

$$\textit{Tangent lines to } \mathcal{X} \textit{ at distinct smooth points are skew.} \tag{1.1}$$

If we assume that $\mathbb{K}$ is the algebraic closure of $GF(q)$, the condition (1.1) means that tangent lines to $\mathcal{X}$ at $GF(q)$–rational points will form a (partial) spread of $PG(3, q)$. We will see, under suitable assumptions, that if $\mathcal{X}$ satisfies condition (1.1), then $\mathcal{X}$ must necessarily be a twisted cubic, giving in this manner a characterization of twisted cubics.

Also an infinite family of curves of $PG(3, \mathbb{K})$, distinct from twisted cubics and satisfying property (1.1), is found.

2 Definitions and Preliminaries

We work over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$.

Let $\mathcal{X} \subset PG(3, \mathbb{K})$ be an integral curve of degree $d \geq 3$. Let $PG(3, \mathbb{K})^*$ be the dual projective space of $PG(3, \mathbb{K})$. Let

$$\mathcal{X}^* = \overline{\{H \in PG(3, \mathbb{K})^* | H \text{ tangent to } \mathcal{X} \text{ at a smooth point } P \in \mathcal{X}\}}$$

be the *dual* of $\mathcal{X}$ and also let $Z(\mathcal{X})$ be the set

$$\overline{\{(P, H) \in \mathcal{X}_{reg} \times PG(3, \mathbb{K})^* | H \text{ tangent to } \mathcal{X} \text{ at the smooth point } P \in \mathcal{X}\}}.$$

It follows that $\mathcal{X}^*$ is the image of $Z(\mathcal{X})$ under the projection $pr_2 : PG(3, \mathbb{K}) \times PG(3, \mathbb{K})^* \to PG(3, \mathbb{K})^*$.

The curve $\mathcal{X}$ is said to be *reflexive* if $Z(\mathcal{X}^*) = Z(X)$ via the identification $PG(3, \mathbb{K})^{**} = PG(3, \mathbb{K})$ [7].

Let $\{b_i\}_{0 \leq i \leq 3}$ be the order sequence of $\mathcal{X}$ [19]. Hence $b_0 = 0$, $b_1 = 1$ and $d \geq b_3 > b_2 > 1$. We have that $\mathcal{X}$ is reflexive if and only if $b_2 = 2$ and $p \neq 2$ [7, 3.5]. The integer b_2 is the order of contact of $\mathcal{X}$ with its tangent line at a generic point of $\mathcal{X}$ [9, Prop. 4]. The integer b_3 is the order of contact of $\mathcal{X}$ with its osculating plane at a generic point of $\mathcal{X}$.

It is known that $b_2 = 2$ and $b_3 = 3$ if the characteristic p of $\mathbb{K}$ is zero. If this is the case, $\mathcal{X}$ is said to be *classical.* [15]. Also, if $p > 0$ and $b_2 > 2$ or $p = 2$, then it can be proven that there exists an integer $e \geq 1$ such that $b_2 = p^e$ [7], and the curve is said to be *non–reflexive.*

Remark 2.1. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$. Fix a general point $P \in \mathcal{X}$ and let $\pi : \mathcal{Y} \to PG(1, \mathbb{K})$ be the morphism induced by the projection of $\mathcal{X}$ onto $PG(1, \mathbb{K})$ from the tangent line to $\mathcal{X}$ at P, which we will denote by $T_P\mathcal{X}$. The morphism π is not separable, i.e. its differential at a generic point is zero if and only if $\mathcal{X}$ is strange ($\mathcal{X}$ is said to be *strange* if all its tangent lines at smooth point pass through a fixed point called the *center* or the *nucleus*). Also $\mathcal{X}$ is strange if and only if its tangent developable is a cone. The morphism π has degree $d - b_2$ and it ramifies at P if and only if $b_3 > b_2 + 1$.

3 The Main Result

In this section we will characterize twisted cubics of $PG(3, \mathbb{K})$ ($PG(3, q)$) as explained in the Introduction.

