WEAKLY JU RINGS

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ABSTRACT. We define and completely explore the so-called *WJU rings*. This class properly encompasses the class of JU rings, introduced and studied by the present author in detail in Toyama Math. J. (2016).

1. Introduction and Main Definition

Everywhere in the text of this paper, all rings are assumed to be associative and unital, containing the identity element 1. Our standard terminology and notations are in agreement with [16]. For instance, for a ring R, the symbol $U(R)$ stands for the unit group of R, $J(R)$ stands for the Jacobson radical of R, $Id(R)$ stands for the set of all idempotents in R, and $Nil(R)$ for the set of all nilpotents in R.

All other specific notions will be stated explicitly below. We start here with the following.

Definition 1 ([4, 5])**.** *A ring R is called weakly exchange if, for any $r \in R$, there exists $e \in Id(rR)$ such that $1 - e \in (1 - r)R$ or $1 - e \in (1 + r)R$.*

Definition 2 ([17])**.** *A ring R is called exchange if, for any $r \in R$, there exists $e \in Id(rR)$ such that $1 - e \in (1 - r)R$.*

Definition 3 ([1])**.** *A ring R is called weakly clean if $R = U(R) \pm Id(R)$.*

Both exchange and weakly clean rings are weakly exchange, but the converse is irreversible. However, for abelian rings, the weakly cleanness and weak exchange property coincide.

Definition 4 ([17])**.** *A ring R is called clean if $R = U(R) + Id(R)$. In addition, if the existing unit and idempotent commute, R is called strongly clean.*

Clean rings are exchange, but the converse is irreversible. However, for abelian rings, the cleanness and exchange property coincide.

Definition 5 ([18])**.** *A ring R is called semi-boolean if $R = J(R) + Id(R)$.*

These rings are termed as *J-clean* by some authors, but we will follow the original name.

Definition 6 ([11])**.** *A ring R is called weakly semi-boolean if $R = J(R) \pm Id(R)$.*

In parallel to the above, these rings could be named as *weakly J-clean.*

Semi-boolean rings are weakly semi-boolean and the latter are clean. None of these two implications is reversible.

Definition 7 ([3])**.** *A ring R is said to be UU, provided the equation $U(R) = 1 + Nil(R)$ holds.*

It was established in [14] and independently in [13], that a ring R is UU if and only if $U(R)$ is a 2-group and $2 \in Nil(R)$.

Definition 8 ([9])**.** *A ring R is said to be WUU, provided the equation $U(R) = \pm 1 + Nil(R)$ holds.*

Certainly, if $2 \in Nil(R)$, UU coincides with WUU. However, if 2 does not belong to $J(R)$, the class of UU rings are properly contained in the class of WUU rings.

Definition 9 ([8])**.** *A ring R is said to be JU, provided the equality $U(R) = 1 + J(R)$ holds.*

The key instrument here is the following new concept.

Definition 10. *We shall say that a ring R is weakly JU or, abbreviated WJU, provided the equality $U(R) = \pm 1 + J(R)$ holds.*

This is obviously tantamount to the relation $U(R/J(R)) = \{-1, 1\}$. In fact, because of the inclusion $1 + J(R) \leq U(R)$, it is well-known that the natural ring epimorphism $\Phi : R \to R/J(R)$ can be restricted to the group surjection $\phi : U(R) \to U(R/J(R))$ with kernel $1 + J(R)$. The resulting isomorphism $U(R)/(1+J(R)) \cong U(R/J(R))$ is fulfilled. If $U(R) = 1+J(R)$ or $U(R) = -1 + J(R) = -(1 + J(R))$, then the claim that $U(R/J(R))$ will contain the only two elements $\{-1, 1\}$ is self-evident. Reciprocally, if $U(R/J(R)) = \{-1, 1\}$, then for any $u \in U(R)$ we have that $\phi(u) = 1$. Hence, $u \in 1 + J(R)$, or $\phi(u) = -1$, i.e., $\phi(-u) = 1$ and hence, $-u \in 1 + J(R)$, that is, $u \in -1 + J(R)$, as required.

Apparently, all JU rings are WJU, while the converse is false. However, if $2 \in J(R)$, then $-1 = 1 - 2 \in 1 + J(R)$ and these two concepts are tantamount.

