Further radii in topological algebras

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Abstract

We introduce two new radii in general topological algebras. The first one, η, plays a role similar to that of the norm in Banach algebras in the sense that the series $\sum x^n$ converges whenever $\eta(x) < 1$. The second one permits, among others, to give new expressions of the spectral radius ρ and the boundedness radius β in a non-commutative locally m-convex algebra. Finally, we show that, in contrast to the locally convex setting, β need not be dominated by ρ in a topological (even F-) algebra with continuous inversion.

1 Introduction

In a Banach algebra $(A, || \;||)$, the series $\sum x^n := \sum_{n=1}^{\infty} x^n$ converges in A whenever $||x|| < 1$ and its limit is nothing but $-x^o$, x^o being the quasi-inverse of x in A. Actually, this is also true [7] in any normed algebra whose set of quasi-invertible elements is open, i.e. which is a Q-algebra in the sense of I. Kaplanski [6]. In some non-normed topological algebras, the spectral radius ρ still plays the role of the norm in the sense that, if $\rho(x) < 1$, then the series above converges. In some other algebras, it is the boundedness radius β which plays this role. However, there exist topological algebras with elements x such that the series diverges although $\rho(x) < 1$ or $\beta(x) < 1$. In section 2, we introduce a new radius in any topological algebra, called radius of ńig-boundedness and denoted by η, in such a way that the series $\sum x^n$ converges for every x with $\eta(x) < 1$. We show by examples that $\rho \neq \eta$ and $\eta \neq \beta$ in general. However, we obtain that η is exactly the maximum of ρ and β. We finally compare η to some known radii introduced by W. Zelazko [9] and studied

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by H. Arizmendi and K. Jaroz [3]. In section 3, we define a second new radius called radius of d́aw-boundedness and denoted by δ. It is known that in a complete commutative locally m-convex algebra A, the spectral radius of an element x is given by the expression $\rho(x) = \sup\{|\chi(x)|, \chi \in M(A)\}$, where $M(A)$ denotes the set of all continuous characters of A. This expression does not hold anymore in general in the non-commutative case since $M(A)$ may be empty. Here, we introduce the notion of a local character at a point $x \in A$. This is any linear functional on A such that $f(x^n) = f(x)^n$ for every $n \in \mathbb{N}$. Then we define the d́aw-boundedness radius $\delta(x)$ of x as being the quantity $\sup\{|f(x)|, f \in M_x\}$, M_x being the set of all continuous local characters at x. We then show that, in a (not necessarily commutative) locally m-convex algebra A, δ coincides with β. If in addition A is complete, δ coincides with ρ, giving new formulas of both the boundedness radius β and the spectral one ρ in a non-commutative locally m-convex algebra.

On the other hand, it is known that β is dominated by ρ in any locally convex algebra A which has either continuous (quasi-) inversion or all its elements bounded [1]. In section 4, we first exhibit an example showing that, without the local convexity, β is no more dominated by ρ even in a commutative and complete metrizable algebra with continuous inversion. Next, we provide two further examples of F-algebras in which $\rho = \beta$, leading to some open problems.

In all what follows A will stand for an associative algebra over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). For arbitrary x and $y \in A$, denote by xoy the Jordan product $x + y - xy$ of x and y. We will say that x is quasi-invertible in A if some $y \in A$ exists such that $xoy = yox = 0$. Such an element y is called the quasi-inverse of x and is denoted by x^o. The spectrum of an element x of A is the set

$$\mathrm{Sp}(x) := \{\lambda \in \mathbb{K} \setminus \{0\} : \tfrac{x}{\lambda} \text{ is not quasi-invertible in } A\} \cup O$$

O being the empty set or the singleton $\{0\}$ according to whether x is invertible in A or not. The spectral radius of x is then defined as

$$\rho(x) := \sup\{|\lambda|, \lambda \in \mathrm{Sp}(x)\}.$$

If τ is a Hausdorff linear topology on A, we will say that (A, τ) is a topological algebra if the multiplication of A is separately continuous with respect to τ. If in addition τ is locally pseudo-convex (resp. p-convex for some $0 < p \leq 1$) [5], then (A, τ) will be called a locally pseudo-convex (resp. p-convex) algebra. In case $p = 1$, we simply say a locally convex algebra (l.c.a. in short). A bounded absolutely p-convex set (i.e. p-disc) is said to be completing or a p-Banach disc if the linear span $A_B := \cup\{rB, r > 0\}$ of B, endowed with the p-homogeneous gauge $|| \, ||_B$ of B is a p-Banach space, where $||y||_B := \inf\{|\mu|^p; \mu \in \mathbb{K} : y \in \mu B\}$, $y \in A_B$. A locally p-convex algebra will be said to be m-complete if every closed bounded and idempotent p-disc is p-Banach. A net $(x_i)_i$ in a topological algebra A is said to converge advertibly if there is some $x \in A$ so that the nets $(x \circ x_i)_i$ and $(x_i \circ x)_i$ converge to 0. A topological algebra is advertibly sequentially complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges in A whenever it converges advertibly. Finally, we will say that a series $\Sigma a_n x^n$ is Cauchy in A if the sequence $\left(s_n := \Sigma_{k=1}^{k=n} a_k x^k\right)_n$ of its partial sums is.

2 Ńig-boundedness in topological algebras

Let x be an element of a topological algebra (A, τ). As in a locally convex algebra, we will say that x is bounded [1] if there exists some $r > 0$ such that the set $\{(\frac{x}{r})^n, n \in \mathbb{N}\}$ is bounded in (A, τ). This is easily seen to be equivalent to the existence of some $r > 0$ so that the sequence $((\frac{x}{r})^n)_{n\in\mathbb{N}}$ converges to 0. Hence the quantities $\beta(x)$, $\beta'(x)$ and $\beta''(x)$ coincide, where

$$\beta(x) := \inf\{r > 0 : ((\tfrac{x}{r})^n)_n \text{ is bounded}\}$$
$$\beta'(x) := \inf\{r > 0 : ((\tfrac{x}{r})^n)_n \text{ tends to } 0\}$$
$$\beta''(x) := \inf\{r > 0 : ((\tfrac{x}{\lambda})^n)_n \text{ tends to 0 for all } \lambda \in \mathbb{K} \text{ with } |\lambda| > r\ \}.$$

with the convention : $\inf \emptyset = +\infty$. This common value is called the boundedness radius of x with respect to (A, τ). This radius satisfies the following properties:

i) $\beta(x) \geq 0$ and $\beta(\lambda x) = |\lambda|\beta(x)$ for any $\lambda \in \mathbb{K}$, here $0\infty = 0$.

