SOME CONNECTIONS BETWEEN BUNKE-SCHICK DIFFERENTIAL K-THEORY AND TOPOLOGICAL $\mathbb{Z}/k\mathbb{Z}$ K-THEORY

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Abstract. The purpose of this note is to prove some results in Bunke-Schick differential K-theory and topological $\mathbb{Z}/k\mathbb{Z}$ K-theory. The first one is an index theorem for the odd-dimensional geometric families of $\mathbb{Z}/k\mathbb{Z}$-manifolds. The second one is an alternative proof of the Freed-Melrose $\mathbb{Z}/k\mathbb{Z}$-index theorem in the framework of differential K-theory.

1. Introduction

In this note we establish some results in Bunke-Schick differential K-theory $\hat{K}_{\mathrm{BS}}$ [7] and topological K-theory with $\mathbb{Z}/k\mathbb{Z}$-coefficients $K^{-1}\mathbb{Z}/k\mathbb{Z}$ [2, Section 5]. We first introduce an index theorem in which the indices take value in $\mathbb{Z}/k\mathbb{Z}$. In order to describe this result, we briefly recall some constructions in $\hat{K}_{\mathrm{BS}}$ and $K^{-1}\mathbb{Z}/k\mathbb{Z}$.

Let X be a smooth compact manifold. Generators of the K-group $\hat{K}_{\mathrm{BS}}(X)$ are constructed out of real differential forms and geometric families over X [7, Definition 2.2]. A geometric family is roughly the data needed to define the index bundle.

Bunke and Schick [7, Subsection 5.9] pointed out the relevance to the notion of *geometric family of* $\mathbb{Z}/k\mathbb{Z}$*-manifolds* over X of a concrete description of the torsion subgroup of $\hat{K}_{\mathrm{BS}}(X)$. An odd-dimensional geometric family of $\mathbb{Z}/k\mathbb{Z}$-manifolds $(\mathcal{W}, \mathcal{E}, \beta)$ consists of an odd-dimensional geometric family $\mathcal{W}$ with boundary, an even-dimensional geometric family $\mathcal{E}$ without boundary, and an isomorphism $\beta : k.\mathcal{E} \to \partial\mathcal{W}$ [7, 2.1.7] from k copies of $\mathcal{E}$ onto the boundary of $\mathcal{W}$. It defines a k-torsion element $\lceil \mathcal{W}, \mathcal{E}, \beta \rceil \in \hat{K}_{\mathrm{BS}}(X)$ [7, Lemma 5.20]. On the other hand, there is a canonical way to construct a class $\lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor \in K^{-1}(X, \mathbb{Z}/k\mathbb{Z})$.

The work of Freed-Melrose [13] has led to the index theorem [13, Corollary 5.4], which expresses the topological index of vector bundles over even-dimensional $\mathbb{Z}/k\mathbb{Z}$-manifolds through the reduced eta invariant of [1]. In the following we discuss a geometric extension of [13, Corollary 5.4] in which $\mathbb{Z}/k\mathbb{Z}$-manifolds is replaced by odd-dimensional families of $\mathbb{Z}/k\mathbb{Z}$-manifolds.

Let $\pi : X \to Y$ be a proper submersion with closed fibers of even relative dimension. Suppose that π carries a smooth K-orientation [7, 3.1.9]. From [7, Section 3] we have an analytical $\mathbb{Z}/2\mathbb{Z}$-graded push-forward map $\hat{\pi}_! : \hat{K}_{\mathrm{BS}}(X) \to \hat{K}_{\mathrm{BS}}(Y)$.

General methods [10, Chapter 1D] show that there is a (topological) direct image $\pi_!^t : K^{-1}(X, \mathbb{Z}/k\mathbb{Z}) \to K^{-1}(Y, \mathbb{Z}/k\mathbb{Z})$.

We may define two differential K-characters ([5]) $Ind_{\mathrm{an}}(\mathcal{W}, \mathcal{E}, \beta)$ and $Ind_{\mathrm{top}}\,(\mathcal{W}, \mathcal{E}, \beta)$ using pairings of $\hat{\pi}_!\lceil \mathcal{W}, \mathcal{E}, \beta \rceil$,$\pi_!^t\lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor$ with K-homology [4]. We prove that

$$Ind_{\mathrm{an}}(\mathcal{W}, \mathcal{E}, \beta) = Ind_{\mathrm{top}}(\mathcal{W}, \mathcal{E}, \beta).$$

In the case when $\mathcal{E}$ is a zero-dimensional geometric family, we get an index theorem in $K^{-1}\mathbb{Z}/k\mathbb{Z}$ for families of Dirac operators. Moreover, if $Y = pt$ and X of odd dimension, we may recover the mod k Index Theorem [3, (8.4)].

The second main result of this note is an alternative approach to the Freed-Melrose $\mathbb{Z}/k\mathbb{Z}$-index theorem ([13, Corollary 5.4]).

2. Background Material

2.1. **Bunke-Schick Differential K-theory.** In this subsection we review $\hat{K}_{\mathrm{BS}}$ and the analytical push-forward construction. We refer the reader to [7, 6, 8] for more details.

Let X be a smooth compact manifold. Let d denote the exterior derivative on the space of real differential forms $\mathbf{\Omega}^*(X)$. Generators of the differential K-group $\hat{K}_{\mathrm{BS}}(X)$ are pairs $(\mathcal{E}, w)$, where $\mathcal{E}$ is a geometric family over X, and $w \in \frac{\mathbf{\Omega}^*(X)}{\mathrm{img}(d)}$ ([7, Definition 2.1]. We have a well-defined notion of isomorphism and sum of generators [7, Definitions 2.5,2.6]. Two generators $(\mathcal{E}_1, w_1)$ and $(\mathcal{E}_2, w_2)$ give rise to the same class in $\hat{K}_{\mathrm{BS}}(X)$ if there is a geometric family $\mathcal{E}'$ such that $(\mathcal{E}_1, w_1) + (\mathcal{E}', 0)$ is paired with $(\mathcal{E}_2, w_2) + (\mathcal{E}', 0)$, two generators $(\mathcal{E}_1', w_1')$ and $(\mathcal{E}_2', w_2')$ are paired ([7, Definition 2.10]) if the disjoint union $\mathcal{E}_1' \sqcup_X \mathcal{E}_2'^{-}$ is tamed ([7, 2.2.2]), and

$$w_1' - w_2' = \eta^{\mathrm{B}}(\mathcal{E}_1' \sqcup_X \mathcal{E}_2'^{-})_t,$$

where η^{B} is the Bunke eta form [6, Subsection 4.4].

