μ-LINDELÖFNESS IN TERMS OF A HEREDITARY CLASS

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Abstract. A hereditary class on a set X is a nonempty collection of subsets of X closed under the hereditary property. In this paper, we define and study the notion of Lindelöfness in generalized topological spaces with respect to a hereditary class called, $\mu\mathcal{H}$-Lindelöf spaces and discuss their properties.

1. Introduction

The Lindelöfness is an important and interesting concept in general topology. This paper will not only study general topology, but also other areas of mathematics. During the last few years several authors have been working to formulate weak notions of open sets. In terms of these open sets those authors have extended and generalized the concept of Lindelöfness. The purpose of the present paper is to introduce and investigate the concept of μ-Lindelöfness by using the notions of generalized topology and hereditary class which are introduced by Császár in [1] and [2], respectively. Also some properties of $\mu\mathcal{H}$-Lindelöfness spaces are obtained. The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many authors such as [2, 9, 12, 17], among others.

2. Preliminaries

Let X be a non-empty set and 2^X denote the power set of X. We call a class $\mu \subseteq 2^X$ a generalized topology [1] (briefly, GT) if $\phi \in \mu$ and arbitrary union of elements of μ belongs to μ. A set X with a GT is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [1, 3]). Let $A \subset X$. A family $\mathcal{C}$ of subsets of X is called a μ-covering of A if $\mathcal{C}$ is a covering of A by μ-open sets [8]. A subset A of X is said to be μ-Lindelöf relative to X if for every μ-covering $\{U_\lambda : \lambda \in \Lambda\}$ of A there exists a countable subfamily

$\{U_\lambda : \lambda \in \Lambda_0\}$ such that it covers A. X is said to be μ-Lindelöf if X is μ-Lindelöf as a subset [15].

A nonempty family $\mathcal{H}$ of subsets of X is called a hereditary class [2] if $A \in \mathcal{H}$ and $B \subset A$ imply that $B \in \mathcal{H}$. Given a generalized topological space (X, μ) with a hereditary class $\mathcal{H}$, for a subset A of X, the generalized local function of A with respect to $\mathcal{H}$ and μ [2] is defined as follows: $A^\star = \{x \in X : U \cap A \notin \mathcal{H} \quad \text{for all} \quad U \in \mu_x\}$, where $\mu_x = \{U : x \in U \quad \text{and} \quad U \in \mu\}$; and the following are defined: $c^\star_\mu(A) = A \cup A^\star$ and the family $\mu^\star = \{A \subset X : X \setminus A = c^\star_\mu(X \setminus A)\}$. If the hereditary class $\mathcal{H}$ satisfies the additional condition: if $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$, then $\mathcal{H}$ is called an ideal on X [10]. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space and briefly we denote it by HGTS. If there is no confusion, we simply write $A^\star$ instead of $A^\star(\mathcal{H}, \mu)$. It is clear that a subset A is $\mu^\star$-closed if and only if $A^\star \subset A$.

Definition 2.1. [1] *Let (X, μ) and (Y, ν) be two GTSs, then a function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-continuous if $U \in \nu$ implies $f^{-1}(U) \in \mu$.*

Definition 2.2. [16] *A function $f : (X, \mu) \to (Y, \nu)$ is (μ, ν)-open (or μ-open) if $U \in \mu$ implies $f(U) \in \nu$.*

Definition 2.3. *Let (X, μ) be a GTS. Then a subset A of X is said to be μ-dense* [6] *if $c_\mu(A) = X$. The space (X, μ) is said to be μ-submaximal* [7] *if every μ-dense subset is μ-open in X.*

Definition 2.4. *Let (X, μ) be a GTS. Then a subset A of X is called a μ-generalized closed set (in short, a μg-closed set)*[13] *if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ where U is μ-open in X. The complement of a μg-closed set is called a μg-open set.*

Definition 2.5. [4] *A GTS (X, μ) is said to be μ-extremally disconnected if the μ-closure of every μ-open set is μ-open.*

Theorem 2.6. [2] *Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X and A a subset of X, then $A^\star \subset c_\mu(A)$.*

Theorem 2.7. [2] *Let (X, μ) be a GTS, $\mathcal{H}$ a hereditary class on X and A be a subset of X. If A is $\mu^\star$-open, then for each $x \in A$ there exist $U \in \mu_x$ and $H \in \mathcal{H}$ such that $x \in U \setminus H \subset A$.*

3. $\mu\mathcal{H}$-Lindelöf Spaces

Definition 3.1. *Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X. A HGTS $(X, \mu, \mathcal{H})$ is said to be $\mu\mathcal{H}$-Lindelöf or μ-Lindelöf with respect to a hereditary class $\mathcal{H}$ if for every μ-covering $\{U_\lambda : \lambda \in \Lambda\}$ of X there exists a countable subset Λ_0 of Λ such that $X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$.*

The following theorem gives a characterization of $\mu\mathcal{H}$-Lindelöfness.