ii) $\beta(x) < +\infty$ if and only if x is bounded.

iii) If $|\lambda| > \beta(x)$, then the sequence $((\frac{x}{\lambda})^n)_n$ converges to 0 and if $|\lambda| < \beta(x)$, the sequence is unbounded.

iv) For every $x \in A$ and $s \in \mathbb{N}$, $\beta(x^s) = \beta(x)^s$. Indeed, if $(\frac{x}{r})^n$ converges to 0, then so does also $(\dfrac{x^s}{r^s})^n$ and then $\beta(x^s) \leq \beta(x)^s$. Conversely, if $\left((\dfrac{x^s}{r^s})^n\right)_n$ converge to 0, then

$$\{(\frac{x}{r})^n, n \in \mathbb{N}\} = \left(\cup_{p=1}^{s-1}(\frac{x}{r})^p\{(\frac{x}{r})^{ms}, m \in \mathbb{N}\}\right) \cup \{(\frac{x}{r})^{ms}, m \in \mathbb{N}\}.$$

Hence $\{(\frac{x}{r})^n, n \in \mathbb{N}\}$ is bounded and then $\beta(x)^s \leq \beta(x^s)$.

v) If A happens to be commutative and its multiplication continuous with respect to τ, then β is submultiplicative, i.e.

$$\beta(xy) \leq \beta(x)\beta(y), \quad \forall x,\ y \in A; \text{ here } 0\infty = 0.$$

Indeed, let x and y be arbitrary in A. The inequality is trivial if $\beta(x)$ or $\beta(y)$ is infinite. Assume then that $r > \beta(x)$ and $s > \beta(y)$, then $((\frac{x}{r})^n)_n$ as well as $((\frac{y}{x})^n)_n$ converge to 0. If V is any 0-neighbourhood. Choose another 0-neighbourhood U such that $UU \subset V$. Then there exists some n_0 such that $(\frac{x}{r})^n \in U$ and $(\frac{y}{s})^n \in U$ whenever $n_0 \leq n$. For such an n, we have $(\frac{xy}{rs})^n = (\frac{x}{r})^n(\frac{y}{s})^n \in UU \subset V$. Hence $\beta(xy) \leq rs$, whereby $\beta(xy) \leq \beta(x)\beta(y)$.
At this point, let us note that if τ is in addition locally convex, then (see [4], Lemma 2.9) β is also subadditive, i.e.

$$\beta(x + y) \leq \beta(x) + \beta(y), \quad \forall x\ y \in A.$$

One can give a further expression of $\beta(x)$ using the gauges of 0-neighborhoods from a local basis at 0. To this aim, let U be a circled absorbent subset of A and P_U its gauge functional. This is the function defined on A by

$$P_U(x) := \inf\{r > 0 : x \in rU\}.$$

Proposition 1 : *Let (A,τ) be a topological algebra, x an element of A and $(U_i)_{i\in I}$ a pseudo-basis of 0-neighborhoods consisting of circled sets. Then*

$$\beta(x) = \sup_{i\in I} \limsup [P_{U_i}(x^n)]^{\frac{1}{n}}.$$

Proof : Set $\alpha(x) := \sup_{i\in I} \limsup_{n} [P_{U_i}(x^n)]^{\frac{1}{n}}$ and let us show that $\beta(x) \le \alpha(x)$ for every $x \in A$. Fix $x \in A$. If $\alpha(x) = +\infty$, then there is nothing to show. Now, suppose that $\alpha(x) < +\infty$. For arbitrary $r > \alpha(x)$, one has $r > \limsup P_{U_i}(x^n)^{\frac{1}{n}}$ for all $i \in I$. Then there is some $n_i \in \mathbb{N}$ such that $r > \sup_{m\ge n_i} P_{U_i}(x^m)^{\frac{1}{m}}$. Let U be an arbitrary 0-neighborhood in (A,τ). There is some finite subset J of I so that $\cap_{j\in J} U_j \subset U$. Fix $n_0 \in \mathbb{N}$ larger that each n_j, $j \in J$. We have

$$\sup_{m\ge n_0} [P_U(x^m)]^{\frac{1}{m}} \le \max_{j\in J} \sup_{m\ge n_j} [P_{U_j}(x^m)]^{\frac{1}{m}} < r,$$

showing that $(\frac{x}{r})^m$ belongs to U for every $m \ge n_0$ and then that $((\frac{x}{r})^n)_n$ tends to 0. Hence $\beta(x) \le r$ and consequently $\beta(x) \le \alpha(x)$. Conversely, fix an arbitrary $x \in A$. If $\beta(x) = +\infty$, the inequality is obvious. Assume then that $\beta(x) < +\infty$ and that $r > \beta(x)$ is arbitrary. Then $(\frac{x}{r})^n$ tends to 0. Hence, for every $i \in I$, there exists $n_i \in \mathbb{N}$ so that $(\frac{x}{r})^m \in U_i$ for every $m \ge n_i$. This shows that $\sup_{m\ge n_i} P_{U_i}(x^m)^{\frac{1}{m}} \le r$. Hence $\limsup P_{U_i}(x^m)^{\frac{1}{m}} \le r$. Since i was arbitrary, $\alpha(x) \le r$ whereby $\alpha(x) \le \beta(x)$.■

In the previous proposition, one can take any pseudo-basis $(U_i)_{i\in I}$ of 0-neighborhoods for an arbitrary linear topology on A having the same bounded sets as τ. Moreover, if each U_i is pseudo-convex and $||\ ||_i$ is its p_i-homogeneous seminorm, $0 < p_i \le 1$, then clearly

$$\beta(x) = \sup_{i\in I} \limsup ||x^n||_i^{\frac{1}{np_i}}.$$

In particular, if each $p_i = p$ for some p, e.g. if (A,τ) is a locally p-convex algebra, then

$$(\beta(x))^p = \sup_{i\in I} \limsup ||x^n||_i^{\frac{1}{n}}.$$

Inspired by the expression $\beta''(x)$, we introduce the

Definition 2 : *Let x be an element of a topological algebra (A,τ). We will say that x is ńig-bounded if there exists some $r > 0$ such that, the series $\sum(\frac{x}{\lambda})^n$ converges in (A,τ) for every $\lambda \in \mathbb{K}$ with $|\lambda| > r$. The radius of ńig-boundedness of x is then defined as*

$$\eta(x) := \inf\{r > 0 : \sum(\frac{x}{\lambda})^n \text{ converges in } (A,\tau),\ \forall \lambda \in \mathbb{K} :\ |\lambda| > r\}$$

again with the convention : $\inf \emptyset = +\infty$.