Moreover, since in a WJU ring it has to be $1+Nil(R/J(R)) \subseteq U(R/J(R)) = \{\pm 1\}$, one observes that either $2 \in Nil(R/J(R))$ and thus, $2^t \in J(R)$ by [10], or by Lemma 2 below it follows that $2 \in J(R)$ or $Nil(R/J(R)) = \{0\}$ and, in particular, $Nil(R) \subseteq J(R)$. Thus, in a WJU ring which is not JU, the factor-ring $R/J(R)$ is always reduced and hence abelian. Moreover, as we shall establish below, in a WJU (weakly clean) ring the containment $6 \in J(R) \setminus Nil(R)$ is possible.

Other non-trivial examples of WJU rings are considered in the sequel (compare with Proposition 4).

On the other hand, it is known that for a ring R, its idempotents *lift* modulo $J(R)$ whenever, for each $r \in R$, the condition $r^2 - r \in J(R)$ implies that there is $f \in Id(R)$ with $f - r \in J(R)$. We shall slightly expand this property to the following one: Let S be an idempotent subset, i.e., $S \subseteq Id(R)$, possessing the zero element 0. If $S = \{0, 1\}$, we shall say that idempotents of R 2-*lift* modulo $J(R)$ whenever, for every $r \in R$, the condition $r^2 - r \in J(R)$ implies that there exists $f \in Id(R)$ such that $f - r \in J(R) + 2S \subseteq J(R) + 2R$ (see [10] as well). Note that $J(R) + 2S$ may not be an ideal of R, whereas $J(R) + 2R$ is.

It is readily seen that the ordinary lifting modulo $J(R)$ guarantees 2-lifting, because $J(R) \subseteq J(R) + 2S$. However, if $2 \in J(R)$, it is then clear that these two lifting do coincide, because $J(R) + 2S = J(R)$. Certainly, if R is semi-primitive (also termed semi-simple in the Jacobson sense), that is, $J(R) = \{0\}$, then $r^2 - r \in \{0\}$ forces that r is an idempotent with $r - r = 0$ and there is nothing to show.

The objective of the current article, in regard to [8, Problem 3], is to discover the structure of WJU rings as well as to find relationships with other classical ring classes as these of exchange rings (see Theorem 2.2 below) and weakly clean rings (see Theorem 2.3 below), respectively.

2. WJU Rings

We first begin here with some element-wise properties. The first one is an important technicality, while the second one is an improvement on [11, Proposition 3.4].

Lemma 1. *If R is a WJU ring and $r \in R$ is an arbitrary element such that $r + J(R)$ or $-r + J(R)$ is an idempotent in $R/J(R)$, then $2r \in J(R)$ or $2r \in 2 + J(R)$ or $2r \in -2 + J(R)$.*

In particular, for any idempotent e in a WJU ring R, $2e \in J(R)$ or $2(1 - e) \in J(R)$.

Proof. For any $r \in R$, assuming the element $r + J(R)$ is an idempotent, it follows that $2(r + J(R)) - (1 + J(R)) = (2r - 1) + J(R) \in U(R/J(R)) = \{-1, 1\}$ and so $2r \in J(R)$ or $2r \in 2 + J(R)$.

Similarly, for any $r \in R$, assuming the element $-(r + J(R)) = -r + J(R)$ is an idempotent, it follows that $2(-r + J(R)) - (1 + J(R)) = (-2r - 1) + J(R) \in U(R/J(R)) = \{-1, 1\}$ and thus, $2r \in J(R)$ or $2r \in -2 + J(R)$.

The second part follows directly, because e being idempotent in a ring R is a guarantor that $e + J(R)$ is also an idempotent in the factor-ring $R/J(R)$. $\square$

Notice that if $2e \in J(R)$ and $2(1-e) \in J(R)$ are fulfilled simultaneously, it is easy to see that $2 = 2e + 2(1-e) \in J(R)$ and thus, the WJU ring will reduce to a JU ring.

Proposition 1. *The following two points are true.*

(i) *Weakly semi-boolean elements are clean. In WJU rings for which* $6 = 0$, *clean elements are weakly semi-boolean.*

(ii) *Weakly clean elements in JU rings are semi-boolean. Weakly clean elements in WJU rings for which* $6 = 0$ *are clean and thus weakly semi-boolean.*

Proof. (i) If $j \in J(R)$ and $e \in Id(R)$, then one may write $j + e = (j + 2e - 1) + (1 - e)$ or $j - e = (j - 1) + (1 - e)$. Since $(2e - 1)^2 = 1$ and so $2e - 1 \in U(R)$, it must be that $j + 2e - 1 \in U(R)$. But also $j - 1 \in U(R)$ and $1 - e \in Id(R)$, as needed.