Theorem 3.1. *Let $\mathcal{X} \subset PG(3, \mathbb{K})$ be a smooth degree d curve such that for a general point $P \in \mathcal{X}$ there is no tangent line to $\mathcal{X}$ at a point $Q \neq P$, with $T_P\mathcal{X} \cap T_Q\mathcal{X} \neq \emptyset$. Then $\mathcal{X} \equiv PG(1, \mathbb{K})$, and either $d = 3$ and $\mathcal{X}$ is a twisted cubic, or $d = p^e + 1$ and $\mathcal{X}$ is projectively equivalent to the rational curve $\mathcal{D}$ with the parametrization*

$$(w_0, w_1) \to (w_0^{p^e+1}, w_0^{p^e} w_1, w_0 w_1^{p^e}, w_1^{p^e+1}). \tag{3.1}$$

Conversely, any two tangent lines to $\mathcal{X}$, where $\mathcal{X}$ is one of the above curves, are skew. If $\mathcal{X}$ is the the curve $\mathcal{D}$, we also require $p^e \geq 4$.

Proof. We divide the proof into three steps.
Step 1. Here we assume that $\mathcal{X}$ has classical order sequence, i.e. $b_i = i$, $i = 2, 3$. Let g be the genus of $\mathcal{X}$. Fix a general point $P \in \mathcal{X}$ and let $\pi : \mathcal{X} \to PG(1, \mathbb{K})$ be the morphism induced by the projection of $\mathcal{X}$ from the line $T_P\mathcal{X}$. Since $b_2 = 2$, π is a degree $d - 2$ morphism (Remark 2.1). By the last assertion of Remark 2.1, for a general point P, the morphism π has differential not zero. Hence the differential of π at a general point of $\mathcal{X}$ is not zero, i.e. π is separable. Thus we may apply the Riemann–Hurwitz formula [6, Cor. 2.4] and obtain that the ramification divisor of π has degree $2d - 6 + 2g$. Since $b_3 = b_2 + 1$ and P is general, every ramification point of π corresponds either to a smooth point Q of $\mathcal{X}$ with $T_P\mathcal{X} \cap T_Q\mathcal{X} \neq \emptyset$, or to a cusp of $\mathcal{X}$. Hence we have $2d - 6 + 2g = 0$. Since $d \geq 3$ and $g \geq 0$, we obtain $g = 0$ and $d = 3$. Hence $\mathcal{X}$ is a twisted cubic, as wanted.

Step 2. Here we assume $b_2 = p^e$ and $b_3 = p^e + 1$. Let g be the genus of $\mathcal{X}$. Fix a general point $P \in \mathcal{X}$ and let $\pi : \mathcal{X} \to PG(1, \mathbb{K})$ be the morphism induced by the projection of $\mathcal{X}$ from the line $T_P(\mathcal{X})$. Since $b_2 = p^e$, π is a morphism of degree $d - b_2$. Since the general tangent line to $\mathcal{X}$ does not intersect $T_P\mathcal{X}$, the morphism π is separable (see Remark 2.1). Hence, we may apply the Riemann–Hurwitz formula and obtain that, counting multiplicities, the ramification divisor of π has degree $2d - 2p^e + 2g - 2$. Since $b_3 = b_2 + 1$ and $\mathcal{X}$ is smooth, every point on the ramification divisor of π corresponds to a point $Q \in \mathcal{X}$, $Q \neq P$, such that $T_P\mathcal{X} \cap T_Q\mathcal{X} \neq \emptyset$ (see Remark 2.1). Hence $2d - 2p^e + 2g - 2 = 0$ and so $d = p^e + 1$ and $g = 0$. It follows that $\mathcal{X} \equiv PG(1, \mathbb{K})$. We choose homogeneous coordinates $x_0, \ldots, x_3$ on $PG(3, \mathbb{K})$ such that $P = (1, 0, 0, 0)$, $T_P\mathcal{X} = \{x_2 = x_3 = 0\}$ and $\{x_3 = 0\}$ is the osculating plane to $\mathcal{X}$ at P. Hence, taking affine coordinates $X_i = x_i/x_0$, $i = 1, 2, 3$, $\mathcal{X}$ has a parametrization $(t, \alpha t^{p^e}, \beta t^{p^e+1})$, with $\alpha \neq 0$, $\beta \neq 0$. Again, passing to homogeneous coordinates, we obtain $\mathcal{X} = \mathcal{D}$, as wanted.

Step 3. Now we assume $b_3 \geq b_2 + 2$. From the Riemann–Hurwitz formula and the assumptions one finds $2g - 2 = 2(d - b_2) + (b_3 - b_2 - 1)$. Hence $2d - 2b_2 - 2 + 2g = b_3 - b_2 - 1$. From $d \geq b_3$, we have that $2(b_3 - b_2) - 2 + 2g \leq b_3 - b_2 - 1$. This is a contradiction as $b_3 - b_2 > 1$.