As for the radius of boundedness, one shows easily :

i. $\eta(x) \ge 0$ and $\eta(\lambda x) = |\lambda|\eta(x)$ for any $\lambda \in \mathbb{K}$ and $x \in A$. Here also $0\infty = 0$.

ii. $\eta(x) < +\infty$ if and only if x is ńig-bounded.
iii. If $|\lambda| > \eta(x)$, then the series $\sum(\frac{x}{\lambda})^n$ converges in (A,τ).

Notice however that, in contrast to β, $\eta(x)$ need not coincide with

$$\eta'(x) := \inf\{r > 0 : \textstyle\sum(\frac{x}{r})^n \text{ converges in } (A,\tau)\}.$$

To give an instance where these are different, take the unital subalgebra A of the field $\mathbb{C}(X)$ of rational functions in one indeterminate X generated by X and $f := \dfrac{1}{1-X}$. Endow A with the topology τ of uniform convergence on the compacta of $[0,1[$. Then (A,τ) is a locally m-convex algebra and the series $\sum X^n$ converges in A, while, for any $|\alpha| > 1$, $\sum(\frac{X}{\alpha})^n$ does not. The sum of the latter series in $C[0,1[$, being the rational function $\dfrac{\alpha}{\alpha - X} - 1$, does not belong to A. This shows that $\eta(X) \neq \eta'(X)$.

Nevertheless, the equality $\eta = \eta'$ holds in a large class of topological algebras containing in particular the complete locally pseudo-convex ones. Recall that a topological (vector space or) algebra (A,τ) is said to be fundamental [2] if every sequence $(x_n)_n$ is Cauchy, whenever there exists some $r > 1$ such that $r^n(x_n - x_{n-1})$ tends to 0. Here, we introduce a more general class of algebras.

Definition 3 : *A topological algebra (A,τ) is said to be Σ-fundamental if the series $\sum(\frac{x}{\alpha})^k$ is Cauchy for every $x \in A$ and every $\alpha \in \mathbb{K}$ with $(x^n)_n$ bounded and $|\alpha| > 1$.*

We also introduce the

Definition 4 : *A topological algebra (A,τ) is said to be pointwise pseudo-m-complete if every $x \in A$ such that $(x^n)_n$ is bounded is contained in some idempotent bounded p-Banach disc $B \subset A$ with $0 < p \leq 1$. If p can be taken the same for all such x, (A,τ) is then called pointwise p-m-complete, and if $p = 1$, we simply drop it.*

It is easily seen that every locally pseudo-convex algebra is fundamental and that every fundamental one is Σ-fundamental. Furthermore, every m-complete locally p-convex algebra is pointwise p-m-complete and every pointwise pseudo-m-complete algebra is Σ-fundamental. On the other hand, if $((A_i,\tau_i))_{i\in I}$ is an inductive system of locally p_i-Banach algebras, $i \in I$, and $A := \cup_{i\in I} A_i$ is its inductive limit, then A, endowed with the inductive limit linear topology of $((A_i,\tau_i))_i$, is a pointwise pseudo-m-complete algebra.

Proposition 5: *In each of the following cases $\eta = \eta'$ on A:*
1. (A,τ) is a pointwise pseudo-m-complete topological algebra.
2. (A,τ) is an advertibly sequentially complete Σ-fundamental topological algebra.

Proof : It is clear that $\eta' \leq \eta$ on A. Moreover if $\eta'(x) = +\infty$, then also $\eta(x) = +\infty$. Now, let $x \in A$ and $\alpha \in \mathbb{K}$ be such that $\eta'(x) \leq |\alpha|$. In the case 1., consider s so that $\eta'(x) < s < |\alpha|$ and $\sum(\frac{x}{s})^n$ converges in (A,τ). By hypothesis, there exist $0 < p \leq 1$

and an idempotent bounded p-Banach disc B containing $\frac{x}{s}$. Then we have

$$\begin{aligned} \|\sum_{k=n}^{m}(\frac{x}{\alpha})^k\|_B &= \|\sum_{k=n}^{m}(\frac{s}{\alpha})^k(\frac{x}{s})^k\|_B \\ &\leq \sum_{k=n}^{m}((\frac{s}{\alpha})^p)^k\|\frac{x}{s}\|_B^k \\ &\leq \sum_{k=n}^{m}(\frac{s}{\alpha})^{pk} \to 0. \end{aligned}$$

showing that $\left(\sum_{k=1}^{n}(\frac{x}{\alpha})^k\right)_n$ is Cauchy in the p-Banach algebra $(A_B, \|\ \|_B)$. Therefore, it converges in A_B and then also in A. This gives $\eta(x) \leq \eta'(x)$ since $|\alpha| > \eta'(x)$ is arbitrary.
In the case 2., the sequence $((\frac{x}{\alpha})^n)_n$ tends to 0 and then is bounded. By our assumption, for $|\lambda| > 1$, the series $\sum(\frac{x}{\lambda\alpha})^k$ is Cauchy. Since $\frac{x}{\lambda\alpha} \circ \left(-\sum_{k=1}^{n}(\frac{x}{\lambda\alpha})^k\right) = (\frac{x}{\lambda\alpha})^{n+1}$ tends to 0 and (A, τ) is advertibly sequentially complete, the series converges in (A, τ). Hence $\eta(x) \leq |\alpha|$ and again $\eta(x) \leq \eta'(x)$ since $|\alpha| > \eta'(x)$ was arbitrary. ■

