Realcompactness and Banach-Stone theorems

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Abstract

For realcompact spaces X and Y we give a complete description of the linear biseparating maps between spaces of vector-valued continuous functions on X and Y in two cases: the spaces of all continuous functions and the spaces of *bounded* continuous functions. With similar techniques we also describe the linear biseparating maps defined between some other families of spaces, in particular spaces of vector-valued uniformly continuous bounded functions.

1 Definitions and notation

All results given in this paper are valid both in the real and complex contexts. $\mathbb{K}$ will denote the scalar field, that is, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Throughout the paper E and F will be $\mathbb{K}$-normed spaces.

For a completely regular space X, $C(X, E)$ and $C_b(X, E)$ denote the spaces of E-valued continuous functions and *bounded* continuous functions on X, respectively. We assume that $C(X, E)$ and $C_b(X, E)$ are endowed with the compact-open topology and the sup norm, respectively. When $E = \mathbb{K}$, $C(X) := C(X, \mathbb{K})$ and $C_b(X) := C_b(X, \mathbb{K})$.

On the other hand, if X is also a complete metric space, $C_b^u(X, E)$ denotes the space of uniformly continuous bounded functions defined on X, taking values in E, which we assume endowed with the sup norm. In this case, $C_b^u(X) := C_b^u(X, \mathbb{K})$.

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Also, if $\mathbf{e} \in E$, then $\widehat{\mathbf{e}}$ denotes the constant function from X to E taking the value $\mathbf{e}$. For $D \subset X$, $\mathrm{cl}\, D$ denotes the closure of D in X. Given $D \subset X$ and $f \in C(X)$, for $\alpha \in \mathbb{K}$, $f \equiv \alpha$ on D means that $f(x) = \alpha$ for every $x \in D$; in the same way, if $f_1, f_2 : X \to E$, $f \equiv g$ on D means that $f_1(x) = f_2(x)$ for every $x \in D$. Also, for $f \in C(X)$, $0 \leq f \leq 1$ means that $f(x) \in \mathbb{R}$ and $0 \leq f(x) \leq 1$ for every $x \in X$.

As for the spaces of linear functions, we will denote by $L'(E,F)$ and by $L(E,F)$ the sets of (not necessarily continuous) linear maps and *continuous* linear maps from E to F, respectively.

Recall that the *strong operator topology* in $L(E,F)$ is the topology defined by the basic set of neighborhoods

$$N(S; D, \epsilon) := \{R \in L(E,F) : \|(R - S)(x)\| < \epsilon \,\forall x \in D\}$$

where D is an arbitrary finite subset of E, and $\epsilon > 0$ arbitrary. Consequently this topology is characterized by the property that $(S_\alpha)_{\alpha \in \Lambda}$ in $L(E,F)$ converges to $S \in L(E,F)$ if and only if $(S_\alpha(x))$ converges to Sx for every $x \in E$ (see [9, p. 476]).

Unless otherwise stated we will assume that $L(E,F)$ is endowed with the strong operator topology.

Throughout the paper the word "homeomorphism" will be synonymous with "surjective homeomorphism".

The contexts. Our results will be valid (with the same proof) for different kinds of spaces. For this reason we first consider different contexts to work in.

From now on we will assume that we are in one of the following six contexts. All definitions, results and comments given in this paper apply to these six contexts unless otherwise stated.

- **Context 1.** X and Y are realcompact. $\mathcal{A}(X,E) = C(X,E)$, $\mathcal{A}(Y,F) = C(Y,F)$, and $\mathcal{B} := C_b(X)$.
- **Context 2.** E and F are *infinite-dimensional*. X and Y are realcompact. $\mathcal{A}(X,E) = C_b(X,E)$, $\mathcal{A}(Y,F) = C_b(Y,F)$, and $\mathcal{B} := C_b(X)$.
- **Context 3.** X and Y are completely regular, and all points of X and Y are G_δ-points. $\mathcal{A}(X,E) = C(X,E)$, $\mathcal{A}(Y,F) = C(Y,F)$, and $\mathcal{B} := C_b(X)$.
- **Context 4.** E and F are *infinite-dimensional*. X and Y are completely regular, and all points of X and Y are G_δ-points. $\mathcal{A}(X,E) = C_b(X,E)$, $\mathcal{A}(Y,F) = C_b(Y,F)$, and $\mathcal{B} := C_b(X)$.
- **Context 5.** X and Y are completely regular and first countable. $\mathcal{A}(X,E) = C_b(X,E)$, $\mathcal{A}(Y,F) = C_b(Y,F)$, and $\mathcal{B} := C_b(X)$.
- **Context 6.** X and Y are *complete* metric spaces. $\mathcal{A}(X,E) = C_b^u(X,E)$, $\mathcal{A}(Y,F) = C_b^u(Y,F)$, and $\mathcal{B} := C_b^u(X)$.