To treat the second reverse part, for any $u \in U(R)$ and $e \in Id(R)$, the sum $u+e$ can be written as $u+e = -1+j+e = j-(1-e)$ for some $j \in J(R)$ or as $u + e = 1 + j + e$. In the first case since $1 - e \in Id(R)$, we are done. The second case is more complicated and by Lemma 1 it has two variants. If $2e \in J(R)$, then one can represent $1 + j + e = (j + 2e) + (1 - e)$, and we are set. If now $2(1 - e) \in J(R)$, then $1 + j + e = (j - 2(1 - e)) + (3 - e) = (j - 2(1 - e)) - (3 + e)$. But $(3 + e)^2 = 9 + 7e = 3 + e$, because $6 = 0$, and we are finished.

(ii) We know that $2 \in J(R)$ and since, for any $u \in U(R)$ and $e \in Id(R)$, we have that $u - e = (u - 2e) + e \in U(R) + Id(R)$ is a clean element, so we employ [8, Proposition 3.2] to get the assertion.

To deal with the second part, as above $u - e = 1 + j - e = (j + 1 - 2e) + e$ is a clean element. On the other side, with Lemma 1 at hand, we derive that either $u - e = -1 + j - e = (j - 2e - 1) + e \in U(R) + Id(R)$ provided $2e \in J(R)$, or $u - e = -1 + j - e = [(j + 2e - 2) + 1] - 3e = [(j + 2e - 2) + 1] + 3e$ provided $2(1 - e) \in J(R)$, where $(3e)^2 = 9e = 3e$, as needed. $\square$

The last statement raises the following two queries.

(1) Is it true in general that clean elements in WJU rings are weakly semi-boolean? In this respect, we remember that clean elements in JU rings are always semi-boolean (for more details see [8, Proposition 3.2]).
(2) Is it true in general that weakly clean elements in WJU rings are clean?

Let us recall now that an element in a ring is called weakly nil-clean if it is the sum or the difference of a nilpotent and an idempotent (see [15] and [2]). If these commute, the element is said to have the strong property (cf. [6]). In this respect, we are now ready to extend [14, Theorem 3.7] as well as [9, Theorem 2.21] in the following way.

Theorem 2.1. *In a WUU ring R with $6 \in J(R)$ the next two items are valid:*

(i) *any element is strongly clean if and only if it is weakly nil-clean with the strong property.*

(ii) *any element is clean if and only if it is weakly nil-clean, provided $6 = 0$.*

Proof. According to [9, Theorem 2.11], one can decompose $R \cong R_1 \times R_2$, where R_1 is a UU ring with $2 \in J(R_1)$, and R_2 is either zero, or a WUU ring without non-trivial idempotents (i.e., a WJU strongly indecomposable ring) such that $3 \in J(R_2)$.

(i) For the "if" part, for all $q \in Nil(R)$ and $e \in Id(R)$ with $qe = eq$, one can write that $q + e = (q + 2e - 1) + (1 - e)$, where $(2e - 1)^2 = 1$ and hence, $q+2e-1$ is a unit because $q(2e-1) = (2e-1)q$. Also, $q-e = (q-1)+(1-e)$, where $q - 1$ is a unit. Since in both presentations $1 - e$ is an idempotent, the elements $q + e$ and $q - e$ are clean, and we are done (see [2] and [6] too).

As for the "only if" part, because of the above decomposition of R, we will identify it with its isomorphic copy $R_1 \times R_2$. And so, if $r = (r_1, r_2) \in R = R_1 \times R_2$ is presentable as $(r_1, r_2) = (u_1, u_2) + (e_1, e_2) = (u_1+e_1, u_2+e_2)$, where the first pair is a unit and the second pair is an idempotent, then it is plainly observed that u_1, u_2 are units and e_1, e_2 are idempotents. In addition, if the commutation $(u_1e_1, u_2e_2) = (u_1, u_2).(e_1, e_2) = (e_1, e_2).(u_1, u_2) = (e_1u_1, e_2u_2)$ is fulfilled, then u_1 commutes with e_1 and, respectively, u_2 commutes with e_2.