Now, whenever the series $\sum(\frac{x}{\lambda})^n$ converges, the sequence $((\frac{x}{\lambda})^n)_n$ obviously converges to 0. Hence $\beta(x) \leq \eta(x)$ for all $x \in A$. Moreover, if $\lambda \in \mathbb{K}$ and $x \in A$ are so that $|\lambda| > \eta(x)$, then the sum $\sum(\frac{x}{\lambda})^n$ enjoys: $\frac{x}{\lambda} \circ \left(-\sum(\frac{x}{\lambda})^n\right) = \left(-\sum(\frac{x}{\lambda})^n\right) \circ \frac{x}{\lambda} = 0$ which shows that $\frac{x}{\lambda}$ is quasi-invertible and $(\frac{x}{\lambda})^o = -\sum(\frac{x}{\lambda})^n$. This gives $\rho(x) \leq \eta(x)$ for every $x \in A$. The three radii may fail to coincide with each other as show the

Examples :

1. Let A be the complex algebra $\mathbb{C}[X]$ of polynomials in one indeterminate X endowed with the topology of uniform convergence on the unit interval $[0, 1]$. Then $\beta(X) = 1$, while the series $\sum(\frac{X}{\lambda})^n$ does not converge for any complex number λ. Hence $\beta(X) < \eta(X)$.

2. Let $A = \mathbb{C}(X)$ be the field of rational functions of the indeterminate X over the complex field $\mathbb{C}$. Endow A with its strongest locally convex linear topology τ^*. Then (A, τ) is a complete locally convex Q-algebra. For $x = X$, the series $\sum(\frac{x}{\lambda})^n$ does not converge for any complex number λ, since otherwise, A will contain a bounded subset of infinite dimension which is not true. Hence $\eta(X) = +\infty$. However, the spectrum of X is empty and then $\rho(X) = 0$. Whence $\rho \neq \eta$.

3. In order to get an example in which $\rho \neq \eta$ and $\beta \neq \eta$ simultaneously, take the product of $\mathbb{C}[X]$ and $\mathbb{C}(X)$ from examples 1 and 2 above with the pointwise operations and the product topology. For instance, $\beta\left((X, 0)\right) = 1$ but $\eta\left((X, 0)\right) = +\infty$ and $\rho\left((0, X)\right) = 0$ while $\eta\left((0, X)\right) = +\infty$.

However, we have:

Proposition 6 : *Let (A, τ) be a topological algebra. Then the following equality holds:*

$$\eta(x) = \max\left(\beta(x), \rho(x)\right), \forall x \in A.$$

Proof : We just have to show that $\eta(x) \leq \max\left(\beta(x), \rho(x)\right)$, for every $x \in A$. Fix $x \in A$. If $\max\left(\beta(x), \rho(x)\right) = +\infty$, the inequality is obvious. Assume now that $\max\left(\beta(x), \rho(x)\right) < +\infty$ and let $\lambda \in \mathbb{K}$ satisfy $|\lambda| > \max\left(\beta(x), \rho(x)\right)$. Then $\dfrac{x}{\lambda}$ is quasi-invertible and $\left(\dfrac{x}{\lambda}\right)^n$ converges to 0. Then from

$$\frac{x}{\lambda} \circ \left(-\sum_{k=1}^{n} (\frac{x}{\lambda})^k \right) = (\frac{x}{\lambda})^{n+1},$$

follows

$$\sum_{k=1}^{n} (\frac{x}{\lambda})^k = -\left((\frac{x}{\lambda})^o \circ (\frac{x}{\lambda})^{n+1} \right).$$

Since $(\frac{x}{\lambda})^{n+1}$ tends to 0 as n tends to infinity, the series $\sum (\dfrac{x}{\lambda})^n$ converges in A and its limit is nothing but $-(\dfrac{x}{\lambda})^o$. Whence $\eta(x) \leq \max\left(\beta(x), \rho(x)\right)$ for every $x \in A$. ■

By the proposition above if $\rho(x) \leq \beta(x)$ (resp. $\beta(x) \leq \rho(x)$) for every $x \in A$, then $\eta = \beta$ (resp. $\eta = \rho$). In [4], it is shown that, in a unital locally convex algebra,

$\rho \leq \beta$ if and only if ($\forall x \in A,\ \beta(x) < 1 \Longrightarrow \sum_{n=0}^{\infty} x^n$ converges)

and

$\beta \leq \rho$ if and only if ($\forall x \in A,\ \rho(x) < 1 \Longrightarrow \sum_{n=0}^{\infty} x^n$ converges).

The following proposition yields further necessary and sufficient conditions for the inequality $\rho \leq \beta$ (resp. $\beta \leq \rho$) to hold in the general setting.

Proposition 7 : *Let (A, τ) be a topological algebra. The conditions 1) to 4) are equivalent and so are also 1') to 4').*

1) $\eta = \beta$ (i.e. $\rho \leq \beta$).

2) The series $\sum x^n$ converges whenever $\beta(x) < 1$.

3) The series $\sum (\dfrac{x}{\alpha})^n$ converges whenever $\beta(x) \leq 1$ and $|\alpha| > 1$.

4) ρ is bounded on idempotent bounded subsets of A.

1') $\eta = \rho$ (i.e. $\beta \leq \rho$).

2') The series $\sum x^n$ converges whenever $\rho(x) < 1$.

3') The series $\sum (\dfrac{x}{\alpha})^n$ converges whenever $\rho(x) \leq 1$ and $|\alpha| > 1$.

4') The set $\{\sum_{k=1}^{n} (\frac{x}{\alpha})^k, n \in \mathbb{N}\}$ is bounded whenever $\rho(x) \leq 1$ and $|\alpha| > 1$.