This means that when we refer to spaces X, Y, $\mathcal{A}(X,E)$, $\mathcal{A}(Y,F)$, $\mathcal{B}$, we assume that all of them are included at the same time in one of the above six contexts.

Remarks.

1. Notice that every first countable completely regular space satisfies the property that all its points are G_δ-points. This means that Contexts 4 and 5 above are not disjoint. In fact, if we are in Context 5 and E and F are infinite-dimensional, then we are also in Context 4. On the other hand, we can be in Context 4 but not in Context 5, if X is a completely regular space consisting of G_δ-points which is not first countable (see for instance [11, 4M]).

2. Some other natural contexts will be analyzed in Remark 4 after Corollary 3.6. We will see that our results cannot be extended in general to them.

Definition 1.1. Given $f \in \mathcal{A}(X, E)$, we define the cozero set of f as

$$c(f) := \{x \in X : f(x) \neq 0\}.$$

Definition 1.2. A map $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is said to be *separating* if it is additive and $c(Tf) \cap c(Tg) = \emptyset$ whenever $f, g \in \mathcal{A}(X, E)$ satisfy $c(f) \cap c(g) = \emptyset$. Besides T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

Equivalently, we see that an additive map $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is separating if $\|(Tf)(y)\| \, \|(Tg)(y)\| = 0$ for all $y \in Y$ whenever $f, g \in \mathcal{A}(X, E)$ satisfy $\|f(x)\| \, \|g(x)\| = 0$ for all $x \in X$.

2 Introduction

(Bi)separating linear maps between spaces of scalar-valued continuous functions have drawn attention of researchers recently. In general, they can be described as weighted composition maps (see for instance [1], [4], [5], [6], [8], [10], [13] and [14]). As a result, automatic continuity for this kind of maps is obtained as a corollary.

As for spaces of vector-valued continuous functions, a similar approach is taken for biseparating linear maps in [12] and [3], where a description as weighted composition maps is obtained when topological spaces X and Y are compact or locally compact.

In this paper, we do not make any assumption of (local) compactness of X and Y. We assume realcompactness (or other properties) instead. Notice at this point that the class of realcompact spaces is fairly large since it includes, apart from that of compact spaces, the class of subsets of Euclidean spaces and even the class of all metric spaces of nonmeasurable cardinal (see for instance [11, p. 232]). When dealing with realcompact spaces we obtain a representation of biseparating linear maps similar to that given in the compact setting. Special mention deserves the fact that this description of biseparating linear maps as weighted compositions apply even when they are just defined between spaces of *bounded* continuous functions $C_b(X, E)$ and $C_b(Y, F)$, that is, in Contexts 2, 4 and 5 (see Theorem 3.5).

On the other hand, we also mention that the requirements of realcompactness on our spaces is in general necessary for the descriptions we provide. If X or Y are not realcompact, the biseparating linear maps may not admit such representations. An example can be given even in a very explicit situation:

Example. Assume that X is not realcompact (for instance $X = W(\omega_1) := \{\sigma : \sigma < \omega_1\}$, where ω_1 denotes the first uncountable ordinal (see [11, 5.12]), and $E = l^2$. Since l^2 is separable, it is realcompact (see [11, 8.2]). Consequently each bounded continuous map $f : X \to l^2$ can be extended to a continuous map $f^{\upsilon X} : \upsilon X \to l^2$ defined in the realcompactification υX of X, which is also bounded. In this way we can define a biseparating map from $C_b(X, E)$ onto $C_b(\upsilon X, E)$. Clearly this map does not admit a description as the one given in Theorem 3.5.