On the other hand, the arithmetic in R_1 is as follows: $r_1 = u_1 + e_1 = q_1 + 1 + e_1 = (q_1 + 2e_1) + (1 - e_1) = (q_1 + 2) - (1 - e_1)$ for some $q_1 \in Nil(R_1)$. Since 2 is a nilpotent in R_1, it follows that $q_1 + 2e_1$ and $q_1 + 2$ are also nilpotent elements (as q_1 and $2e_1$ do commute, provided q_1 and e_1 are commutable), so that the above presentation is a strongly (weakly) nil-clean decomposition of r_1.

On the other hand, the arithmetic in R_2 is as follows: $r_2 = u_2 + e_2$ amounts to either u_2 or to $u_2 + 1$, because $e_2 \in \{0, 1\}$. Furthermore, since $u_2 = 1 + q_2$ or $u_2 = -1 + q_2$ for some $q_2 \in Nil(R_2)$, we have the following four cases.

- $r_2 = q_2 + 1 \in Nil(R_2) + Id(R_2)$;
- $r_2 = q_2 - 1 \in Nil(R_2) \in Nil(R_2) - Id(R_2)$;
- $r_2 = q_2 = q_2 + 0 = q_2 - 0 \in Nil(R_2) \pm Id(R_2)$;

- $r_2 = q_2 + 2 = (q_2 + 3) - 1 \in Nil(R_2) - Id(R_2)$ since $3 \in Nil(R_2)$.

Finally, one checks that the fourth presentation of the element r, namely $r = (q_1 + 2e_1, q_2) + (1 - e_1, 1)$, $r = (q_1 + 2, q_2) - (1 - e_1, 1)$, $r = (q_1 + 2, q_2) - (1 - e_1, 0)$ and $r = (q_1 + 2, q_2 + 3) - (1 - e_1, 1)$, give its strongly weakly nil-clean decomposition, as required.

(ii) All arguments are analogous to these of (i). In fact, about the "if" part, we observe that the equalities $q + e = (q + 2e - 1) + (1 - e)$ and $q - e = (q - 1) + (1 - e)$ take the form $q + e = (q - 1) + (1 - e) = q - e$ in R_1 because $2 = 0$. Concerning R_2, they have the kinds $q + e = (q - 1) + (1 - e)$ or $q + e = (q + 1) + (1 - e)$ and $q - e = (q - 1) + (1 - e)$, where e is either 0 or 1. These are obviously clean decompositions.

Conversely, in order to show validity of the "only if" half, taking into account that $2 = 0$ in R_1 and $3 = 0$ in R_2, we obtain the weakly nil-clean presentations of the element r like these: $r = (q_1, q_2) + (1 - e_1, 1)$, $r = (q_1, q_2) - (1 - e_1, 1)$, $r = (q_1, q_2) - (1 - e_1, 0)$ and $r = (q_1, q_2) - (1 - e_1, 1)$, as wanted. □

The following technicality appeared in [10]. We will now give a new and more transparent verification.

Lemma 2. *If R is a ring and c is its central element such that $c^n \in J(R)$ for some $n \in \mathbb{N}$, then $c \in J(R)$.*

Proof. Since $c^n \in J(R)$, it follows that $(c + J(R))^n = J(R)$, whence $c + J(R)$ is a central nilpotent in $R/J(R)$. But this forces $c + J(R) = J(R)$, that is, $c \in J(R)$, because the quotient $R/J(R)$ does not contain non-trivial central nilpotent elements. □

Proposition 2. *Let R be a weakly clean WJU ring. Then either $6 \in J(R)$ or $10 \in J(R)$.*

Proof. Write $3 = u + e$ or $3 = u - e$ for some $u \in U(R)$ and $e \in J(R)$. But every element u of $U(R)$ can be written as $j \pm 1$ for some $j \in J(R)$ and thus we have four variants $3 = (j + 1) + e$, or $3 = (j - 1) + e$, or $3 = (j + 1) - e$, or $3 = (j - 1) - e$. We will consider for completeness these four cases separately.

Case 1. $3 = (j + 1) + e$ gives that $(2 - j)^2 = 2 - j$, i.e., $4 - 4j + j^2 = 2 - j$. This means that $2 = -j^2 + 3j \in J(R)$.