Proof : Under the assumption 1), if $\beta(x) < 1$, then also $\rho(x) < 1$ and either x is quasi-invertible and $(x^n)_n$ converges to 0. But $\sum_{k=1}^{n} x^k = -x^o - x^{n+1} + x^o x^{n+1}$

converges to $-x^o$ and 2) follows. 3) derives obviously from 2). As to 3) $\Longrightarrow$ 4), let B be an idempotent bounded subset of A. Then, $\beta(x) \leq 1$ for every $x \in B$. By 3), for arbitrary α with $|\alpha| > 1$, the series $\sum(\dfrac{x}{\alpha})^n$ converges. Hence $\frac{x}{\alpha}$ is quasi-invertible and then $\rho(x) \leq 1$. Whence ρ is bounded on B. To show 4) $\Longrightarrow$ 1), let $x \in A$ be given. If $\beta(x) = +\infty$, nothing is to be proved. Now, if $\beta(x) < r$, then ρ is bounded on the idempotent bounded set $\{(\frac{x}{r})^n, n \in \mathbb{N}\}$. Therefore, $\rho((\frac{x}{r})^n) \leq c$ for some $c > 0$ and then $\rho(x)^n \leq cr^n$. Since n is arbitrary, $\rho(x) \leq r$, whereby $\rho(x) \leq \beta(x)$. Now, a similar proof shows that 1′) $\Longrightarrow$ 2′) $\Longrightarrow$ 3′), while 4′) derives obviously from 3′). As to 4′) $\Longrightarrow$ 1′), let $x \in A$ be given. If $\rho(x) = +\infty$, nothing is to be proved. Now, assume that $\rho(x) < r$ but $\beta(x) > r$. Then, for $\alpha \in \mathbb{K}$ such that $\beta(\frac{x}{r}) > |\alpha| > 1$, the set $\{\sum_{k=1}^n (\frac{x}{\alpha r})^k, n \in \mathbb{N}\}$ is bounded and $((\frac{x}{\alpha r})^k)_k$ is unbounded. Let U and V be circled 0-neighborhoods such that, for every $k \in \mathbb{N}$, $(\frac{x}{\alpha r})^{m_k} \notin kU$ for some $m_k \geq k$ and $V + V \subset U$. Since $\{\sum_{k=1}^n (\frac{x}{\alpha r})^k, n \in \mathbb{N}\}$ is bounded, there exists $c > 0$ with $\displaystyle\sum_{k=1}^n (\frac{x}{\alpha r})^k \in cV$, $n \in \mathbb{N}$. In particular, for $n \geq c$,

$$(\frac{x}{\alpha r})^{m_n} = \sum_{k=1}^{m_n} (\frac{x}{\alpha r})^k - \sum_{k=1}^{m_n - 1} (\frac{x}{\alpha r})^k \in cV + cV \subset cU \subset nU.$$

This is a contradiction. ∎

The boundedness of the set $\{\sum_{k=1}^n (\frac{x}{\alpha})^k, n \in \mathbb{N}\}$ whenever $\beta(x) \leq 1$ and $|\alpha| > 1$ need not be equivalent to 1) - 4) even in a normed algebra. Indeed, in the algebra $\mathbb{C}[X]$ with the norm $||P|| = \sup_{t \in [0,1]} |P(t)|$, the set $\{\sum_{k=1}^n (\frac{P}{\alpha})^k, n \in \mathbb{N}\}$ is bounded whenever $\beta(P) \leq 1$ and $|\alpha| > 1$. However, such a series does not converge for any non-constant P.

The following proposition yields general instances where $\rho \leq \beta$ so that $\eta = \beta$.

Proposition 8 : *Let (A, τ) be a topological algebra. Then $\rho \leq \beta$ holds whenever (A, τ) is either pointwise pseudo-m-complete or advertibly sequentially complete and Σ-fundamental.*

Proof : It is clear that $\rho(x) \leq \beta(x)$ whenever $\beta(x) = +\infty$. Assume next that $x \in A$ and $\alpha \in \mathbb{K}$ are such that $\beta(x) < |\alpha|$. In the first situation, take $\beta(x) < s < |\alpha|$ and consider an idempotent bounded p-Banach disc B containing $\frac{x}{s}$. As in the proof of Proposition 5, the series $\sum(\frac{x}{\alpha})^k$ converges in $(A_B, ||\ ||_B)$ and then also in (A, τ). In the second situation, the sequence $\displaystyle\left(\sum_{k=1}^n (\frac{x}{\alpha})^k\right)_n$ is either Cauchy and advertibly convergent, then it converges. In both cases, we get $\eta(x) \leq \beta(x)$. Hence also $\rho(x) \leq \beta(x)$. ∎

We end this section with a comparison of η to some other radii. For an element x of a topological algebra (A, τ), several radii were introduced in [9] among which

$$r_6(x) := \inf\{r > 0 : \ \exists (a_n)_n \subset \mathbb{K} \text{ with } R((a_n)_n) = r \text{ and } \sum_{n \geq 1} a_n x^n \text{ converges in } A\}$$

$$r_7(x) := \inf\{r > 0 : \forall (a_n)_n \subset \mathbb{K} \text{ with } R((a_n)_n) = r, \sum_{n\geq 1} a_n x^n \text{ converges in } A\}.$$

Here $R((a_n)_n)$ designates the radius of convergence of the series $\sum a_n z^n$. Obviously, one has $\beta \leq r_6 \leq r_7$ in general. Moreover, we get :

Proposition 9 : *Let (A, τ) be a topological algebra. Then $r_6 \leq \eta$ on A. Moreover, $\eta \leq r_7$ whenever (A, τ) is either pointwise pseudo-m-complete or advertibly sequentially complete and Σ-fundamental.*

Proof : Let $x \in A$ be given. If $\eta(x) = +\infty$, then obviously $r_6(x) \leq \eta(x)$. Otherwise, let $r \geq \eta(x)$. Then the series $\sum (\dfrac{x}{r})^n$ converges and then, by the very definition of r_6, $r_6(x) \leq r$. This gives $r_6 \leq \eta$ on the whole of A. As to r_7, let $x \in A$ be arbitrary. If $r_7(x) = +\infty$, then there is nothing to prove. Assume then that $r_7(x)$ if finite and that $|\alpha| > r_7(x)$. Then there is some s such that $r_7(x) < s < |\alpha|$ and, for every series $\sum a_n z^n$ whose radius of convergence is s, $\sum a_n x^n$ converges in A. In particular, $\sum (\dfrac{x}{s})^n$ converges. Again, as in the proof of Proposition 5, the series $\sum (\frac{x}{\alpha})^n$ converges in (A, τ). Hence $|\alpha| \geq \eta(x)$. Since $|\alpha| > r_7(x)$ is arbitrary, $\eta(x) \leq r_7(x)$. ∎