In a much more general setting, (not necessarily linear) biseparating maps are studied in [2] for a large family of spaces. Applied to our Contexts 1, 2, 5 and 6, it is proved there that the existence of a biseparating map from $\mathcal{A}(X, E)$ onto $\mathcal{A}(Y, F)$ leads to the existence of a homeomorphism $h : Y \to X$.

As for Contexts 3 and 4, the existence of a biseparating map between $\mathcal{A}(X, E)$ and $\mathcal{A}(Y, F)$ ensures the existence of a homeomorphism h from the realcompactification of Y onto the realcompactification of X. On the other hand, as in [11, 9.7], since the only G_δ-points in the realcompactifications of X and Y are those of X and Y, respectively, we conclude that h is a homeomorphism from Y onto X.

So in all our Contexts 1–6, we have a homeomorphism $h : Y \to X$. Among the properties of this map h (called *support map*) we have the following:

Lemma 2.1. *([2, Lemma 4.4]) Let $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ be a biseparating map. Suppose that $h(y) = x$ for some $y \in Y$, and that $f \in \mathcal{A}(X, E)$ satisfies $f \equiv 0$ on a neighborhood of x. Then $Tf \equiv 0$ on a neighborhood of y.*

Corollary 2.2. *([2, Corollary 3.3]) Assume that we are in Context 6. If $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a biseparating map, then $h : Y \to X$ is a uniform homeomorphism (that is, both h and h^{-1} are uniform maps).*

3 Representation of linear biseparating maps

We start this section with two lemmas.

Lemma 3.1. *Let $\alpha, \beta \in \mathbb{R}$ satisfy $0 < \alpha < \beta$. Suppose that $f : X \to [0, +\infty)$ belongs to $\mathcal{B}$, and that the sets $U := \{x \in X : f(x) < \alpha\}$ and $V := \{x \in X : f(x) < \beta\}$ are both nonempty and different. Then there exists $g \in \mathcal{B}$ such that $0 \le g \le 1$, $g \equiv 1$ on U, and $g \equiv 0$ on $X - V$.*

Proof. Let us define $f_1 : X \to [0, +\infty)$ as $f_1(x) := \min\{f(x), \beta\}$, and $f_2 : X \to [0, \infty)$ as $f_2(x) := \max\{f_1(x), \alpha\}$. It is easy to check that $f_1, f_2 \in \mathcal{B}$. On the other hand, $f_2(x) \in [\alpha, \beta]$ for every $x \in X$, $f_2 \equiv \alpha$ on U, and $f_2 \equiv \beta$ on $X - V$. Finally we define

$$g(x) := \frac{\beta - f_2(x)}{\beta - \alpha}$$

for every $x \in X$, and we are done. ∎

Lemma 3.2. *Let $x_0 \in X$ and $f \in \mathcal{A}(X, E)$ satisfy that $f(x_0) = 0$, and that f is not constant on any neighborhood of x_0. Then there exists a strictly decreasing sequence (λ_n) in $(0, +\infty)$ convergent to 0 such that all elements in the sequence*

$$(\{x \in X : \|f(x)\| < \lambda_n\})$$

are different.

Proof. For $\alpha \in \mathbb{R}$, $\alpha > 0$, let us define $U_\alpha := \{x \in X : \|f(x)\| < \alpha\}$. Notice that to prove the lemma, it is enough to show that if $\alpha > 0$, then there exists $\beta \in \mathbb{R}$ with $0 < \beta < \alpha/2$ such that the sets U_α and U_β are different. Suppose on the contrary that if $0 < \beta < \alpha/2$, then $U_\beta = U_\alpha$. This implies clearly that, if $0 < \beta < \alpha/2$ and $\|f(x)\| < \alpha$, then $\|f(x)\| < \beta$. Taking into account that $U_\beta \subset U_{\alpha/2} \subset U_\alpha$, we deduce that if $\|f(x)\| < \alpha$, then $f(x) = 0$, that is, $f \equiv 0$ on U_α. This contradicts the assumption that f is not constant on any neighborhood of x_0. ∎

Thanks to these basic results, we can prove the following proposition.