The group $\hat{K}_{\mathrm{BS}}(X)$ is $\mathbb{Z}/2\mathbb{Z}$-graded ([7, Definition 2.4]). Moreover, it has a $\mathbb{Z}/2\mathbb{Z}$-graded ring structure $\hat{K}_{\mathrm{BS}}(X) \otimes \hat{K}_{\mathrm{BS}}(X) \stackrel{\cup}{\to} \hat{K}_{\mathrm{BS}}(X)$ [7, Definition 4.1].

From [7, 2.4.5, 2.4.6, Lemma 4.3] we have the exact sequences of rings:

$$0 \to \frac{\mathbf{\Omega}^{*-1}(X)}{\mathbf{\Omega}_0^{*-1}(X)} \xrightarrow{a} \hat{K}^*_{\mathrm{BS}}(X) \xrightarrow{i} K^*(X) \to 0,$$

$$0 \to \hat{K}^f(X) \hookrightarrow \hat{K}^*_{\mathrm{BS}}(X) \xrightarrow{R} \mathbf{\Omega}_0^*(X) \to 0, \tag{2.1}$$

where

- $\mathbf{\Omega}_0^*(X)$ is the group of forms on X with integer periods, $K^*(X)$ is the K-theory of X,
- $a(w) = [\emptyset, -w]$, $i(\mathcal{E}, w) = index(\mathcal{E})$,
- $R(\mathcal{E}, w) = \Omega(\mathcal{E}) - dw$ with $\Omega(\mathcal{E})$ is the geometric Chern form of $\mathcal{E}$ [7, 2.2.4], and $\hat{K}^f(X) = \ker(R)$.

If $\mathcal{E}$ is an even-dimensional geometric family with $\mathbb{Z}/2\mathbb{Z}$-graded kernel bundle $K^{\mathcal{E}} = K_+^{\mathcal{E}} \oplus K_-^{\mathcal{E}}$ [7, 5.3.1], then $index(\mathcal{E}) = [K_+^{\mathcal{E}}] - [K_-^{\mathcal{E}}]$.

Recall from [7, Definition 5.19] that a geometric family of $\mathbb{Z}/k\mathbb{Z}$-manifolds over X is a triple $(\mathcal{W}, \mathcal{E}, \beta)$, where $\mathcal{W}$ is a geometric family with boundary, $\mathcal{E}$ is a geometric family without boundary, and $\beta : k.\mathcal{E} \to \partial\mathcal{W}$ is an isomorphism of geometric families over X. Its corresponding class in $\hat{K}^f(X)$ is $\lceil \mathcal{W}, \mathcal{E}, \beta \rceil := [\mathcal{E}, -\frac{1}{k}\Omega(\mathcal{W})]$ ([7, Definition 5.19]).

Let Y be a smooth compact manifold, and let $\pi : X \to Y$ be a proper submersion with closed fibers, of even relative dimension. Suppose that π is topologically K-oriented [7, Definition 3.2]. Fix a representative of a smooth K-orientation $o(\pi)$ [7, Definition 3.5], consisting of a geometric refinement of the $Spin^c$-structure on the vertical tangent bundle $T^{\mathrm{v}}X$, and a differential form $\sigma(o) \in \mathbf{\Omega}^{odd}(X)$. The $\mathbb{Z}/2\mathbb{Z}$-graded push-forward map $\hat{\pi}_! : \hat{K}_{\mathrm{BS}}(X) \to \hat{K}_{\mathrm{BS}}(Y)$ ([7, 3.2.3]) evaluated at a generator $(\mathcal{E}, w)$ (whose underlying proper submersion is p) is given by

$$\hat{\pi}_![\mathcal{E}, w] = [\pi_!^\lambda \mathcal{E}, \int_{X/Y} \hat{A}^c(o(\pi)) \wedge w + \tilde{\Omega}(\lambda, \mathcal{E}) + \int_{X/Y} \sigma(o) \wedge R(\mathcal{E}, w)] \tag{2.2}$$

(which does not depend on $\lambda \in]0, \infty[$), where $\pi_!^\lambda \mathcal{E}$ is a certain geometric family [7, 3.2.1] (whose underlying submersion is $\pi \circ p$), $\hat{A}^c(o(\pi))$ is the even-form in [7, 3.1.11], and

$$\tilde{\Omega}(\lambda, \mathcal{E}) := \int_{]0,\lambda[\times Y/Y} \Omega(\mathcal{H}) \tag{2.3}$$

with $\mathcal{H} = (id_{]0,\infty[} \times \pi)_!(]0, \infty[\times \mathcal{E})$ together with an appropriate vertical metric.

2.2. **Topological K-theory with $\mathbb{Z}/k\mathbb{Z}$-coefficients.** In this subsection we briefly recall the definition of $K^{-1}(X, \mathbb{Z}/k\mathbb{Z})$ and the construction of $\pi_!^t : K^{-1}(X, \mathbb{Z}/k\mathbb{Z}) \to K^{-1}(Y, \mathbb{Z}/k\mathbb{Z})$. We refer to [2, Section 5] and [10, Chapter 1D] for the details.

From [2, Proposition 5.5], the K-group $K^{-1}(X,\mathbb{Z}/k\mathbb{Z})$ is generated by triples (E,F,α), where E,F are complex vector bundles over X, and $\alpha : kE \to kF$ is an isomorphism.

Furthermore, if X is $Spin^c$ of odd dimension, there is a (topological) direct image $Ind_k : K^{-1}(X,\mathbb{Z}/k\mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ ([2, Section 5]).

Let us explicitly state the construction of the integration along the fiber $\pi_!^t : K^{-1}(X,\mathbb{Z}/k\mathbb{Z}) \to K^{-1}(Y,\mathbb{Z}/k\mathbb{Z})$. Fix an embedding $i : X \hookrightarrow \mathbb{R}^{2d}$, and define the embedding $\imath := i\times\pi : X \hookrightarrow \mathbb{R}^{2d}\times Y$. Let v be the normal bundle associated to $\imath$. The homomorphism $\pi_!^t$ is the composite

$$K^{-1}(X,\mathbb{Z}/k\mathbb{Z}) \overset{Th}{\to} K^{-1}(X^v, pt, \mathbb{Z}/k\mathbb{Z}) \overset{c}{\to} K^{-1}(Y^{\mathbb{R}^{2d}\times Y}, pt, \mathbb{Z}/k\mathbb{Z}) \overset{\mathcal{D}}{\to} K^{-1}(Y,\mathbb{Z}/k\mathbb{Z});$$

here,

- X^H denotes the Thom space of a vector bundle H over X, Th is a Thom isomorphism,
- c is the homomorphism induced by the collapsing map $Y^{\mathbb{R}^{2d}\times Y} \to X^v$, and
- $\mathcal{D}$ is a desuspension map.