Case 2. $3 = (j - 1) + e$ gives that $(4 - j)^2 = 4 - j$, i.e., $16 - 8j + j^2 = 4 - j$. This means that $12 = -j^2 + 7j \in J(R)$. So, $6^2 = 3.12 \in J(R)$, whence Lemma 2 yields that $6 \in J(R)$.

Case 3. $3 = (j + 1) - e$ gives that $(j - 2)^2 = j - 2$, i.e., $j^2 - 4j + 4 = j - 2$. This means that $6 = -j^2 + 5j \in J(R)$.

Case 4. $3 = (j-1)-e$ gives that $(j-4)^2 = j-4$, i.e., $j^2-8j+16 = j-4$. This means that $20 = -j^2 + 9j \in J(R)$. Thus, $10^2 = 5.20 \in J(R)$ and hence, Lemma 2 implies that $10 \in J(R)$. □

Proposition 3. *Let R be a WJU ring. Then*

(a) $3 \in U(R) \iff 2 \in J(R)$.
(b) $2 \in U(R) \iff 3 \in J(R)$.

Proof. (a) Since $1 + J(R) \subseteq U(R)$, the "$\Leftarrow$" implication follows directly.

As for the converse "$\Rightarrow$" implication, writing $3 = 1 + z$ or $3 = -1 + z$ for some $z \in J(R)$, one can derive that $2 = z$ or that $4 = 2^2 = z$. Next, Lemma 2 applies to get that 2 lies in $J(R)$, which we desired.

(b) As above, 3 belonging to $J(R)$ yields that 2 belongs to $U(R)$. Conversely, writing $2 = 1+z$ or $2 = -1+z$, we deduce that either $1 = z$, which is impossible, or $3 = z$, as wanted. □

Corollary 1. *If R is a WJU ring in which $10 \in J(R)$, then $2 \in J(R)$.*

Proof. Since $-10 \in J(R)$ and $1+J(R) \subseteq U(R)$, it follows that $-9 \in U(R)$, that is, $3^2 = 9 \in U(R)$. That is why, $3 \in U(R)$ which by Proposition 3 (a) leads to $2 \in J(R)$, as claimed. □

Corollary 2. *In a weakly clean WJU ring R, the inclusion $6 \in J(R)$ is valid.*

Proof. It follows immediately in view of Proposition 2 jointly with Corollary 1. □

Remark 1. The same technique could be applied in [12] to simplify the proofs.

We continue with some useful connections between certain classes of rings.

Proposition 4. *Weakly semi-boolean rings are WJU rings.*

Proof. Letting $u \in U(R)$ we write that $u = z \pm e$ for some $z \in J(R)$ and $e \in Id(R)$. Since $U(R) + J(R) = U(R)$, it follows that $u - z = \pm e \in U(R)$. Thus, $e = 1$ and $u = z \pm 1$, as needed. □

The next result is essential.

Theorem 2.2. *Suppose R is a ring. Then the following four conditions are equivalent:*

(1) *R is exchange WJU.*
(2) *R is clean WJU.*
(3) *R is weakly semi-boolean.*

(4) *all idempotents of R lift modulo $J(R)$ and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.*

Proof. The equivalence (3) $\iff$ (4) was proved in [11]. The implication (3) $\Rightarrow$ (2) follows combining [18] and Proposition 4, whereas the implication (2) $\Rightarrow$ (1) is elementary. To show that (1) $\Rightarrow$ (4), we foremost apply [17] to infer that all the idempotents of R are lifted modulo $J(R)$. After that, one observes with the aid of [17] that $R/J(R)$ is also exchange as well as with the comments associated to above that $U(R/J(R)) = \{-1, 1\}$. Therefore, $R/J(R)$ is a WUU ring and an application of [9, Corollary 2.15 (i)] allows us to conclude that the conditions for the factor-ring $R/J(R)$ are really satisfied. □

The next statement is basic.

Theorem 2.3. *Suppose that R is a ring. Then R is a weakly clean WJU ring if and only if all idempotents of R 2-lift modulo $J(R)$ and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.*

Proof. "**Necessity**". Since a homomorphic image of a weakly clean ring is again a weakly clean ring, we have that $R/J(R)$ is weakly clean too. Moreover, due to our comments above, $U(R/J(R)) = \{\pm 1\}$, whence $R/J(R)$ is a weakly clean WUU ring. So, with [12, Theorem 2.7] at hand, we get the isomorphic conditions for $R/J(R)$.