The viceversa comes from [10, 21.1.9] and the following remark. ∎

Remark 3.2. Fix integers p,e,d with p prime, $e > 0$ and $p^e < d < 2p^e$. Let $\mathcal{X} \subset PG(3, \mathbb{K})$ be any integral degree d curve with order sequence $\{b_i\}_{0 \leq i \leq 3}$ and $b_2 = p^e$. Fix $P, Q \in X_{reg}$ with $T_P\mathcal{X} \neq T_Q\mathcal{X}$; for a general point $P \in \mathcal{X}$ this is the case for every $Q \in \mathcal{X}_{reg}$.

Assume $T_P\mathcal{X} \cap T_Q\mathcal{X} \neq \emptyset$ and let H be the plane spanned by $T_P\mathcal{X}$ and $T_Q\mathcal{X}$. Since $T_P\mathcal{X} \cap \mathcal{X}$ (resp. $T_Q\mathcal{X} \cap \mathcal{X}$) contains at least a 0–dimensional subscheme of length p^e with P (resp. Q) as support, and $d < 2p^e$, this is impossible. The possibility $d = 2p^e - 1$ does not occur because in this case we would find a plane intersecting the curve in a 0–dimensional subscheme of length at least $2p^e$, which is more than the degree of the curve. In particular, if $d \leq 2p^e - 1$ (but also in several other cases), we are sure that for a general point $P \in \mathcal{X}$ no tangent line to a smooth point of $\mathcal{X}$ may intersect $T_P\mathcal{X}$.

Remark 3.3. The proof of Theorem 3.1 works in the same way if instead of assuming that $\mathcal{X}$ is smooth, we assume only that the normalization map $f : \mathcal{Y} \to \mathcal{X}$ is unramified, i.e. $\mathcal{X}$ has no cusps, or equivalently, that for every $A \in$Sing($\mathcal{X}$) (if any) all the formal branches of $\mathcal{X}$ at A are smooth. Note that the normalization map may be unramified even if some of these formal branches have the same tangent line (e.g. if A is a tacnode or a higher order tacnode of $\mathcal{X}$).

However, it would be interesting to have the analogue of Theorem 3.1 for singular curves. In this case, the Hasse–Weil bound [11, 2.9] for the number N of $GF(q)$–rational points, gives

$$N \leq q + 1 + 2g\sqrt{q},$$

and so if our curve $\mathcal{X}$ satisfies property (1.1) one could obtain s–span of $PG(3, q)$, with $s > q + 1$.

Remark 3.4. Here we show the existence of a large number of space curves satisfying all the assumptions of Remark 3.2.

All the possible order sequences of projective curves are "known".

A sequence $\{b_i\}_{0 \leq i \leq 3}$ is the order sequence of a curve if and only if the p–adic criterion, stated for instance in the introduction of [8] is satisfied ; for the proof of the necessity of the p–adic criterion, see [19, Cor. 1.9]; for the existence part when the p–adic criterion is satisfied use a a monomial curve $t \to (t^{b_0}, \ldots, t^{b_N})$ as in the introduction of [8].

The example just given shows that for every prime p, for every integer $e \geq 0$ and for every integer $b_3 > 2$ such that the order sequence $\{b_i\}_{1 \leq i \leq 3}$ satisfies the p–adic criterion, we may find a rational singular curve of degree b_3 with $b_2 = p^e$. For instance we may take $b_3 = b_2 + 1$. If $b_3 \leq 2p^e - 2$, this is an example satisfying all the assumptions of Remark 3.2. Notice that we find singular curves with the same order sequence and degree as the smooth curve D considered in Theorem 3.1.

Remark 3.5. Assume $\mathbb{K} = GF(q)$, $q = 2^h$. Then

$$\mathcal{C}(2^n) = \{P(t) = (t^{m+1}, t^m, t, 1), t \text{ in } GF(q) \cup \{\infty\}\},$$

with $m = 2^n$ is a $(q + 1)$–arc of $PG(3, q)$ if and only if $(n, h) = 1$[10]. Also, $\mathcal{C}(2^n)$ is a twisted cubic if and only if $n = 1$ or $n = h - 1$.

Regarding $\mathcal{C}(2^n)$ as curve (over the algebraic closure of $GF(q)$) we obtain another example for Theorem 3.3. On the other hand, from [3, Lemma 5] the set of tangent lines to $\mathcal{C}(2^n)$ is a $(q + 1)$–span and form a regulus of a hyperbolic quadric [10].