Proposition 3.3. *Suppose that $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is biseparating and that*

$$f(h(y)) = 0$$

for some $f \in \mathcal{A}(Y, F)$ and $y \in Y$. Then $(Tf)(y) = 0$.

Proof. By Lemma 2.1, the result holds if $f \equiv 0$ on a neighborhood of $h(y)$. Thus we can assume that f is not constant on any neighborhood of $h(y)$. According to Lemma 3.2, we can construct a strictly decreasing sequence of positive numbers (λ_n) convergent to 0 such that the sequence of sets

$$(\{x \in X : \|f(x)\| < \lambda_n\})$$

does not contain two equal elements. Taking a subsequent if necessary, we may assume that $\lambda_n \leq 1/n^3$ for every $n \in \mathbb{N}$. Define

$$U_n := \{x \in X : \|f(x)\| < \lambda_n\}$$

for each $n \in \mathbb{N}$. Notice that, by definition of U_n, we have $\operatorname{cl} U_{n+1} \subset U_n$, $n \in \mathbb{N}$. It is also easy to check that

$$\{x \in X : 0 < \|f(x)\| < \lambda_2\} = \left(\bigcup_{n\in\mathbb{N}} (U_{4n} - \operatorname{cl} U_{4n+3})\right) \cup \left(\bigcup_{n\in\mathbb{N}} (U_{4n-2} - \operatorname{cl} U_{4n+1})\right).$$

It is clear from the fact that f is not constant on any neighborhood of $h(y)$ that $h(y) \in \operatorname{cl}\{x \in X : 0 < \|f(x)\| < \lambda_2\}$, so we have at least one of the following two possibilities:

$$h(y) \in V_1 := \operatorname{cl} \bigcup_{n\in\mathbb{N}} (U_{4n} - \operatorname{cl} U_{4n+3})$$

or

$$h(y) \in V_2 := \operatorname{cl} \bigcup_{n\in\mathbb{N}} (U_{4n-2} - \operatorname{cl} U_{4n+1}).$$

We assume without loss of generality that $h(y)$ belongs to V_1. Notice that in this case

$$h(y) \in \operatorname{cl} \bigcup_{n \geq k} (U_{4n} - \operatorname{cl} U_{4n+3})$$

for every $k \in \mathbb{N}$.

By Lemma 3.1, we can construct a sequence (f_n) in $\mathcal{B}$ such that, for every $n \in \mathbb{N}$, $0 \leq f_n \leq 1$, $c(f_n) \subset U_{4n-1}$, and $f_n \equiv 1$ on U_{4n}.

Next we are going to see that $g := \sum_{n=1}^{\infty} f_n f$ belongs to $\mathcal{A}(X, E)$. First, it is easy to check that each $f_n f$ belongs to $C_b(X, E)$ if we are in Contexts 1 – 5, and belongs to $C_b^u(X, E)$ if we are in Context 6. Suppose now that $\bar{E}$ denotes the completion of the normed space E. We have that, when endowed with the sup norm, $C_b(X, \bar{E})$ and $C_b^u(X, \bar{E})$ are Banach spaces. On the other hand, since $\lambda_n \leq 1/n^3$, then

$$\sup_{x \in X} \|f_n f(x)\| \leq 1/n^2.$$

We deduce that $g \in C_b(X, \bar{E})$ in Contexts $1-5$, and $g \in C_b^u(X, \bar{E})$ in Context 6, that is, $g \in \mathcal{A}(X, \bar{E})$. To prove that $g \in \mathcal{A}(X, E)$, we just have to see that $g(x) \in E$ for every $x \in X$. For, take $x \in X$: if $f(x) = 0$, then $g(x) = 0 \in E$; if $f(x) \neq 0$, then there exists a finite number of $n \in \mathbb{N}$ with $\|f(x)\| > \lambda_n$, so there is a finite number of $n \in \mathbb{N}$ with $x \in U_{4n-1}$; this clearly implies that there exists $n_0 \in \mathbb{N}$ such that $f_n(x) = 0$ for every $n > n_0$, and consequently