We shall call a vector bundle E geometric, if E is a Hermitian vector bundle equipped with a unitary connection.

Let $(\mathrm{E},\mathrm{F},\alpha)$ be a generator of $K^{-1}(X,\mathbb{Z}/k\mathbb{Z})$ where E and F are geometric vector bundles and α is a unitary isomorphism (not required to preserve connections). According to [7, 2.1.4], the $\mathbb{Z}/2\mathbb{Z}$-graded geometric bundle $\mathrm{E}\oplus\mathrm{F}$ with grading $\mathrm{diag}(1,-1)$ defines a zero-dimensional geometric family $\mathcal{F}(\mathrm{E}\oplus\mathrm{F})$. Using the isomorphism $k.\mathcal{F}(\mathrm{E}\oplus\mathrm{F}) \cong \mathcal{F}(k(\mathrm{E}\oplus\mathrm{F}))$, we may define a geometric family of $\mathbb{Z}/k\mathbb{Z}$-manifolds by setting

$$(k\mathrm{E}\dot{\times}[0,1], \mathcal{F}(\mathrm{E}\oplus\mathrm{F}), id),$$

where $k\mathrm{E}\dot{\times}[0,1]$ is the geometric family whose proper submersion is the projection $X\times[0,1] \to X$ and the twisting vector bundle is the product $k\mathrm{E}\times[0,1]$ with the identification $k\mathrm{E}\times\{1\} \overset{\alpha}{\sim} k\mathrm{F}\times\{1\}$.

In the following, we identify $(\mathrm{E},\mathrm{F},\alpha)$ with its associated family of $\mathbb{Z}/k\mathbb{Z}$-manifolds.

2.3. Pairings of $\hat{K}^{\mathbf{ev},f}$, $K^{-1}\mathbb{Z}/k\mathbb{Z}$ with Geometric K-homology.

Here, we explicitly give analytical and topological pairings

$$\widetilde{\eta} : K_{\mathrm{odd}}^{\mathrm{geo}}(X) \otimes \hat{K}^{\mathrm{ev},f}(X) \to \mathbb{R}/\mathbb{Z},$$
$$\langle\cdot,\cdot\rangle : K_{\mathrm{odd}}^{\mathrm{geo}}(X) \otimes K^{-1}(X,\mathbb{Z}/k\mathbb{Z}) \to \mathbb{R}/\mathbb{Z},$$

where $K_*^{\mathrm{geo}}(X)$ stands for the geometric K-homology group of X [4, Section 5].

Let $x := (P, \mathrm{H}, f)$ be an odd geometric K-cycle over X [4, Definition 5.1]. Here, P is a closed odd-dimensional $Spin^c$-manifold, H is a geometric vector bundle over P (trivially $\mathbb{Z}/2\mathbb{Z}$-graded), and $f : P \to X$ is a smooth map. Let $y := (\mathcal{E}, w)$ and $z := (\mathrm{E}, \mathrm{F}, \alpha)$ be generators of $\hat{K}^{\mathrm{ev}, f}(X)$ and $K^{-1}(X, \mathbb{Z}/k\mathbb{Z})$. Let $q : P \to pt$ be the map to a point. We set

$$\widetilde{\eta}(x, y) := \hat{q}_! \left([\mathcal{F}(\mathrm{H}), 0] \cup f^* y\right) \in \hat{K}^{\mathrm{odd}}_{\mathrm{BS}}(pt) = \mathbb{R}/\mathbb{Z} \tag{2.4}$$

$$\langle x, z\rangle := i_k \left(Ind_k \left([\mathrm{H} \otimes f^*\mathrm{E}, \mathrm{H} \otimes f^*\mathrm{F}, id \otimes \alpha]\right)\right), \tag{2.5}$$

where f^*y is the pull-back under f [7, 2.3.2] and $i_k : \mathbb{Z}/k\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}$ is the embedding which sends $1 + k\mathbb{Z}$ to $\frac{1}{k}$.

Proposition 2.1. *The assignments $(x, y) \mapsto \widetilde{\eta}(x, y)$ and $(x, z) \mapsto \langle x, z\rangle$ factor through well-defined pairings*

$$K^{geo}_{odd}(X) \otimes \hat{K}^{ev,f}(X) \xrightarrow{\widetilde{\eta}} \mathbb{R}/\mathbb{Z}$$
$$K^{geo}_{odd}(X) \otimes K^{-1}(X, \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{R}/\mathbb{Z}.$$

Proof. It is obvious that $\widetilde{\eta}$ and $\langle\cdot,\cdot\rangle$ are bi-additive.

From [7, 2.3.2, Lemma 3.14], $\widetilde{\eta}(x, \cdot)$ is well-defined. Let us show that $\widetilde{\eta}(x, y)$ does not depend on the choice of a representative of $[x] \in K^{\mathrm{geo}}_{\mathrm{odd}}(X)$. As noted in [4, Definition 5.7], the equivalence relation on $K^{\mathrm{geo}}_*(X)$ is generated by the relations of bordism, direct sum, and vector bundle modification.

Suppose that $\mathcal{W} := (W, \mathrm{G}, g)$ is a K-chain which bounds x [4, Definition 5.5]. We equip $W \to pt$ with a smooth K-orientation $o(W)$ as in [7, 5.8.2]. By [7, Proposition 5.18, Lemma 4.3], we have

$$\begin{aligned}\widetilde{\eta}(\partial\mathcal{W}, y) &= -a\left(\int_W \hat{A}^c(o(W)) R\left((\mathcal{F}(\mathrm{G}), 0) \cup g^*y\right)\right)\\ &= -a\left(\int_W \hat{A}^c(o(W)) \mathrm{Ch}(\nabla^{\mathrm{G}}) g^* R(y)\right) = 0.\end{aligned}$$

Then $\widetilde{\eta}(x, \cdot)$ depends only on the bordism class of x.

We will rewrite the pairing $\widetilde{\eta}$ in order to show that $\widetilde{\eta}(\cdot, y)$ does not depend on the relations of disjoint sum and vector bundle modification.