Remark 3.6. All strange curves in $PG(n, \mathbb{K})$, $n \geq 3$ are completely described in [1]. In particular, [1] contains a complete description of all space curves (without any restriction on their singularities) and such that their tangent developable is a quadric cone.

Moreover, the methods of [1] give the corresponding result for a smooth quadric surface. We will write explicitly this description.

Let $\mathcal{H} = PG(1, \mathbb{K}) \times PG(1, \mathbb{K}) \subset PG(3, \mathbb{K})$ be a smooth quadric surface (hyperbolic quadric) and let $\pi : PG(1, \mathbb{K}) \times PG(1, \mathbb{K}) \to PG(1, \mathbb{K})$ be the projection onto the first factor. We will use bihomogeneous coordinates (w_0, w_1, z_0, z_1) on $PG(1, \mathbb{K}) \times PG(1, \mathbb{K})$, i.e. we will use homogeneous coordinates (w_0, w_1) on the first factor and homogeneous coordinates (z_0, z_1) on the second factor.

Every curve $\mathcal{X} \subset PG(1, \mathbb{K}) \times PG(1, \mathbb{K})$ (even not irreducible or unreduced) has a bidegree, say (a, b) (see [6, Chapter III ex. 5.6]), and $\mathcal{X}$ may be described by an equation (unique up to a non–zero multiplicative constant) $f(w_0, w_1, z_0, z_1) = 0$ with f a homogeneous polynomial of degree a in the variables w_0, w_1 and of degree b in the variables z_0, z_1. The curve $\mathcal{X}$ is union of disjoint lines if and only if $ab = 0$. From now on we assume that $\mathcal{X}$ has no multiple component. Every tangent line to a smooth point of $\mathcal{X}$ is contained in the quadric $\mathcal{H}$ as a line of the form $\pi^{-1}(P)$, $P \in PG(1, \mathbb{K})$, if and only if the restriction of π to every irreducible component D of $\mathcal{X}$ is not separable. In particular, $p :=$char$(\mathbb{K}) > 0$. Let f be the bihomogeneous equation of $\mathcal{X}$.

The proof of [2, Sec. 3, Cor. 1], shows that this is the case if and only if every monomial of f contains both w_0 and w_1 with exponents divisible by p. Fix an integer $e \geq 1$ and set $r := p^e$. If all these exponents are divisible by r, then the tangent line to $\mathcal{X}$ at every smooth point Q of $\mathcal{X}$ has order of contact at least r with $\mathcal{X}$ at Q, i.e. (assuming $\mathcal{X}$ irreducible) $\mathcal{X}$ has $b_1 \geq r$. If $\mathcal{X}$ is irreducible and r is the maximal integer with that property, then indeed $r = b_1$.

Example 3.7. Here we assume char$(\mathbb{K})= p > 0$.

Recall that the Hirzebruch surface F_1 has an embedding into $PG(4, K)$ as a minimal degree rational normal scroll [6, V, Cor. 2.19], and that the unique section A of F_1 with self–intersection -1 is sent by this embedding into a plane conic A'.

Let S be the cubic surface with a double line obtained by projecting the smooth rational scroll $F_1 \subset PG(4, \mathbb{K})$ from a general point of the plane spanned by A'. Hence S is ruled by lines and we will describe explicitly all integral space curves with S as tangent developable.

The description here is related to the description given in [1, 2.0] for a similar problem.

Any such curve is the image by the linear projection $F_1 \to S$ of an integral curve $Y \subset F_1$ such that all the lines of the ruling $\pi : F_1 \to PG(1, \mathbb{K})$ are tangent to Y.

We will describe "the equations" of all such curves Y. Fix homogeneous coordinates x_0, x_1 on the base $PG(1, \mathbb{K})$ and take another variable, say w (the coordinate along the fibers taking as origin of the fiber the point of intersection of the fiber with A). Give weight 1 to the variables x_0 and x_1, and weight -1 to the variable w. Every curve of F_1 is described by a unique polynomial (up to a multiplicative constant) $f(x_0, x_1, w)$ such that there is an integer $t \geq 0$ with the property that for every monomial $\lambda x_0^a x_1^b w^c$ of f with $\lambda \neq 0$ we have $a + b - c = t$, i.e. every monomial appearing in f has weight t.