Theorem 3.2. *The following are equivalent for a HGTS $(X,\mu,\mathcal{H})$:*

(1) *$(X,\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf;*
(2) *For any family $\{F_\lambda : \lambda \in \Lambda\}$ of μ-closed sets of X such that $\cap\{F_\lambda : \lambda \in \Lambda\} = \phi$, there exists a countable subset Λ_0 of Λ such that $\cap\{F_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of μ-closed sets of X such that $\cap\{F_\lambda : \lambda \in \Lambda\} = \phi$. Then $\{X \setminus F_\lambda : \lambda \in \Lambda\}$ is a μ-covering of X. By (1) $(X,\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $X \setminus \cup\{X \setminus F_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$. This implies that $\cap\{F_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$.
(2) $\Rightarrow$ (1): Let $\{U_\lambda : \lambda \in \Lambda\}$ be any μ-covering of X, then $\{X \setminus U_\lambda : \lambda \in \Lambda\}$ is a family of μ-closed sets and $\cap\{X \setminus U_\lambda : \lambda \in \Lambda\} = \phi$. Hence, there exists a countable subset Λ_0 of Λ such that $\cap\{X \setminus U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$. This implies that $\cap\{X \setminus U_\lambda : \lambda \in \Lambda_0\} = X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$. This shows that $(X,\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf. □

Theorem 3.3. *Let (X,μ) be a GTS and $\mathcal{H}$ be a hereditary class on X. Then, the following statements hold.*

(1) *If a HGTS $(X,\mu^*, \mathcal{H})$ is $\mu^*\mathcal{H}$-Lindelöf, then $(X,\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf.*
(2) *If a HGTS $(X,\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf and the class $\mathcal{H}$ is closed under countable union, then the HGTS $(X,\mu^*,\mathcal{H})$ is $\mu^*\mathcal{H}$-Lindelöf.*

Proof. (1): The proof follows directly from the fact that every μ-closed set is μ^*-closed set.
(2): Suppose that $\mathcal{H}$ is closed under countable union and X is $\mu\mathcal{H}$-Lindelöf. Given $\{U_\lambda : \lambda \in \Lambda\}$ a $\mu^\star$-covering of X, then for each $x \in X$, $x \in U_{\lambda_x}$ for some $\lambda_x \in \Lambda$. By Theorem 2.6, there exist $V_{\lambda_x} \in \mu_x$ and $H_{\lambda_x} \in \mathcal{H}$ such that $x \in V_{\lambda_x} \setminus H_{\lambda_x} \subset U_{\lambda_x}$. Since the family $\{V_{\lambda_x} : \lambda_x \in \Lambda\}$ is a μ-covering of X, it follows that there exists a countable subset Λ_0 of Λ such that $H = X \setminus \cup\{V_{\lambda_x} : \lambda_x \in \Lambda_0\} \in \mathcal{H}$. Since $\mathcal{H}$ is closed under countable union, then $\cup\{H_{\lambda_x} : \lambda_x \in \Lambda_0\} \in \mathcal{H}$. Hence, $H \cup [\cup\{H_{\lambda_x} : \lambda_x \in \Lambda_0\}] \in \mathcal{H}$. Observe that $X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \subset H \cup [\cup\{H_{\lambda_x} : \lambda_x \in \Lambda_0\}] \in \mathcal{H}$. By the heredity property of the class H we have $X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$ and therefore, $(X,\mu^*,\mathcal{H})$ is $\mu^*\mathcal{H}$-Lindelöf. □

Remark 3.4. *To show that the assumption $\mathcal{H}$ is closed under countable union in (2) of Theorem 3.3 is required in the hypotheses, we consider the following example.*

Example 3.5. *Let $\mathbb{R}$ be the set of real numbers and μ be the generalized topology defined as*

$$\mu = \{A \subset \mathbb{R} : A \quad is \quad an \quad uncountable \quad set\} \cup \{\phi\}.$$

The hereditary class on $\mathbb{R}$ *is defined as*

$$\mathcal{H} = \{\mathbb{R} \setminus A : A \in \mu\}.$$

Observe that $\mathcal{H}$ *is not an ideal. To show that* $\mathbb{R}$ *is* $\mu\mathcal{H}$*-Lindelöf, let* $\{U_\lambda : \lambda \in \Lambda\}$ *be any* μ*-covering of* $\mathbb{R}$*. Any countable subfamily* $\{U_\lambda : \lambda \in \Lambda_0\}$*, where* $\Lambda_0 \subset \Lambda$*. We obtain,* $\mathbb{R} \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \subset \mathbb{R} \setminus U_\lambda \in \mathcal{H}$*. It follows that* $\mathbb{R}$ *is* $\mu\mathcal{H}$*-Lindelöf. Observe that for each* $x \in \mathbb{R}$ *we have* $(\mathbb{R} \setminus \{x\})^\star \subset \mathbb{R} \setminus \{x\}$*. Thus,* $\{x\}$ *is* $\mu^\star$*-open for each* $x \in \mathbb{R}$*. It then follows that* $\{\{x\} : x \in \mathbb{R}\}$ *is a* μ^**-covering of* $\mathbb{R}$*. Suppose now that there exist countable* $x_1, x_2, \ldots, x_n, \ldots \in \mathbb{R}$ *such that* $\mathbb{R} \setminus \bigcup_{i=1}^{\infty} \{x_i\} \in \mathcal{H}$*. But this is impossible. Therefore,* $\mathbb{R}$ *is not* $\mu^*\mathcal{H}$*-Lindelöf.*

Given a generalized topological space (X, μ), we denote by $\mathcal{H}_c$ the hereditary class of countable subsets of X. The following proposition is obvious and thus the proof are omitted.