Let $r : M \to X$ be the proper submersion induced from $\mathcal{E}$. Let f^*M be the pull-back of the family of manifolds M along f, and let $p_P : f^*M \to P$ and $p_M : f^*M \to M$ be the projections. Let $\mathcal{S}^c_r$ and $\mathcal{S}^c_P$ denote the geometric spinor bundles associated to the $Spin^c$-structures on $T^{\mathrm{v}}M$ and TP. We will use L to denote the twisting bundle of $\mathcal{E}$ [6, 4.3.2]. Since $index(q^1_!(\mathcal{F}(\mathrm{H}) \times_P f^*\mathcal{E})) \in K^1(pt) = \{0\}$, we can choose a taming $(q^1_!(\mathcal{F}(\mathrm{H}) \times_P f^*\mathcal{E}))_t$. From (2.2), [7, Lemma 3.11], and [6, Definition 4.16], we obtain $\widetilde{\eta}(x,y)$ in terms of the reduced eta invariant of [1], $\bar{\eta}(\mathcal{D})$, as follows:

$$\begin{aligned}
\widetilde{\eta}(x,y) &= [\left(f^*M \to pt, p_P^*(\mathrm{H}\otimes\mathcal{S}_P^c)\otimes p_M^*(\mathrm{L}\hat{\otimes}\mathcal{S}_r^c)\right), \int_P \hat{A}^c(o(P))\mathrm{Ch}(\nabla^{\mathrm{H}})f^*w \\
&\quad + \tilde{\Omega}(\lambda, (\mathcal{F}(\mathrm{H})\times_P f^*\mathcal{E}))] \\
&= [\emptyset, -\eta^B\left(f^*M \to pt, p_P^*(\mathrm{H}\otimes\mathcal{S}_P^c)\otimes p_M^*(\mathrm{L}\hat{\otimes}\mathcal{S}_r^c)\right)_t \\
&\quad + \int_P \hat{A}^c(o(P))\wedge\mathrm{Ch}(\nabla^{\mathrm{H}})f^*w + \tilde{\Omega}(\lambda, (\mathcal{F}(\mathrm{H})\times_P f^*\mathcal{E}))] \\
&\overset{\lambda\to 0}{=} [\emptyset, \bar{\eta}(\mathcal{D}^{p_P^*\mathrm{H}\otimes p_M^*\mathrm{L}}) + \int_P \hat{A}^c(o(P))\mathrm{Ch}(\nabla^{\mathrm{H}})f^*w] \\
&= a\left(-\bar{\eta}(\mathcal{D}^{p_P^*\mathrm{H}\otimes p_M^*\mathrm{L}}) - \int_P \hat{A}^c(o(P))\mathrm{Ch}(\nabla^{\mathrm{H}})f^*w\right) \\
&= -\bar{\eta}(\mathcal{D}^{p_P^*\mathrm{H}\otimes p_M^*\mathrm{L}}) - \int_P \hat{A}^c(o(P))\mathrm{Ch}(\mathrm{H})f^*w \bmod \mathbb{Z}. \qquad (2.6)
\end{aligned}$$

From [5, Proposition 5] and $\int_P \hat{A}^c(o(P))\mathrm{Ch}(\mathrm{H})f^*w \bmod \mathbb{Z} = \bar{f}_w(P,\mathrm{H},f)$, where $\bar{f}_w$ is the differential K-character in [5, Examples], we get $\widetilde{\eta}(\cdot,y)$ is invariant under the relations of disjoint sum and vector bundle modification.

Let us show that $\langle\cdot,[\mathrm{E},\mathrm{F},\alpha]\rangle$ is well-defined. Assume that E and F are geometric vector bundles and α is a unitary isomorphism. Since the geometric family $\mathcal{F}(\mathrm{H})\times_P f^*\mathcal{F}(\mathrm{E}\oplus\mathrm{F})$ has zero-dimensional fibers, we have $\tilde{\Omega}(1,\mathcal{F}(\mathrm{H}\otimes f^*(\mathrm{E}\oplus\mathrm{F})),q) = 0$. Let $CS(k\nabla^E, \alpha^*k\nabla^F) \in \frac{\mathbf{\Omega}^{\mathrm{odd}}(X)}{\mathrm{img}(d)}$ denote the Chern-Simons class of $(k\nabla^E, k\nabla^F, \alpha)$ [16, (4)] and let $SF(k\mathcal{D}^{\mathrm{E}}, k\mathcal{D}^{\mathrm{F}})$ denote the spectral flow from $k\mathcal{D}^{\mathrm{E}}$ to $k\mathcal{D}^{\mathrm{F}}$ [3, Section 7]. By [15, (4.59)] and [3, Proposition (8.3), Theorems (3.4), (8.4)], we calculate

$$\begin{aligned}
&\widetilde{\eta}(x, \lceil(\mathrm{E},\mathrm{F},\alpha)\rceil) = \hat{q}_![\mathcal{F}(\mathrm{H}\otimes f^*(\mathrm{E}\oplus\mathrm{F})), -\frac{1}{k}\Omega((\mathrm{H}\otimes f^*k\mathrm{E})\dot{\times}[0,1])] \\
&= -(\bar{\eta}(\mathcal{D}^{\mathrm{H}\otimes f^*\mathrm{E}}) - \bar{\eta}(\mathcal{D}^{\mathrm{H}\otimes f^*\mathrm{F}})) \\
&\quad + \frac{1}{k}\int_P \hat{A}^c(o(P))\left(\int_0^1 \mathrm{Ch}(tk\nabla^{\mathrm{H}\otimes f^*\mathrm{F}}\right. \\
&\qquad \left. + (1-t)(id\otimes\alpha)^*k\nabla^{\mathrm{H}\otimes f^*\mathrm{E}} + dt\partial_t)\right) \bmod \mathbb{Z} \\
&= \bar{\eta}(\mathcal{D}^{\mathrm{H}\otimes f^*\mathrm{F}}) - \bar{\eta}(\mathcal{D}^{\mathrm{H}\otimes f^*\mathrm{E}}) \\
&\quad - \frac{1}{k}\int_P \hat{A}^c(o(P))CS(k\nabla^{\mathrm{H}\otimes f^*\mathrm{E}}, (id\otimes\alpha)^*k\nabla^{\mathrm{H}\otimes f^*\mathrm{F}}) \bmod \mathbb{Z} \\
&= \frac{1}{k}SF(k\mathcal{D}^{\mathrm{H}\otimes f^*\mathrm{E}}, k\mathcal{D}^{\mathrm{H}\otimes f^*\mathrm{F}}) \bmod \mathbb{Z} = \langle x, [\mathrm{E},\mathrm{F},\alpha]\rangle. \qquad (2.7)
\end{aligned}$$

Now, let y be another representative of $[x]$. Then

$$\langle y, [\mathrm{E}, \mathrm{F}, \alpha]\rangle = \widetilde{\eta}(y, \lceil(\mathrm{E}, \mathrm{F}, \alpha)\rceil) = \widetilde{\eta}(x, \lceil(\mathrm{E}, \mathrm{F}, \alpha)\rceil) = \langle x, [\mathrm{E}, \mathrm{F}, \alpha]\rangle.$$