About the lifting property, every element r of R can be written as $r = (j+1)+e$, or $r = (j-1)+e$, or $r = (j+1)-e$, or $r = (j-1)-e$. We will consider for completeness these four cases separately.

<u>Case 1</u>. $r = (j+1)+e = j+(1+e)$. Then $r^2 - r + J(R) = (1+e)^2 - (1+e) + J(R) = 2e + J(R)$ and so $r^2 - r \in J(R)$ yields that $2e \in J(R)$. Now $(1-e) - r = 1 + e - r - 2e = -j - 2e \in J(R)$, as required.

<u>Case 2</u>. $r = (j-1)+e = j+(e-1)$. Then $r^2 - r + J(R) = (e-1)^2 - (e-1) + J(R) = 2 - 2e + J(R)$ and thus, $r^2 - r \in J(R)$ implies that $2 - 2e \in J(R)$. Now $(1-e) - r = (1-e) - (e-1) - j = 2 - 2e - j \in J(R)$, as required.

<u>Case 3</u>. $r = (j+1)-e = j+(1-e)$. Then $r^2 - r + J(R) = (1-e)^2 - (1-e) + J(R) = J(R)$. Now $(1-e) - r = -j \in J(R)$, as required.

<u>Case 4</u>. $r = (j-1)-e = j-(1+e)$. Then $r^2 - r + J(R) = (1+e)^2 + (1+e) + J(R) = 2 + 4e + J(R) = 2 - 2e + 6e + J(R) = 2 - 2e + J(R)$, because an appeal to Corollary 2 enables us that $6 \in J(R)$. Therefore, $r^2 - r \in J(R)$ leads to $2 - 2e \in J(R)$ and now $(1-e) - r = 1 - e + 1 + e - j = 2 - j \in J(R) + 2$, as required.

"**Sufficiency**". If $R/J(R)$ is Boolean, then it is clean. Also, $2 \in J(R)$ which is a guarantor that idempotents lift modulo $J(R)$, so that by [17] we detect that R is clean.

If now $R/J(R) \cong \mathbb{Z}_3$, it follows that R is local and thus clean.

If now $R/J(R) \cong B \times \mathbb{Z}_3$, one has that, for any $r \in R$, either $r + J(R)$ or $-(r + J(R)) = -r + J(R)$ is an idempotent and hence, $r^2 - r \in J(R)$ or $r^2 + r = (-r)^2 - (-r) \in J(R)$. In the first case, there exists $f \in Id(R)$ such that $f - r \in J(R)$ or $f - r \in J(R) + 2$ and so $r = z + f = (z+1) - (1-f) \in U(R) - Id(R)$ or $r = z + f - 2 = (z-1) - (1-f) \in U(R) - Id(R)$ for some $z \in J(R)$, as needed. In the second case, there exists $f \in Id(R)$ such that $f - (-r) = f + r \in J(R)$ or $f + r \in J(R) + 2$ and thus, $r = y - f = (y-1) + (1-f) \in U(R) + Id(R)$ or $r = y - f + 2 = (y+1) + (1-f) \in U(R) + Id(R)$ for some $y \in J(R)$, as needed. These two equalities ensure that R is really weakly clean, as promised.

On the other side, one simply checks that $U(R/J(R)) \cong U(B) = \{1\}$, or that $U(R/J(R)) \cong U(\mathbb{Z}_3) = \{-1, 1\}$, or that $U(R/J(R)) \cong U(B \times \mathbb{Z}_3) = U(B) \times U(\mathbb{Z}_3) = \{\pm 1\}$. As observed above, these isomorphisms along assure that R is WJU, as expected. □

The next assertion is major.

Corollary 3. *Suppose that R is a ring for which $6 = 0$. Then the following three points are equivalent:*

(1) *R is a weakly clean WJU ring.*
(2) *all idempotents of R lift modulo $J(R)+2R$ and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.*
(3) *R is a clean WJU ring.*
(4) *all idempotents of R lift modulo $J(R)$ and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.*

Proof. "(3) ⇒ (1)" and "(4) ⇒ (2)" are straightforward.

"(1) ⇒ (2)". It follows by a combination of Theorem 2.3 and [10, Lemma 2.1].