$$g(x) = f_1(x) f(x) + f_2(x) f(x) + \ldots + f_{n_0}(x) f(x) \in E.$$

Notice also that, for $n_0 \in \mathbb{N}$, if $x_0 \in U_{4n_0} - \operatorname{cl} U_{4n_0+3}$, then $x_0 \notin U_{4(n_0+k)-1}$ for every $k \in \mathbb{N}$. Consequently, since $c(f_{n_0+k}) \subset U_{4(n_0+(k-1))+3}$, then $f_{n_0+k}(x_0) = 0$ for every $k \in \mathbb{N}$. On the other hand, it is clear that x_0 belongs to the sets $U_4, U_8, \ldots, U_{4n_0}$, which implies that

$$1 = f_1(x_0) = f_2(x_0) = \ldots = f_{n_0}(x_0).$$

We deduce from the definition of g that $g(x_0) = n_0 f(x_0)$. Then we conclude that $g \equiv nf$ on $U_{4n} - \operatorname{cl} U_{4n+3}$ for each $n \in \mathbb{N}$.

Next suppose that $(Tf)(y) = \mathbf{e}_0 \in F$, $\mathbf{e}_0 \neq 0$, and $(Tg)(y) = \mathbf{e}_1 \in F$. Consider $n_1 \in \mathbb{N}$ with

$$n_1 \|\mathbf{e}_0\| / 2 > \|\mathbf{e}_1\| + 1,$$

and an open neighborhood $U(y)$ of y in Y such that $h(U(y)) \subset U_{4n_1}$ and

$$\|(Tf)(y')\| > \|\mathbf{e}_0\| / 2$$

for every $y' \in U(y)$.

Taking into account that h is a homeomorphism and that $h(y)$ belongs to $\operatorname{cl} \bigcup_{n \geq k} (U_{4n} - \operatorname{cl} U_{4n+3})$ for every $k \in \mathbb{N}$, we can see that there exists $k \in \mathbb{N}$, $k \geq n_1$, such that

$$h(U(y)) \cap (U_{4k} - \operatorname{cl} U_{4k-3}) \neq \emptyset.$$

Then if for $y_1 \in U(y)$, $h(y_1)$ belongs to $U_{4k} - \operatorname{cl} U_{4k-3}$, we have that $g - kf$ is constantly equal to zero in a neighborhood of $h(y_1)$. We deduce from Lemma 2.1 that $Tg \equiv kTf$ on a neighborhood of y_1, which implies that

$$\|(Tg)(y_1)\| = k \|(Tf)(y_1)\| > \|\mathbf{e}_1\| + 1.$$

Since this can be done for every neighborhood of y, this behavior implies that Tg is not continuous, which is not possible. We conclude that $(Tf)(y) = 0$. ■

To prove the next theorem, we need the following result, which can be found in [2, Claim 1 in the proof of Theorem 3.1].

Lemma 3.4. *Suppose that $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a biseparating map having $h : Y \to X$ as its support map. Then $h^{-1} : X \to Y$ is the support map of $T^{-1} : \mathcal{A}(Y, F) \to \mathcal{A}(X, E)$.*

Theorem 3.5.

- *Suppose that $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a linear biseparating map. Then there exist a homeomorphism $h : Y \to X$ and a map $J : Y \to L'(E, F)$ such that*

  $$(Tf)(y) = (Jy)(f(h(y)))$$

  for every $f \in \mathcal{A}(X, E)$ and $y \in Y$. The map J satisfies that Jy is bijective for every $y \in Y$.