$\square$

3. The First Main Result

Let $(\mathcal{W}, \mathcal{E}, \beta)$ be an odd-dimensional geometric family of $\mathbb{Z}/k\mathbb{Z}$-manifolds over X. Let $(\mathcal{D}_x)_{x\in X}$ denote the family of Dirac operators associated to $\mathcal{E}$. We assume that $\dim(\ker(\mathcal{D}_x))$ is constant. This condition can always be satisfied ([6, 9.2.4]). So, we can form the $\mathbb{Z}/2\mathbb{Z}$-graded geometric index bundle $\mathcal{K}^{\mathcal{E}} = \mathcal{K}^{\mathcal{E}}_{+} \oplus \mathcal{K}^{\mathcal{E}}_{-}$ [7, 5.3.1]. Let $K^{\mathcal{E}} = K^{\mathcal{E}}_{+} \oplus K^{\mathcal{E}}_{-}$ be the topological $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle induced from $\mathcal{K}^{\mathcal{E}}$. In $K^0(Y)$ we have

$$[kK^{\mathcal{E}}_{+}] - [kK^{\mathcal{E}}_{-}] = index(k.\mathcal{E}) = index(\partial\mathcal{W}) = 0.$$

A unitary isomorphism $\alpha : k(\mathcal{K}^{\mathcal{E}}_{+} \oplus \mathbf{1}^{\ell}) \to k(\mathcal{K}^{\mathcal{E}}_{-} \oplus \mathbf{1}^{\ell})$, for some trivial vector bundle $\mathbf{1}^{\ell}$ (of rank ℓ), can be induced by a taming $(\partial\mathcal{W})_t$ ([7, 2.2.2]). We set

$$\lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor := [K^{\mathcal{E}}_{+} \oplus \mathbf{1}^{\ell}, K^{\mathcal{E}}_{-} \oplus \mathbf{1}^{\ell}, \alpha] \in K^{-1}(X, \mathbb{Z}/k\mathbb{Z}).$$

Let $\pi : X \to Y$ be a proper submersion with closed fibers, of even relative dimension. Suppose that π has a smooth K-orientation represented by $o(\pi)$. We define

$$Ind_{\mathrm{an}}(\mathcal{W}, \mathcal{E}, \beta) := \widetilde{\eta}\left(\cdot, \hat{\pi}_{!}\lceil \mathcal{W}, \mathcal{E}, \beta \rceil\right),$$
$$Ind_{\mathrm{top}}(\mathcal{W}, \mathcal{E}, \beta) := \langle \cdot, \pi^{t}_{!}\lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor\rangle,$$

($\in \mathrm{Hom}(K^{\mathrm{geo}}_{\mathrm{odd}}(Y), \mathbb{R}/\mathbb{Z}) \cong \hat{K}^{f,\mathrm{ev}}(Y)$ [7, Proposition 2.25,(10)]).

Proposition 3.1. *The following identity holds.*

$$Ind_{an}(\mathcal{W}, \mathcal{E}, \beta) = Ind_{top}(\mathcal{W}, \mathcal{E}, \beta). \tag{3.1}$$

Proof. Let $x = [N, \mathrm{F}, f]$ for some generator (N, F, f) of $K^{\mathrm{geo}}_{\mathrm{odd}}(Y)$. According to [14], we can assume that $\mathrm{F} = \mathbf{1}_N$. From definitions (2.4) and (2.5), we pull everything back to N and we can assume Y is an arbitrary closed odd-dimensional $Spin^c$-manifold. Thus, (3.1) is equivalent to

$$\widetilde{\eta}([Y], \hat{\pi}_{!}\lceil \mathcal{W}, \mathcal{E}, \beta \rceil) = \langle [Y], \pi^{t}_{!}\lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor\rangle, \tag{3.2}$$

where $[Y] \in K^{\mathrm{geo}}_{\mathrm{odd}}(Y)$ is the fundamental class of Y.

Let X have the $Spin^c$-structure which is induced from combining those on $T^{\mathrm{v}}X$ and TY. There is a homomorphism $\pi^{!} : K^{geo}_{\mathrm{odd}}(Y) \to K^{geo}_{\mathrm{odd}}(X)$

which is dual to the integration along the fiber $\pi_!^t$, and we have $\pi^![Y] = [X]$. Then

$$\begin{aligned}\langle [Y], \pi_!^t \lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor \rangle &= \langle \pi^![Y], \lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor \rangle = \langle [X], \lfloor \mathcal{W}, \mathcal{E}, \beta \rfloor \rangle \\ &= \frac{1}{k} SF(k\mathcal{D}^{\mathcal{K}_+^{\mathcal{E}} \oplus \mathbf{1}^\ell}, k\mathcal{D}^{\mathcal{K}_-^{\mathcal{E}} \oplus \mathbf{1}^\ell}) \bmod \mathbb{Z}.\end{aligned}$$

Fix a representative $o(Y)$ of a differential $Spin^c$-structure on TY, and let $o(X)$ be the composite $o(Y) \circ o(\pi)$ [7, Definition 3.21]. Let $q_Y : Y \to pt$ be the map to a point. By [7, Theorem 3.23] and $\tilde{\Omega}(1, \mathcal{E}, \pi)(= \frac{1}{k}\tilde{\Omega}(1, \partial\mathcal{W}, \pi))$ is exact from (2.3), we calculate

$$\begin{aligned}\widetilde{\eta}([Y], \hat{\pi}_! \lceil \mathcal{W}, \mathcal{E}, \beta \rceil) &= (\hat{q}_Y)_!(\hat{\pi}_! \lceil \mathcal{W}, \mathcal{E}, \beta \rceil) = (\hat{q}_X)_![\mathcal{E}, -\frac{1}{k}\Omega(\mathcal{W})] \\ &= [(q_X^1)_!\mathcal{E}, -\frac{1}{k}\int_X \hat{A}^c(o(X))\Omega(\mathcal{W})] \\ &= -\bar{\eta}(\mathcal{D}^E) + \frac{1}{k}\int_X \hat{A}^c(o(X))\Omega(\mathcal{W}) \bmod \mathbb{Z}.\end{aligned}$$

Here, E is the twisting bundle of $\mathcal{E}$.