The curve $Y : f(x_0, x_1, w) = 0$ is tangent to all lines of the ruling if and only if c is divisible by p. More precisely, if we want that the the linear projection C of Y into $PG(3, \mathbb{K})$ has $b_2 = p^e$, we just assume that every c appearing in this way is divisible by p^e, and that p^e is the maximal power of p with this property. Let $f : F_1 \to PG(2, \mathbb{K})$ be the blowing–down of A. Every curve Y just described, i.e. every curve Y tangent at each smooth point to the fibers of the ruling, has as image $f(Y)$ a strange plane curve of degree t with the point $f(A)$ as center. Viceversa, every such strange curve is the image of a unique curve Y such that, every fiber of the ruling of F_1 is tangent to Y.

Example 3.8. Here we assume $p :=$char$(\mathbb{K}) > 0$.

As in [1, 2.0], we may extend the previous example and obtain for every integer $a \geq 4$ a singular rational ruled surface $S \subset PG(3, \mathbb{K})$ with deg$(S)= a$, and such that there are perfectly described (in terms of equations) curves $C \subset S$ with S as tangent developable. Such surface S will be a projection of a smooth minimal degree rational normal scroll $S' \subset PG(a + 1, \mathbb{K})$.

As abstract surface, S' is isomorphic to a Hirzebruch surface F_e and all integers e, with $e - a$ even and $0 \leq e \leq a - 2$, may occur in this way. As in [1, 2.0] and the above example, we introduce coordinates x_0, x_1 and w, and give weight 1 to x_0 and x_1 and weight $-c$ to w; we fix an integer $t > 0$ and consider polynomials $f(x_0, x_1, w)$ in which each monomial $\lambda x_0^a x_1^b w^c$ of f with $\lambda \neq 0$ of f has weight t; then $f = 0$ is one of such curves if and only if, for each such monomial, p divides c.

In particular, we obtain large families of singular curves whose tangent developable has degree four.

Remark 3.9. Here we assume char$(\mathbb{K})= 0$.

We will check that the rational normal curve is the only integral curve $\mathcal{X} \subset PG(3, \mathbb{K})$ such that its tangent developable, say S, has degree four.

Fix any such $\mathcal{X}$. Since an integral plane curve of degree four has at most three singular points, the degree of the one–dimensional part of Sing(S) is at most three.

Thus to check that $\mathcal{X}$ is a rational normal curve, it is sufficient to check that S is singular along $\mathcal{X}$. Let $\pi : \mathcal{Y} \to \mathcal{X}$ be the normalization. Set $L := \pi^*(\mathcal{O}_{\mathcal{X}}(1))$. Let $P^1(L)$ the principal bundle of order 1 of L in the sense of [16]. The rank 2 vector bundle $P^1(L)$ fits into an exact sequence

$$0 \to \Omega_{\mathcal{Y}} \otimes L \to P^1(L) \to L \to 0. \tag{3.9}$$

Hence, if g is the genus of $\mathcal{X}$ and d the degree of $\mathcal{X}$, we have deg$(P^1(L))= 2g-2+2d$. Let F be the projectivization of $P^1(L)$. There is a rational map (everywhere defined except over the cusps of $\mathcal{X}$, i.e. over the cuspidal locus of π) $\alpha : F \to PG(3, \mathbb{K})$ whose image is S and sending the fibers of the ruling $f : F \to \mathcal{X}$ into the lines tangent to $\mathcal{X}$; this is the reason for the classical formula deg$(S)= 2g - 2 + 2d - \kappa$, where κ is the number of cusps (counting multiplicities) [5, p. 454].

Furthermore, there is an embedding, say β, of $\mathcal{Y}$ in F, induced by the surjection in (3.9). Since a general line of S is tangent to $\mathcal{X}$, we see that either all fibers of f are tangent to $\beta(\mathcal{Y})$ or f has differential of rank at most 1 at each point of $\beta(\mathcal{Y})$ and hence S is singular along $\mathcal{X} = \alpha(\beta(\mathcal{Y}))$. Alternatively, by the Lefschetz principle we may assume $\mathbb{K} = \mathbb{C}$, the field of complex numbers.

Take a complex variable u on $\mathbb{C}$ and a local parametrization $\alpha : \Delta \to \mathbb{C}^3$ (Δ the unit disk of $\mathbb{C}$). Then, the parametrization of S is given by $x(u,v) : U \to \mathbb{C}^3$ (U an open neighborhood of $0 \in \mathbb{C}^2$) with $x(u,v) = \alpha(u) + \alpha'(u)v$ whose Jacobian determinant vanishes when $v = 0$, i.e. at the points sent onto the curve $\mathcal{X}$; for more details, see e.g. [14, pp. 216-217]. The same proof gives that if the ground field has characteristic zero, no integral space curve has tangent developable of degree two or three.

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