- *On the other hand, if we are in Context 6, then h is also a uniform homeomorphism.*

Proof. For each $y \in Y$, we define a linear map $Jy : E \to F$ as $(Jy)(\mathbf{e}) = (T\widehat{\mathbf{e}})(y)$. It is clear that, if $h : Y \to X$ is the support map, then for every $y \in Y$ and $f \in \mathcal{A}(X, E)$, $f(h(y)) = \widehat{f(h(y))}(h(y))$, and by Proposition 3.3, $(Tf)(y) = (T\widehat{f(h(y))})(y)$, that is,

$$(Tf)(y) = (Jy)(f(h(y))).$$

Obviously Jy is linear. Next we prove that each $Jy : E \to F$ is bijective. Notice first that since T^{-1} is also biseparating, the above representation can be applied to T^{-1}. On the other hand, by Lemma 3.4 the support map of T^{-1} is h^{-1}. This implies in particular that there exists $K : X \to L'(F, E)$ such that, for every $g \in \mathcal{A}(Y, F)$ and $x \in X$,

$$(T^{-1}g)(x) = (Kx)(g(h^{-1}(x)).$$

Fix $y \in Y$ and $\mathbf{f} \in F - \{0\}$. Let $x = h(y)$. Now take $g \in \mathcal{A}(Y, F)$ with $g(y) = \mathbf{f}$. Then it is clear that $\mathbf{f} = g(y) = (T(T^{-1}g))(y)$, that is,

$$\begin{aligned} \mathbf{f} &= (Jy)((T^{-1}g)(x)) \\ &= (Jy)((Kx)(g(h^{-1}(x)))) \\ &= (Jy)((Kx)(g(y))) \\ &= (Jy)((Kx)(\mathbf{f})). \end{aligned}$$

This implies that $(Jy)(Kx)$ is the identity map on F. In the same way we can prove that $(Kx)(Jy)$ is the identity map on E. Consequently, Jy is bijective.

Finally, if we are in Context 6, the fact that h is a uniform homeomorphism follows from Corollary 2.2. ∎

Next we are going to see that, when we deal with finite-dimensional E, some properties regarding continuity can be obtained. The following result follows immediately from Theorem 3.5.

Corollary 3.6. *Suppose that $T : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is biseparating and linear, and that E is finite-dimensional. Then E and F have the same dimension.*

Remarks.

1. As a consequence of the last corollary, when E is finite-dimensional, the map $J : Y \to L'(E,F)$ given in Theorem 3.5 attains values in $L(E,F)$.

2. The existence of a nonlinear biseparating map $T : C(X,E) \to C(Y,F)$ does not imply in general that E and F are isomorphic as vector spaces, even if they are finite-dimensional. Consider for instance $E := \mathbb{K}$ and $F = \mathbb{K}^2$. Take a Hamel base $\mathfrak{U} = \{a_i : i \in I\}$ of E as a $\mathbb{Q}$-linear space, where $\mathbb{Q}$ is the field of rational numbers. Clearly $\mathfrak{V} := \{(a_i, 0) : i \in I\} \cup \{(0, a_j) : j \in I\}$ is a Hamel base of F as a $\mathbb{Q}$-linear space. Also it is easy to see that $\mathfrak{U}$ and $\mathfrak{V}$ have the same cardinal and there exists a bijective map $v : \mathfrak{U} \to \mathfrak{V}$. Then we can extend v by $\mathbb{Q}$-linearity to a bijection defined in the whole space E. Now suppose that $X = \{x\} = Y$. We clearly have that $T : C(X,E) \to C(Y,F)$ defined as $T\widehat{\mathbf{e}} := \widehat{v(\mathbf{e})}$, for every $\mathbf{e} \in E$, is a biseparating map (which obviously is not $\mathbb{K}$-linear).

3. It is quite natural to try to see if Theorem 3.5 can be given in new contexts like the following (which are a mixture of Contexts 2 and 4 and Contexts 1 and 3, respectively, where we replace the property that all points in X are G_δ-points with the more general of being first countable):

- **Context 7.** E and F are infinite-dimensional. X is first countable and completely regular, and Y is realcompact. $\mathcal{A}(X,E) = C_b(X,E)$, $\mathcal{A}(Y,F) = C_b(Y,F)$, and $\mathcal{B} := C_b(X)$.
- **Context 8.** X is first countable and completely regular, and Y is realcompact. $\mathcal{A}(X,E) = C(X,E)$, $\mathcal{A}(Y,F) = C(Y,F)$, and $\mathcal{B} := C_b(X)$.