"(2) ⇒ (4)". First of all, notice that for all $r \in R$, the conditions $r^2 - r \in J(R)$ or $r^2 + r \in J(R)$ are always fulfilled, because $r + J(R)$ or $-r + J(R) = -(r + J(R))$ are idempotents. So, actually, if idempotents lift modulo $J(R) + 2R$, then they can be lifted modulo $J(R)$. In fact, as shown in Lemma 1, $2R + J(R) = 2T + J(R)$, where $T = \{0, 1, -1\}$. And so, for any $e \in Id(R)$ and any $r \in R$, we obtain that $e - r \in J(R) + 2R$ ensures that $e - r = x$ or $e - r = x + 2$ or $e - r = x - 2$ for some $x \in J(R)$. Consequently, in the first situation we are finished, while in the second and third situations one deduces that

$$(-2 - e) - r = x - 2e \in J(R), (-2 - e)^2 = -2 - e$$

since $r^2 - r + J(R) = (e-2)^2 - (e-2) + J(R) = 2e + J(R)$, and hence, $r^2 - r \in J(R)$ assures that $2e \in J(R)$;

$$3e - r = x - (2 - 2e) \in J(R), (3e)^2 = 3e$$

since $r^2 - r + J(R) = (e+2)^2 - (e+2) + J(R) = 4e + 2 + J(R) = 2 - 2e + J(R)$, whence $r^2 - r \in J(R)$ insures that $2 - 2e \in J(R)$. This substantiates our claim about lifting.

"(4) $\Rightarrow$ (3)". Since $R/J(R)$ is clean, we apply [17] to get that R is clean. That R is a WJU ring follows in the same manner as in Theorem 2.3. □

What can be derived from this corollary is that weakly clean WJU rings for which $6 = 0$ are clean.

The next affirmation is pivotal.

Theorem 2.4. *Suppose R is a ring. If R is a weakly exchange WJU ring, then either idempotents lift modulo $J(R)$ and $R/J(R) \cong B$, where B is a Boolean ring, or idempotents lift modulo $J(R)+2R$ and one of the following holds: $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$.*

In particular, if $6 = 0$ in R, the converse holds.

Proof. From [5] it follows that the quotient $R/J(R)$ is again a weakly exchange ring. Likewise, as commented before, $U(R/J(R)) = \{\pm 1\}$. Consequently, either $2 \in Nil(R/J(R))$ or $Nil(R/J(R)) = \{0\}$. In the first case, $2^n \in J(R)$ for some $n \in \mathbb{N}$ and an appeal to Lemma 2 leads us to $2 \in J(R)$. In accordance with [5], we furthermore conclude that R is exchange, whence Theorem 2.2 produces the claim. In the second case, $R/J(R)$ being a reduced factor-ring, and hence an abelian ring, allows us to infer with the aid of [4] that $R/J(R)$ is weakly clean and, therefore, Theorem 2.3 accomplished with [10, Lemma 2.1] can be applied to obtain the claim.

The second part follows immediately by Corollary 3. □

<u>Remark 2</u>. What can be concluded is that if R is a ring for which $6 = 0$, then R is weakly exchange WJU if, and only if, R is either exchange WJU or weakly clean WJU. In that aspect, does it follow that R is a weakly exchange WJU ring if, and only if, either idempotents lift modulo $J(R)$ and $R/J(R) \cong B$, where B is a Boolean ring, or idempotents lift modulo $J(R) + 2R$ and one of the following holds: $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$? In regard to Theorem 2.3 the answer is perhaps "no" even in the case where R is abelian, because as it was shown in [4] then weakly exchange rings have to be weakly clean.

Note also that it was proved in [5] that if $R/J(R)$ is weakly exchange (respectively, weakly clean) and idempotents are lifted modulo $J(R)$, then R remains weakly exchange (respectively, weakly clean).

Besides, in the last result we used the fact that if P is a ring with $U(P) = \{\pm 1\}$, then either $Nil(P) = \{0\}$ or $2 \in Nil(P)$. Indeed, for example, in $\mathbb{Z}_4 = \{0, 1, 2, 3 \mid 4 = 0\}$ we have that $U(\mathbb{Z}_4) = \{\pm 1\} = \{1, 3\}$ and $2 \in Nil(\mathbb{Z}_4)$.