Let $\left(k.(\mathcal{E} \sqcup_X \mathcal{F}(\mathcal{K}^{\mathcal{E}} \oplus \mathbf{1}^\ell)^-)\right)_t$ be the taming induced by the isomorphisms α,β. From [7, Theorem 3.12], [6, 4.2.1, Theorem 4.13], and the definition of η^B [6, Definition 4.16], we have

$$\begin{aligned}&\lim_{\lambda \to 0} \eta^B\left((q_X^\lambda)_!(k.(\mathcal{E} \sqcup_X \mathcal{F}(\mathcal{K}^{\mathcal{E}} \oplus \mathbf{1}^\ell)^-))\right)_t \\ &= \int_X \hat{A}^c(o(X))\eta^B(\partial\mathcal{W} \sqcup_X \mathcal{F}(k(\mathcal{K}^{\mathcal{E}} \oplus \mathbf{1}^\ell))^-)_t \\ &= -\int_X \hat{A}^c(o(X))\Omega(\mathcal{W}) - \int_X \hat{A}^c(o(X))\eta^B(\mathcal{F}(k(\mathcal{K}^{\mathcal{E}} \oplus \mathbf{1}^\ell))_t \\ &= -\int_X \hat{A}^c(o(X))\Omega(\mathcal{W}) - \int_X \hat{A}^c(o(X))CS(k\nabla^{\mathcal{K}_+^{\mathcal{E}} \oplus \mathbf{1}^\ell}, \alpha^* k\nabla^{\mathcal{K}_-^{\mathcal{E}} \oplus \mathbf{1}^\ell}),\end{aligned}$$

and on the other hand,

$$\begin{aligned}&\lim_{\lambda \to 0} \eta^B\left((q_X^\lambda)_!(k.(\mathcal{E} \sqcup_X \mathcal{F}(\mathcal{K}^{\mathcal{E}} \oplus \mathbf{1}^\ell)^-))\right)_t \\ &= -k\bar{\eta}(\mathcal{D}^E) + k(\bar{\eta}(\mathcal{D}^{\mathcal{K}_+^{\mathcal{E}} \oplus \mathbf{1}^\ell}) - \bar{\eta}(\mathcal{D}^{\mathcal{K}_-^{\mathcal{E}} \oplus \mathbf{1}^\ell})).\end{aligned}$$

Then

$$\begin{aligned}\widetilde{\eta}([Y], \hat{\pi}_! \lceil \mathcal{W}, \mathcal{E}, \beta \rceil) &= \bar{\eta}(\mathcal{D}^{\mathcal{K}_-^{\mathcal{E}} \oplus \mathbf{1}^\ell}) - \bar{\eta}(\mathcal{D}^{\mathcal{K}_+^{\mathcal{E}} \oplus \mathbf{1}^\ell}) \\ &\quad - \frac{1}{k}\int_X \hat{A}^c(o(X))CS(k\nabla^{\mathcal{K}_+^{\mathcal{E}} \oplus \mathbf{1}^\ell}, \alpha^* k\nabla^{\mathcal{K}_-^{\mathcal{E}} \oplus \mathbf{1}^\ell}) \bmod \mathbb{Z} \\ &= \frac{1}{k} SF(k\mathcal{D}^{\mathcal{K}_+^{\mathcal{E}} \oplus \mathbf{1}^\ell}, k\mathcal{D}^{\mathcal{K}_-^{\mathcal{E}} \oplus \mathbf{1}^\ell}) \bmod \mathbb{Z},\end{aligned}$$

which implies that (3.2) holds. □

Remark 3.2. *The formula* (3.1) *may be considered as a geometric extension of the Freed-Melrose $\mathbb{Z}/k\mathbb{Z}$-index theorem* [13, Corollary 5.4] *to the odd-dimensional geometric families of $\mathbb{Z}/k\mathbb{Z}$-manifolds.*

Let X be a closed manifold of finite fundamental group $\pi_1(X)$. Let θ be a unitary representation of $\pi_1(X)$. Denote the flat vector bundle over X defined by θ, equipped with a Hermitian metric and a flat connection compatible with the metric, by V_θ. We choose $k \in \mathbb{N}^*$ and a unitary isomorphism $\alpha : kV_\theta \to \mathbf{1}^{kr}$.

Let

$$\pi_!^a[V_\theta, \mathbf{1}^r, \alpha] \in K^{-1}(Y, \mathbb{Z}/k\mathbb{Z})$$

such that

$$\lceil \pi_!^a[V_\theta, \mathbf{1}^r, \alpha] \rceil = \hat{\pi}_! \lceil (V_\theta, \mathbf{1}^r, \alpha) \rceil.$$

Proposition 3.3. *We have*

$$\pi_!^a[V_\theta, \boldsymbol{1}^r, \alpha] = \pi_!^t[V_\theta, \boldsymbol{1}^r, \alpha].$$

Proof. From (3.2) and (2.7), we get

$$\langle [x], \pi_!^a[V_\theta, \mathbf{1}^r, \alpha] - \pi_!^t[V_\theta, \mathbf{1}^r, \alpha] \rangle = 0 \quad (\text{for all } [x] \in K^{geo}_{\text{odd}}(Y)). \tag{3.3}$$

We consider the $\mathbb{R}/\mathbb{Z}$-pairing [3, (5.2)] with the identification $K^1(TY) \cong K^{\text{geo}}_{\text{odd}}(Y)$ obtained by duality and the Thom isomorphism. It is perfect as a direct consequence of the universal coefficient theorem for $\mathbb{R}/\mathbb{Z}$ K-theory together with $\mathbb{R}/\mathbb{Z}$ is divisible, and its torsion part coincides with $\langle \cdot, \cdot \rangle$ by [3, Theorem 8.4] and the construction [3, Section 5:(i)-(iv)]. Thus, (3.3) yields $\pi_!^a[V_\theta, \mathbf{1}^r, \alpha] = \pi_!^t[V_\theta, \mathbf{1}^r, \alpha]$. □

Remark 3.4.

- *From* [7, Lemma 3.20, Theorem 3.23, and Proposition 5.18]*, the assignment $\pi \mapsto \pi_!^t$ is natural, functorial under the composition of smooth K-oriented proper submersions, and bordism invariant.*
- *Let $\hat{K}_{FL}(X)$ be the Freed-Lott differential K-group of X, and let $\bar{\pi}_!^a, \bar{\pi}_!^t : \hat{K}_{FL}(X) \to \hat{K}_{FL}(Y)$ denote, respectively, the analytical and topological index homomorphisms* [12]*.*
  We set
  $$\overline{(V_\theta, \boldsymbol{1}^r, \alpha)}$$
  $$:= (V_\theta, \nabla^{V_\theta}, \frac{1}{k} CS(\nabla^{kV_\theta}, \alpha^* \nabla^{\boldsymbol{1}^{kr}})) - (\boldsymbol{1}^r, \nabla^{\boldsymbol{1}^r}, 0) \in \hat{K}_{FL}(X).$$
  We will identify $\overline{(V_\theta, \boldsymbol{1}^r, \alpha)}$ with $[V_\theta, \boldsymbol{1}^r, \alpha]$. From [7, 5.3.5], [12, Definition 3-11]*, and the variational formula of the Bismut-Cheeger*

eta form in the proof of [17, Proposition 3]*, it is not hard to see that*

$$\pi_!^a[V_\theta, \boldsymbol{1}^r, \alpha] = \bar{\pi}_!^a\overline{(V_\theta, \boldsymbol{1}^r, \alpha)}.$$