The answer to this question in both contexts is negative, as it can be seen in the example given in the Introduction (notice in particular that the set X in the example is pseudocompact, and the set Y is compact, which means that $C_b(X,E) = C(X,E)$ and $C_b(Y,F) = C(Y,F)$).

4. Finally, we can ask if in Context 4 the fact that E and F are infinite-dimensional is necessary to obtain Theorem 3.5. If we drop all requirements on the dimension of E and F, then we are in the following context.

- **Context 9.** X and Y are completely regular, and all points of X and Y are G_δ-points. $\mathcal{A}(X,E) = C_b(X,E)$, $\mathcal{A}(Y,F) = C_b(Y,F)$, and $\mathcal{B} := C_b(X)$.

It turns out that Theorem 3.5 is no longer valid in Context 9. For, take $X = \mathbb{N}$ and $Y = \mathbb{N} \cup \{\sigma\}$, where $\sigma \in \beta\mathbb{N} \setminus \mathbb{N}$. Clearly each $f \in C_b(X)$ admits a continuous extension $f' : Y \to \mathbb{K}$. The map $T : C_b(X) \to C_b(Y)$ sending each $f \in C_b(X)$ into its extension $f' \in C_b(Y)$ is certainly biseparating and linear, but X and Y are not homeomorphic (see [11, 4M]).

4 Continuity and biseparating maps

In this section we give some corollaries concerning automatic continuity of biseparating maps, and a general description of the continuous linear biseparating maps.

Corollary 4.1. *Suppose that $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is biseparating and linear, and that E is finite-dimensional. Then the map $J : Y \to L(E, F)$ is continuous.*

Proof. We have that E is finite-dimensional and, by Corollary 3.6, so is F. Since we are dealing with the strong operator topology in $L(E, F)$, to prove that J is continuous it is enough to show that, for every $\mathbf{e} \in E$ and $y_0 \in Y$, if $(y_i)_{i\in I}$ is a net in Y which converges to $y_0 \in Y$, then $((Jy_i)(\mathbf{e}))_{i\in I}$ converges to $(Jy_0)(\mathbf{e})$. Notice that, by the definition of J, this is equivalent to prove that $((T\widehat{\mathbf{e}})(y_i))$ converges to $(T\widehat{\mathbf{e}})(y_0)$. This convergence follows from the fact that $T\widehat{\mathbf{e}}$ is continuous, and we are done. ∎

Remark. When E and F are finite-dimensional, so is $L(E, F)$. As a consequence, all linear topologies in $L(E, F)$ coincide, and then the strong operator topology and that of the norm are the same. This implies that, if E is finite-dimensional and we consider the *normed* space $L(E.F)$, then $J : Y \to L(E, F)$ is also continuous.

Corollary 4.2. *Suppose that $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is biseparating and linear, and that E is finite-dimensional. Then T (and T^{-1}) is continuous.*

Proof. Assume that we are in Contexts 1 or 3. To prove that T is continuous, take a compact subset K of Y, and $\epsilon > 0$. Since $J : Y \to L(E, F)$ is continuous (see Remark after Corollary 4.1), then

$$\sup_{y\in K} \|Jy\| < +\infty.$$

Suppose that for every $x \in h(K)$,

$$\|f(x)\| < \frac{\epsilon}{\sup_{y\in K} \|Jy\|}.$$

Now take $y_0 \in K$. Since $(Tf)(y_0) = Jy_0(f(h(y_0)))$, it is straightforward to see that

$$\begin{aligned} \|(Tf)(y_0)\| &\leq \|Jy_0\| \, \|f(h(y_0))\| \\ &< \|Jy_0\| \frac{\epsilon}{\sup_{y\in K} \|Jy\|} \\ &\leq \epsilon. \end{aligned}$$

Also, since h is a homeomorphism, $h(K)$ is a compact subset of X, and the fact that T is continuous follows.