Recall that a ring R is said to be *nil-good* in [7, 8] provided that any its element is nilpotent or a sum of a unit and a nilpotent. It was proved there that R is a nil-good JU ring if, and only if, $J(R)$ is nil and $R/J(R) \cong \mathbb{Z}_2$. We are now in a position to arrive at the following generalization.

Theorem 2.5. *A ring R is nil-good WJU if, and only if, $J(R)$ is nil and either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.*

Proof. "$\Rightarrow$". Utilizing [7, Proposition 2.5], we have that $J(R)$ is nil and so $J(R) \subseteq Nil(R)$. Then, it is not too hard to verify that $J(R) + Nil(R) = Nil(R)$. Given $r \in R$, we write that $r \in Nil(R)$ or $r \in U(R) + Nil(R) = \pm 1 + J(R) + Nil(R) = \pm 1 + Nil(R) \subseteq U(R)$. We furthermore assert that R is local, that is, $R/J(R)$ is a division quotient. To that goal, assuming $r \notin J(R)$, there is $a \in R$ such that $1 - ar \notin U(R)$. So, $1 - ar \in Nil(R)$, i.e., $ar \in 1 + Nil(R) \subseteq U(R)$. Similarly, there exists $b \in R$ with $rb \in U(R)$. These two containments imply together that $r \in U(R)$, which substantiates our assertion. However, $(R/J(R)) \setminus \{0\} = U(R/J(R)) = \{\pm 1\}$ which assures the desired two isomorphisms.

"$\Leftarrow$". That R is nil-good follows immediately from [7, Corollary 2.9]. Moreover, $U(R/J(R)) = \{\pm 1\} \cong U(R)/(1 + J(R))$ which directly ensures that $U(R) = \pm 1 + J(R)$, as required. $\square$

Now we have accumulated all the information necessary to establish the following two assertions.

Proposition 5. *If R is a WJU ring and $e \in Id(R)$, then eRe is a WJU ring.*

Proof. If $u \in U(eRe)$ is an arbitrary element with inverse $v \in eRe$, then $u + 1 - e \in U(R)$ with inverse $v + 1 - e$. So, we may write $u + 1 - e = 1 + z$ or $u + 1 - e = -1 + z$ for some $z \in J(R)$. In the first case $z = u - e \in J(R) \cap eRe = J(eRe)$ by virtue of [8, Lemma 3.4], so that $u = e + z \in e + J(eRe)$.

In the other case, multiplying both sides of the equality $u + 1 - e = -1 + z$ by e from the left and from the right, one deduces that $eu = -e + ez$ and $ue = -e + ze$. But it is not too hard to see that $eu = ue = u$ and so $ez = ze$. Furthermore, $ze = ze.e = eze \in eJ(R)e = J(eRe)$, where the last equality follows owing to [16]. Finally, $u = -e + eze \in -e + J(eRe)$ which is what we wanted. $\square$

Proposition 6. *If R is a non-zero ring, then for all $n \in \mathbb{N}$ the full matrix $n \times n$ ring $\mathbb{M}_n(R)$ is not WJU.*

Proof. Since $\mathbb{M}_2(R)$ is isomorphic to a corner ring of $\mathbb{M}_n(R)$, in conjunction with Proposition 5 it suffices to show that $\mathbb{M}_2(R)$ is not WJU. To that purpose, we consider the matrix unit $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. But $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ cannot lie in the Jacobson radical, because it is a unit with inverse $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Moreover, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ does also not belong in the Jacobson radical being a unit with inverse $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. We, consequently, conclude that $\mathbb{M}_2(R)$ is not WJU. □

We close the work with three questions of some interest.

Problem 1. Find a criterion when the triangular matrix $n \times n$ ring $\mathbb{T}_n(R)$ is a JU ring or a WJU ring.

Problem 2. Describe *UJI* rings R with the property

$$U(R) = Inv(R) \cap Z(R) + J(R),$$

where $Inv(R)$ is the set of all involutions in $U(R)$ and $Z(R)$ is the center of R.

Problem 3. Describe weakly exchange WUU rings. Are they weakly clean WUU? Does it follow that $6 \in J(R)$ or $10 \in J(R)$ in such a ring R?

Corrections. In [9], on p. 113, line 2 the phrase "$3 + e$ is a nilpotent" should be written and read as "$3+e$ *is an idempotent*". Moreover, on p. 70, line -4 in [8] the word "which" should be read as "whose".

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