In the example after Definition 2, the series $\sum X^n$ converges to $f - 1$ which belongs to A. Then $r_6(X) \leq 1$. However, $\eta(X) = +\infty$ since $\sum \left(\dfrac{X}{\alpha}\right)^n$ does not converge for any $\alpha \neq 1$. Hence $\eta \neq r_6$ in general. Now, if B is the unital subalgebra of $\mathbb{C}(X)$ generated by X and the functions $f_\alpha = \dfrac{1}{\alpha - X}$, $|\alpha| > 1$, then $\eta(X) = 1$, for the series $\sum (\dfrac{X}{\alpha})^n$ converges in B to $f_\alpha - X$. But, for $|\alpha| > 1$, the series $\sum \dfrac{1}{n}\left(\dfrac{X}{\alpha}\right)^n$, having $|\alpha|$ as radius of convergence, does not converge in B, for $x \mapsto -\mathrm{Log}(1 - \frac{x}{\alpha})$ is not a rational function. This proves that $r_7(X) = +\infty$ and then that $r_7 \neq \eta$ in general.

3 Ḋaw-boundedness radius

In this section we introduce the d́aw-boundedness radius δ and use it to deduce new expressions of β and ρ in non-commutative locally m-convex algebras.

Let then (A, τ) be a topological algebra and A' (resp. A^+, A^*) its continuous (resp. bounded, algebraic) dual. Let x be an element of A. A non-zero functional $f \in A^*$ is said to be a local character at x if it satisfies $f(x^n) = f(x)^n$ for every $n \in \mathbb{N}$. The set of all such functionals is denoted by M_x^*. Similarly, M_x and M_x^+ denote the sets of all local characters at x which belong to A' and A^+, respectively.

Definition 10 : *The d́aw-boundedness radius of x with respect to (A, τ) is the quantity*

$$\delta(x) := \sup\{|f(x)|; f \in M_x\}.$$

Of course, one can also consider the bounded and the algebraic d́aw-boundedness radii as being respectively $\delta^+(x) := \sup\{|f(x)|, f \in M_x^+\}$ and $\delta^*(x) := \sup\{|f(x)|,$

$f \in M_x^*\}$.

We gather the properties of the d́aw-boundedness radii in the following:

Proposition 11 : *Let (A, τ) be a topological algebra and $x \in A$. Then*
1. $\delta(x) \leq \delta^+(x) \leq \delta^*(x)$.
2. For every subalgebra B of A containing x, $\delta_A^*(x) = \delta_B^*(x)$. Moreover, if τ is locally convex, then also $\delta_A(x) = \delta_B(x)$.
3. $\delta^+(x) \leq \beta(x)$.
4. $\delta^*(x) < +\infty$ if and only if x is algebraic.
5. $\eta(x) \leq \delta^*(x)$.
6. If (A, τ) is a locally m-convex algebra, then $\beta(x) = \delta(x)$.
7. If (A, τ) is either a sequentially advertibly complete or a pointwise m-complete locally m-convex algebra, then $\rho(x) = \delta(x) = \delta^+(x)$.

Proof : 1. is obvious.
2. The first equality derives from the fact that every $f \in M_x^*(B)$ extends to A by linearity. The second one is due to Hahn-Banach theorem, for every element of $M_x(B)$ extends to an element of $M_x(A)$.
For 3., let $x \in A$ be given. If $\beta(x) = +\infty$, there is nothing to show. Otherwise assume that $r > \beta(x)$ and $f \in M_x^+$. Then f is bounded on the bounded set $\{(\frac{x}{r})^n, n \in \mathbb{N}\}$ by some $M > 0$. This gives $|f(x)|^n \leq Mr^n$ for every $n \in \mathbb{N}$, which leads to $|f(x)| \leq r$ and therefore to $\delta^+(x) \leq \beta(x)$.
4. If x is algebraic and P is a polynomial with $P(x) = 0$, then for every $f \in M_x^*$, $f(P(x)) = P(f(x)) = 0$. Hence $f(x)$ is a zero of P. But P has only finitely many zeros. Whence $\delta^*(x)$ is finite. Now, if x is not algebraic, consider for every $\alpha \in \mathbb{K}$ a linear functional f_α on A assigning to x^n the value α^n, $n \in \mathbb{N}$. The functional f_α belongs to M_x^* and then $|\alpha| \leq \delta^*(x)$. Since α is arbitrary, $\delta^*(x)$ is infinite.
5. Let $\lambda \in \mathrm{Sp}(x)$ be given. Since $P(\mathrm{Sp}(x)) \subset \mathrm{Sp}(P(x))$ for every polynomial P, the assignment $x^n \to \lambda^n$ extends to a well-defined character χ on the subalgebra $\mathbb{K}[x]$ of A generated by x. Consider now any linear functional χ_λ on A whose restriction to $\mathbb{K}[x]$ coincides with χ. Then χ_λ belongs to M_x^*. Since λ is arbitrary in $\mathrm{Sp}(x)$ and $\chi_\lambda(x) = \lambda$, we obtain $\rho(x) \leq \delta^*(x)$. In order to show that $\eta(x) \leq \delta^*(x)$, it suffices to establish $\beta(x) \leq \delta^*(x)$. If x fails to be algebraic, by 3., $\delta^*(x) = +\infty$ and then $\eta(x) \leq \delta^*(x)$. Now, if x is algebraic, then $\mathbb{K}[x]$ is a (finite dimensional) Banach algebra and $\beta(x)$ is nothing but $\rho_{\mathbb{K}[x]}(x)$. According to 2. and the fact that $\rho(x) \leq \delta^*(x)$, we have $\beta(x) = \beta_{\mathbb{K}[x]}(x) = \rho_{\mathbb{K}[x]}(x) \leq \delta^*_{\mathbb{K}[x]}(x) = \delta^*(x)$.
6. Assume that (A, τ) is a locally m-convex algebra. We just have to show that $\beta(x) \leq \delta(x)$. For this purpose, let us first notice that, if A is commutative, then $\beta(x) = \sup\{|\chi(x)|, \chi \in M(A)\}$, for this is true in the completion $\hat{A}$ of A, $M(A) = M(\hat{A})$ and $\beta_A(x) = \beta_{\hat{A}}(x)$. Now, let us return back to the general case. Consider again the subalgebra $E := \mathbb{K}[x]$. This is a commutative algebra and then $\beta(x) = \sup\{|\chi(x)|, \chi \in M(E)\}$. But, by Hahn-Banach theorem, every $\chi \in M(E)$ can be extended to some $f \in M_x$. This shows that $\beta(x) \leq \delta(x)$.
7. is a consequence of 3. and 6. together with Proposition 8 since a locally m-convex algebra is Σ-fundamental. ∎