Then, [12, Theorem 6-2] *yields*

$$\pi_!^t[V_\theta, \boldsymbol{1}^r, \alpha] = \bar{\pi}_!^t\overline{(V_\theta, \boldsymbol{1}^r, \alpha)}.$$

4. The Second Main Result

Let (M, N, α) be an even-dimensional compact $Spin^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold ([11, Definition (1.7)]). Here, $\alpha : \partial M \to N$ is the induced map from an orientation preserving diffeomorphism $\partial M = \sqcup_{i=1}^k (\partial M)_i \to k.N$. We equip $M \to pt$ with a smooth K-orientation $o(M)$ as in [7, 5.8.2]. Let $(\mathrm{E}, \mathrm{F}, \beta)$ be a geometric $\mathbb{Z}/k\mathbb{Z}$-vector bundle over (M, N, α). More precisely, E and F are two geometric vector bundles over M and N, respectively, and $\beta : \mathrm{E}|_{\partial M} \to k\alpha^*\mathrm{F}$ is a unitary isomorphism which preserves the unitary connection.

Let $(\mathbb{S}^{n,k}, \mathbb{S}^{n-1}, \alpha')$ be the $\mathbb{Z}/k\mathbb{Z}$-manifold obtained by removing k open balls B^n from the n-sphere $\mathbb{S}^n$ with α' induced from $Id_{\mathbb{S}^{n-1}}$. Fix a $\mathbb{Z}/k\mathbb{Z}$-embedding $(\imath, \jmath) : (M, N, \alpha) \hookrightarrow (\mathbb{S}^{n,k}, \mathbb{S}^{n-1}, \alpha')$ with n even, i.e. $\imath : M \hookrightarrow \mathbb{S}^{n,k}$ and $\jmath : N \hookrightarrow \mathbb{S}^{n-1}$ are two embeddings such that $\alpha' \circ \imath|_{\partial M} = \jmath \circ \alpha$. There is a (topological) direct image $(\imath, \jmath)_!(\mathrm{E}, \mathrm{F}, \beta) := (\imath_!\mathrm{E}, \jmath_!\mathrm{F}, \tilde{\beta})$ which lies in the reduced K-theory $\widetilde{K}(\mathbb{S}^{n,k}, \mathbb{S}^{n-1})$. The topological $\mathbb{Z}/k\mathbb{Z}$-index of $(\mathrm{E}, \mathrm{F}, \beta)$ is given by

$$ind_k(\mathrm{E}, \mathrm{F}) := [(\imath, \jmath)_!(\mathrm{E}, \mathrm{F}, \beta)] \in \mathbb{Z}/k\mathbb{Z} = \widetilde{K}(\mathbb{S}^{n,k}, \mathbb{S}^{n-1}).$$

It is independent of $(\imath, \jmath)$ with respect to the topological $\mathbb{Z}$-index.

Proposition 4.1. ([11],[13]) *The following identity holds.*

$$ind_k(\mathrm{E}, \mathrm{F}) = \int_M \hat{A}^c(o(M))\, Ch(\nabla^{\mathrm{E}}) - k\bar{\eta}(\mathcal{D}_N^{\mathrm{F}}) \ mod\ k\mathbb{Z}.$$

Proof. Let $(\mathcal{S}_M^c, \mathcal{S}_N^c)$ be the $\mathbb{Z}/k\mathbb{Z}$ geometric spinor bundle associated to the $Spin^c$-structure of (M, N). We denote, by $f(\mathrm{E}, \mathrm{F})$, the geometric family of $\mathbb{Z}/k\mathbb{Z}$-manifolds over pt

$$f(\mathrm{E}, \mathrm{F}) := ((M \to pt, \mathrm{E} \otimes \mathcal{S}_M^c), (N \to pt, \mathrm{F} \otimes \mathcal{S}_N^c), \beta)\,.$$

From (2.6), we have

$$\lceil f(\mathrm{E}, \mathrm{F}) \rceil = i_k\left(\int_M \hat{A}^c(o(M))\mathrm{Ch}(\nabla^{\mathrm{E}}) - k\bar{\eta}(\mathcal{D}_N^{\mathrm{F}}) \ \mathrm{mod}\ k\mathbb{Z}\right). \tag{4.1}$$

We will identify $\imath_!\mathrm{E}, \jmath_!\mathrm{F}$ with $\mathbb{Z}/2\mathbb{Z}$-graded geometric vector bundles, where the geometric structures are defined as in the proof of [12, Lemma 4-3].

Let us first prove the following Riemann-Roch property:

$$\lceil f(\mathrm{E},\mathrm{F})\rceil = \lceil f(\imath_!\mathrm{E}, \jmath_!\mathrm{F})\rceil. \tag{4.2}$$

We may regard $f(\mathrm{E},\mathrm{F})$ as a geometric $\mathbb{Z}/k\mathbb{Z}$-cycle of Deleey on pt [9, Definition 2.1]. Let $f(\mathrm{E},\mathrm{F})_V$ denote the modification of $f(\mathrm{E},\mathrm{F})$ by a geometric $Spin^c$ $\mathbb{Z}/k\mathbb{Z}$-vector bundle $V \to (M,N)$ [9, Definition 2.5, Remark 2.6]. More precisely, if $V = (V_M, V_N)$ and $\mathcal{E}(\mathrm{E}) := (M \to pt, \mathrm{E} \otimes \mathcal{S}^c_M)$ then

$$f(\mathrm{E},\mathrm{F})_V = (\mathcal{E}(\mathrm{E})_{V_M}, \mathcal{E}(\mathrm{F})_{V_N}, \beta)\,,$$

where $\mathcal{E}(\mathrm{E})_{V_M}$ and $\mathcal{E}(\mathrm{F})_{V_N}$ are the modifications of the geometric K-chains $\mathcal{E}(\mathrm{E})$ and $\mathcal{E}(\mathrm{F})$ by V_M,V_N ([4, Definition 5.6]).

Inspired by [18, Lemma 2.3.4], there is a $\mathbb{Z}/k\mathbb{Z} - bordism$ $z := ((Q,P), (\mathrm{G},\mathrm{H}), (Q,P) \to pt)$ [9, Definition 2.4] between $f(\mathrm{E},\mathrm{F})_V$ and $f(\imath_!\mathrm{E},\jmath_!\mathrm{F})_W$ for certain geometric $Spin^c$ $\mathbb{Z}/k\mathbb{Z}$-vector bundles $V \to (M,N)$ and $W \to (\mathbb{S}^{n,k}, \mathbb{S}^{n-1})$. We equip $Q, P \to pt$ with smooth K-orientations $o(Q)$,$o(P)$ as in [7, 5.8.2], and we equip the $\mathbb{Z}/k\mathbb{Z}$-vector bundle (G,H) with a geometric structure which extends that induced from $f(\mathrm{E},\mathrm{F})_V$ and $f(\imath_!\mathrm{E},\jmath_!\mathrm{F})_W$.