Assume that we are in Contexts 2, 4, 5 or 6. We have that, given $\mathbf{e} \in E$,

$$\sup_{y\in Y} \|(Jy)(\mathbf{e})\| = \sup_{y\in Y} \|(T\widehat{\mathbf{e}})(y)\| < \infty.$$

Since Jy belongs to $L(E,F)$ for every $y \in Y$, by the Uniform Boundedness Principle (see for instance [7, Theorem 15.2]) we conclude that there exists $M > 0$ with $\sup_{y \in Y} \|Jy\| < M$.

Now if $f \in \mathcal{A}(X,E)$ satisfies $\|f\| \leq 1$, then

$$\|(Tf)(y)\| = \|(Jy)(f(h(y))\| \leq \|Jy\| < M$$

for every $y \in Y$. We conclude that T is continuous. ∎

In the case of continuous linear biseparating maps, we can say more, even if E is not necessarily finite-dimensional.

Corollary 4.3. *Suppose that $T : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is a continuous linear biseparating map. Then the map J given in Theorem 3.5 takes values in $L(E,F)$. Also $J : Y \to L(E,F)$ is continuous.*

Proof. We first have to prove that Jy is continuous for each $y \in Y$. But this is clear if we are in Contexts 2, 4, 5 or 6, because given any $\mathbf{e} \in E$ and $y \in Y$,

$$\|(Jy)(\mathbf{e})\| = \|(T\widehat{\mathbf{e}})(y)\| \leq \|T\| \, \|\mathbf{e}\| .$$

On the other hand, if we are in Contexts 1 or 3, we have that, by definition of continuity, for $y \in Y$, there exist a compact subset K of X and $M > 0$ such that, whenever $\sup_{x \in K} \|f(x)\| \leq 1$, then $\|(Tf)(y)\| \leq M$. As above, this implies that

$$\|(Jy)(\mathbf{e})\| = \|(T\widehat{\mathbf{e}})(y)\| \leq M \, \|\mathbf{e}\| .$$

On the other hand, the map $J : Y \to L(E,F)$ is continuous, where $L(E,F)$ is endowed with the strong operator topology. This can be proved as in Corollary 4.1. ∎

Remarks.

1. A natural question is whether in Corollary 4.3 the word "biseparating" could be replaced by just "separating and bijective". In general, even assuming continuity of the operator T, a representation as a weighted composition map is not possible. In fact, in [12, Example 4.2], the authors provide an example of a *separating* linear continuous map which is bijective but does not admit a representation as a weighted composition map; namely, in their example, $X = \{0\}$, $Y = \{0,1\}$, $E = \mathbb{R}^2$ endowed with the sup norm, and $F = \mathbb{R}$, and $T : C(X,E) \to C(Y,F)$ is defined as $T(a,b) := g \in C(Y,F)$, with $g(0) = a$ and $g(1) = b$. For some other interesting results concerning separating continuous linear maps the reader is referred to that paper.

2. When E is infinite-dimensional, a linear biseparating map need not be continuous, as it is easy to see in the following example. Consider $E := c_0$, the space of sequences converging to zero, and $X = \{x\} = Y$. Take $\mathfrak{U}$ a Hamel base of c_0 such that every element of $\mathfrak{U}$ has norm one. Consider $\mathfrak{V} := \{u_n : n \in \mathbb{N}\}$ a countable subset of $\mathfrak{U}$ and define $Tu := u$ if $u \in \mathfrak{U} - \mathfrak{V}$ and $Tu_n = nu_n$ for $u_n \in \mathfrak{V}$. It is clear that, by identifying each map in $C(\{x\}, c_0)$ with its image, T can be extended by

linearity to a (clearly biseparating) bijective map $T : C(\{x\}, c_0) \to C(\{x\}, c_0)$ which is not continuous.

3. In the same way, if X is not realcompact, it is possible to give linear biseparating maps $T : C(X) \to C(Y)$ which are not continuous (see for instance [5]).

4. A very interesting question, but not easy to deal with, is whether in Corollary 4.2, the same conclusion can be reached if "biseparating" is replaced by "separating and bijective". This would lead to the more general question of when a separating bijective linear map is biseparating. Some partial results are known (see for instance [1] and [6]), but the general question remains unsolved even for the case $E = \mathbb{K} = F$.

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