In spite of Proposition 11, δ may fail to be dominated by ρ although the algebra is a commutative complete locally convex Q-algebra. This occurs, for instance, in the field $\mathbb{C}(X)$ with its strongest locally convex topology. Actually $\rho(X) = 0$ while $\delta(X) = +\infty$. Indeed, for every $\alpha \in \mathbb{C}$, let $f_\alpha : \mathbb{C}[X] \to \mathbb{C}$ be the linear functional assigning to X^n the scalar α^n, $n \in \mathbb{N}$. This is a continuous character on $\mathbb{C}[X]$ with the induced topology. Then f_α extends to an element in M_X. This yields $\delta(X) \geq |\alpha|$ for every α. Whence $\delta(X) = +\infty$.

The foregoing example shows that, for $f \in M_x$, $f(x)$ need not belong to the spectrum of x. Hence f need not be a character on A.

4 Spectral and boundedness radii in F-algebras

It is known that in a locally convex algebra with continuous inversion, the boundedness radius β is dominated by the spectral one ρ. We start this section by showing by an example that the local convexity cannot be released, although the algebra is metrizable and complete (i.e. an F-algebra).

Example Let F be the algebra of all Lebesgue measurable functions on $X := [0, 1]$ with values in $\mathbb{C}$. Endow E with the topology of convergence in measure and consider the quotient algebra $E := F/\mathcal{N}$, where $\mathcal{N}$ is the ideal of F consisting of all functions vanishing almost everywhere. Then E is a commutative unital F-algebra with continuous inversion [8]. Recall that a basis for the neighborhoods of the origin is given by the sets of the form

$$N(k, \epsilon) := \{f \in E : \mu(\{x \in [0, 1] : |f(x)| \geq k\}) < \epsilon\}$$

k and ϵ being arbitrary positive numbers.

Proposition 12 : *In the algebra E, the boundedness radius is nothing but the essential norm*

$$||f|| := \inf\{r > 0 : \mu(\{x \in X : |f(x)| \geq r\}) = 0\},$$

while the spectral radius is given by

$$\rho(f) = \sup\{|\lambda| : \mu(\{x \in X : f(x) = \lambda\}) > 0\}.$$

Proof : Let $f \in E$ be given. If $||f|| = +\infty$, then $\beta(f) \leq ||f||$. Now, if $||f|| < r < +\infty$, then $\mu(\{x \in X : |f(x)| \geq r\}) = 0$. Hence, $\mu(\{x \in X : (\frac{|f(x)|}{r})^n \geq 1\}) = 0$ for every $n \in \mathbb{N}$. Therefore, $\mu(\{x \in X : k(\frac{|f(x)|}{r})^n \geq k\}) = 0 < \epsilon$, for all $n \in \mathbb{N}$ and arbitrary k and ϵ. This means that $k\{(\frac{f}{r})^n, n \in \mathbb{N}\} \subset N(k, \epsilon)$. Hence $((\frac{f}{r})^n)_n$ is bounded and $r \geq \beta(f)$, whereby $\beta(f) \leq ||f||$. Since f is arbitrary, we get $\beta \leq ||\ ||$ on E. Conversely, let f again be arbitrary in E. If $\beta(f) = +\infty$, then $||f|| \leq \beta(f)$. Now, if $\beta(f) < r < +\infty$, then $(\frac{f}{r})^n$ tends to 0 as n tends to infinity. Hence, for every $k > 0$, there exists $n_k \in \mathbb{N}$ so that $(\frac{f}{r})^n \in N(1, \frac{1}{k})$, whenever $n \geq n_k$. This means that $\mu(\{x \in X : (\frac{|f(x)|}{r})^n \geq 1\}) < \frac{1}{k}$. But $\{x \in X : (\frac{|f(x)|}{r})^n \geq 1\} = \{x \in X : |f(x)| \geq r\}$. Hence $\mu(\{x \in X : |f(x)| \geq r\} \leq \frac{1}{k}$, for all k, whereby $r \geq ||f||$ and the equality $\beta =$

$||\ ||$ follows. Concerning the spectral radius, let $f \in E$ be given. For f^{-1} to belong to E, it is necessary and sufficient that $z(f) := \{x \in X : f(x) = 0\}$ be of measure 0. Hence the spectrum of f is given by $\mathrm{Sp}(f) := \{\lambda \in \mathbb{C} : \mu(\{x \in X : f(x) = \lambda\}) > 0\}$. This shows that $\rho(f) := \sup\{|\lambda| : \mu(\{x \in X : f(x) = \lambda\} > 0\}$. ■

If f is the function defined by $f(x) = \frac{1}{x}$, then $\rho(f) = 0$ while $\beta(f) = +\infty$. This shows that β is not dominated by ρ on the algebra E above.

In the locally convex setting the pointwise m-completeness implies $\rho \leq \beta$ (see Proposition 8). When one deals with F-algebras, one disposes instead of local convexity of either a stronger completeness condition and the metrizability. However, we do not know whether or not $\rho \leq \beta$ on any arbitrary F-algebra. On the other hand, on any locally convex algebra whose elements are all bounded, $\beta \leq \rho$ [1]. We also ignore whether or not this still holds on an arbitrary F-algebra whose elements are all bounded. In the following we provide two examples of such F-algebras, but in which ρ even coincides with β.