From [9, Definition 2.4, Remark 1.9], and by gluing together geometric K-chains along their common boundaries, we have

$$\partial\mathcal{E}(\mathrm{G}) \cong \left(\mathcal{E}(\mathrm{E})_{V_M} \sqcup \mathcal{E}(\imath_!\mathrm{E})^-_{W_{\mathbb{S}^{n,k}}}\right) \underset{\partial}{\cup} (k.\mathcal{E}(\mathrm{H})) \tag{4.3}$$

$$\partial\mathcal{E}(\mathrm{H}) \cong \mathcal{E}(\mathrm{F})_{V_N} \sqcup \mathcal{E}(\jmath_!\mathrm{F})^-_{W_{\mathbb{S}^{n-1}}}, \tag{4.4}$$

where $\partial\mathcal{E}(\mathrm{G})$ and $\mathcal{E}(\mathrm{G})^-$ are the boundary and the opposite of $\mathcal{E}(\mathrm{G})$, and $\cong$ stands for an isomorphism between two geometric families (over pt) [7, 2.1.7].

By (4.3), and because the K-homological Chern character is invariant under the relation of modification [5, Proposition 2] and the form $\hat{A}^c(o(Q))\mathrm{Ch}(\nabla^G)$ is closed, we get

$$\begin{aligned}\int_M \hat{A}^c(o(M))\mathrm{Ch}(\nabla^{\mathrm{E}}) - \int_{\mathbb{S}^{n,k}} \hat{A}^c(o(\mathbb{S}^{n,k}))\mathrm{Ch}(\nabla^{\imath_!\mathrm{E}}) + k\int_P \hat{A}^c(o(P))\mathrm{Ch}(\nabla^{\mathrm{H}}) \\ = \int_{\partial Q} (\hat{A}^c(o(Q))\mathrm{Ch}(\nabla^{\mathrm{G}}))|_{\partial Q} = \int_Q d(\hat{A}^c(o(Q))\mathrm{Ch}(\nabla^{\mathrm{G}})) = 0.\end{aligned} \tag{4.5}$$

Using (4.1), together with [5, Lemma 1], we have $\lceil f(\mathrm{E},\mathrm{F})_V\rceil = \lceil f(\mathrm{E},\mathrm{F})\rceil$, and so we may assume that z is a geometric $\mathbb{Z}/k\mathbb{Z}$-bordism between $f(\mathrm{E},\mathrm{F})$

and $f(\imath_!\mathrm{E}, \jmath_!\mathrm{F})$. Now, (4.4) [7, Proposition 5.17] and (4.5) yield

$$\lceil f(\imath_!\mathrm{E}, \jmath_!\mathrm{F})\rceil = [(\mathbb{S}^{n-1} \to pt, \jmath_!\mathrm{F} \otimes \mathcal{S}^c_{\mathbb{S}^{n-1}}), -\frac{1}{k}\int_{\mathbb{S}^{n,k}} \hat{A}^c(o(\mathbb{S}^{n,k}))\mathrm{Ch}(\nabla^{\imath_!\mathrm{E}})]$$
$$= [N \to pt, \mathrm{F} \otimes \mathcal{S}^c_N, -\frac{1}{k}\int_{\mathbb{S}^{n,k}} \hat{A}^c(o(\mathbb{S}^{n,k}))\mathrm{Ch}(\nabla^{\imath_!\mathrm{E}}) + \int_P \hat{A}^c(o(P))\mathrm{Ch}(\nabla^{\mathrm{H}})]$$
$$= \lceil f(\mathrm{E}, \mathrm{F})\rceil.$$

Let $s = (s_1, s_2) : (\mathbb{S}^{2,k}, \mathbb{S}^1) \hookrightarrow (\mathbb{S}^{n,k}, \mathbb{S}^{n-1})$ be the canonical embedding. By the Thom isomorphism $s_! : \widetilde{K}(\mathbb{S}^{2,k}, \mathbb{S}^1) \cong \widetilde{K}(\mathbb{S}^{n,k}, \mathbb{S}^{n-1})$, $[(\imath, \jmath)_!(\mathrm{E}, \mathrm{F}, \beta)]$ is the direct image $[s_!(\bar{\mathrm{E}}, \bar{\mathrm{F}}, \bar{\beta})]$ of a certain $\mathbb{Z}/k\mathbb{Z}$-vector bundle $(\bar{\mathrm{E}}, \bar{\mathrm{F}}, \bar{\beta})$ over $(\mathbb{S}^{2,k}, \mathbb{S}^1)$. We compute, using (4.2) [11, Proposition 1.14] at the marked step, and $ind_k(\mathrm{E}, \mathrm{F})$ is independent of the embedding $(\imath, \jmath)$

$$\lceil f(\mathrm{E,F})\rceil = \lceil f(\imath_!\mathrm{E}, \jmath_!\mathrm{F})\rceil = \lceil f(s_{1!}\bar{\mathrm{E}}, s_{2!}\bar{\mathrm{F}})\rceil = \lceil f(\bar{\mathrm{E}}, \bar{\mathrm{F}})\rceil \stackrel{!}{=} i_k\left(ind_k(\bar{\mathrm{E}}, \bar{\mathrm{F}})\right)$$
$$= i_k\left(ind_k(s_!(\bar{\mathrm{E}}, \bar{\mathrm{F}}))\right) = i_k\left(ind_k((\imath, \jmath)_!(\mathrm{E}, \mathrm{F}))\right) = i_k\left(ind_k(\mathrm{E}, \mathrm{F})\right).$$

□

Remark 4.2. *From* (4.2) *together with Zhang's description of* $\bar{\eta}(\mathcal{D}^{\jmath_!\mathrm{F}}_{\mathbb{S}^{n-1}})$ [19, Theorem 2.2]*, we obtain the following geometric formula for* $ind_k(\mathrm{E}, \mathrm{F})$*:*

$$ind_k(\mathrm{E}, \mathrm{F}) = \int_{\mathbb{S}^{n,k}} \hat{A}^c(o(\mathbb{S}^{n,k}))\, Ch(\nabla^{\imath_!\mathrm{E}}) + k\int_{\mathbb{S}^{n-1}} \hat{A}^c(o(\mathbb{S}^{n-1}))\gamma \ mod\ k\mathbb{Z}.$$

Here, γ *is a certain Chern-Simons current* ([19, (2.18)])*.*

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