Examples : 1. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\varphi(t) := \dfrac{t}{1+t}$ and μ the Lebesgue measure on $\mathbb{R}$. Consider the set E of all continuous functions $f : \mathbb{R} \to \mathbb{K}$ such that $\lim\limits_{|x|\to\infty} f(x) = 0$ and $||f||_\varphi := \displaystyle\int_{\mathbb{R}} \frac{|f|}{1+|f|} d\mu < +\infty$. This is a vector space and, endowed with the pointwise multiplication, it becomes an algebra over $\mathbb{K}$. Indeed, if $f, g \in E$, then $\lim\limits_{|x|\to\infty} (fg)(x) = 0$ and $||f\dfrac{g}{||g||_u}||_\varphi \leq ||f|| < +\infty$, where $||\ ||_u$ denotes the uniform norm. Hence $f\dfrac{g}{||g||_u}$ belongs to E and then so does also fg. Now equip E with the F-norm

$$||f|| := \max\left(||f||_u, ||f||_\varphi\right).$$

Since $||f\dfrac{g}{||g||_u}|| \leq ||f||$ for all $f, g \in E$, the multiplication of E is separately continuous and $(E, ||\ ||)$ is a topological algebra. Actually E is even an F-algebra. Indeed, if $(f_n)_n$ is a Cauchy sequence in $(E, ||\ ||)$, then so is it also either in $C_0(\mathbb{R})$ with the uniform norm $||\ ||_u$ and in the Orlicz space $(L_\varphi(\mathbb{R}), ||\ ||_\varphi)$. Hence $(f_n)_n$ converges uniformly to some $f \in C_0(\mathbb{R})$ and $(\varphi \circ |f_n|)_n$ converges in $L^1(\mathbb{R})$ to some $h \in L^1(\mathbb{R})$. But then $(\varphi \circ |f_{n_k}|)_k$ converges almost everywhere to h for some subsequence $(f_{n_k})_k$ of $(f_n)_n$. By continuity of φ, $h = \varphi \circ |f|$ almost everywhere. Whence $f \in E$ and $(f_n)_n$ converges to f in $(E, ||\ ||)$.
We claim that $\rho = \beta = ||\ ||_u$. Indeed, if $f \in E$ is quasi-invertible in $C(\mathbb{R})$, its quasi-inverse is given by $f^o = \dfrac{f}{f-1}$. Since f is continuous and vanishes at infinity, $||f||_u < 1$. Therefore, $|1 - f(t)| > \delta$ for some $\delta > 0$ and then f^o belongs to E. Hence f is quasi-invertible in E if and only if $f(x) \neq 1$ for every $x \in \mathbb{R}$. Whereby $\rho = ||\ ||_u$. As to β, notice that for every $r > ||f||_u$ and $\epsilon > 0$, owing to the Lebesgue's dominated convergence theorem, there exists some n_0 such that $\displaystyle\int_{|t|>n_0} \frac{\frac{|f|}{r}}{1+\frac{|f|}{r}} d\mu < \frac{\epsilon}{2}$.

Then, for every $m \in \mathbb{N}$,

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{|(\frac{f}{r})^m|}{1+|(\frac{f}{r})^m|} d\mu &= \int_{[-n_0,n_0]} \frac{|(\frac{f}{r})^m|}{1+|(\frac{f}{r})^m|} d\mu + \int_{|t|>n_0} \frac{|(\frac{f}{r})^m|}{1+|(\frac{f}{r})^m|} d\mu \\
&\leq \int_{[-n_0,n_0]} |(\frac{f}{r})^m| d\mu + \int_{|t|>n_0} \frac{|\frac{f}{r}|}{1+|\frac{f}{r}|} d\mu \\
&\leq 2n_0 \left(\frac{||f||_u}{r}\right)^m + \frac{\epsilon}{2} < \epsilon \text{ for } m \text{ large enough.}
\end{aligned}
$$

Hence $\beta(f) \leq r$ whereby $\beta \leq || \, ||_u$. Conversely, if $r > \beta(f)$ and $|f(x_0)| > r$ for some $x_0 \in \mathbb{R}$, then $\left((\frac{f}{r})^m\right)_m$ cannot be bounded in $(C_0(\mathbb{R}), || \, ||_u)$ and then also in E. Whence $|| \, ||_u \leq \beta$.

2. For every $0 < p < 1$, consider the p-Banach space $(\ell_p, || \, ||_p)$, where

$$\ell_p := \{x := (x_n)_{n\in\mathbb{N}} \subset \mathbb{K} : ||x||_p := \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}.$$

Let ℓ_0 be the intersection of all such ℓ_p spaces. Endow ℓ_0 with the topology τ given by the family $(|| \, ||_{\frac{1}{p}})_{p\in\mathbb{N}}$ of pseudo-seminorms. Then (ℓ_0, τ) is an F-space ([5], p. 121). Moreover $||xy||_{\frac{1}{p}} \leq ||x||_{\frac{1}{p}} ||y||_{\frac{1}{p}}$ for every $x, y \in \ell_0$ and $p \in \mathbb{N}$. Then (ℓ_0, τ) is a locally m-pseudo-convex algebra and in particular an F-algebra. Now, if $x \in \ell_0$, $||x||_u := \sup\{|x_n|, n \in \mathbb{N}\}$ and $r > ||x||_u$, then $|\frac{x_n}{r}|^{\frac{m}{p}} \leq |\frac{x_n}{r}|^{\frac{1}{p}}$ for every $m \in \mathbb{N}$. Then $||(\frac{x}{r})^m||_{\frac{1}{p}} \leq ||\frac{x}{r}||_{\frac{1}{p}} < +\infty$ and $((\frac{x}{r})^m)_m$ is then bounded in ℓ_0. This yields $\beta(x) \leq ||x||_u$ for all $x \in \ell_0$. Conversely, if $\beta(x) < r$, then $((\frac{x}{r})^m)_m$ converges to 0. In particular, for every $n \in \mathbb{N}$, $(\frac{x_n}{r})^{\frac{m}{p}}$ tends to 0. This holds only if $|x_n| < r$. Consequently, $||x||_u \leq r$ and then $|| \, ||_u \leq \beta$. To see that $\rho = || \, ||_u$, just notice that $x \in \ell_0$ is quasi-invertible if and only if $x_m \neq 1$ for every $m \in \mathbb{N}$.

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