Proposition 3.6. *Let* (X, μ) *be a* GTS *and* $\mathcal{H}$ *be a hereditary class on* X*. Then the following statements are equivalent:*

(1) (X, μ) *is* μ*-Lindelöf;*
(2) $(X, \mu, \mathcal{H}_c)$ *is* $\mu\mathcal{H}_c$*-Lindelöf;*
(3) $(X, \mu, \{\phi\})$ *is* $\mu\{\phi\}$*-Lindelöf.*

Corollary 3.7. *If the HGTS* $(X, \mu, \mathcal{H}_c)$ *is* $\mu\mathcal{H}_c$*-compact, then* (X, μ) *is* $\mu\mathcal{H}_c$*-Lindelöf.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a μ-covering of X. Since $(X, \mu, \mathcal{H}_c)$ is $\mu\mathcal{H}_c$-compact, there exists a finite subset Λ_0 of Λ such that $X \setminus \{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}_C$ This shows that X has a countable subcover and the proof is completed. □

Proposition 3.8. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two hereditary classes on a GTS* (X, μ) *with* $\mathcal{H}_1 \subseteq \mathcal{H}_2$*. If* $(X, \mu, \mathcal{H}_1)$ *is* $\mu\mathcal{H}_1$*-Lindelöf, then* $(X, \mu, \mathcal{H}_2)$ *is* $\mu\mathcal{H}_2$*-Lindelöf.*

Proof. Suppose that $(X, \mu, \mathcal{H}_1)$ is $\mu\mathcal{H}_1$-Lindelöf. Let $\{U_\lambda : \lambda \in \Lambda\}$ be any μ-covering of X. There exits a countable subset Λ_0 of Λ such that $X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}_1 \subset \mathcal{H}_2$. This implies that $X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}_2$. Hence, X is $\mu\mathcal{H}_2$-Lindelöf. □

Remark 3.9. *The intersection of any two hereditary classes on a nonempty set* X *is a heredity class. To prove this, let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be any two hereditary classes on* X*. If* $A \in \mathcal{H}_1 \cap \mathcal{H}_2$ *and* $B \subset A$ *then* $B \subset A \in \mathcal{H}_1$ *and* $B \subset A \in \mathcal{H}_2$*. By the hypotheses on* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *we have* $B \in \mathcal{H}_1$ *and* $B \in \mathcal{H}_2$*. It then follows that* $B \in \mathcal{H}_1 \cap \mathcal{H}_2$*.*

Corollary 3.10. *If $(X, \tau, \mathcal{H}_1 \cap \mathcal{H}_2)$ is $\mu(\mathcal{H}_1 \cap \mathcal{H}_2)$- Lindelöf, then $(X, \mu, \mathcal{H}_1)$ is $\mu\mathcal{H}_1$-Lindelöf and $(X, \mu, \mathcal{H}_2)$ is $\mu\mathcal{H}_2$-Lindelöf.*

A subset A of GTS is said to be μ-semi-open if $A \subset c_\mu(i_\mu(A))$ and we denote by $\sigma(\mu)$ the class of all μ-semi-open sets [3].

Definition 3.11. *A HGTS $(X, \mu, \mathcal{H})$ is said to be $\mu\mathcal{H}$-semi-Lindelöf if for every μ-semi-open cover $\{U_\lambda : \lambda \in \Lambda\}$ of X there exists a countable subset Λ_0 of Λ such that $X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$.*

Proposition 3.12. *Every $\mu\mathcal{H}$-semi-Lindelöf space is $\mu\mathcal{H}$-Lindelöf.*

Proof. The proof is obvious since every μ-open set is μ-semi-open. □

Proposition 3.13. *If (X, μ) is μ-submaximal and extremally disconnected, then $\mu\mathcal{H}$-Lindelöf and $\mu\mathcal{H}$-semi-Lindelöf are equivalent.*

Proof. The proof comes immediately from the fact that in the extremely disconnected μ-submaximal space $\mu = \sigma(\mu)$. □

The following lemma is very useful in studying the preservation of $\mu\mathcal{H}$-Lindelöfness by certain classes of functions.

Lemma 3.14. [5] *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a function. If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H}) = \{f(E) : E \in \mathcal{H}\}$ is a hereditary class on Y.*

Theorem 3.15. *If $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ is a (μ, ν)-continuous surjection and $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf, then $(Y, \nu, f(\mathcal{H}))$ is $\nu f(\mathcal{H})$-Lindelöf.*

Proof. Let $\{V_\lambda : \lambda \in \Lambda\}$ be a ν-covering of Y. Then $\{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is a μ-covering of X and hence, there exists a countable subset Λ_0 of Λ such that $X \setminus \cup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\} \in \mathcal{H}$. Since f is a surjective function, by Lemma 3.14 we have $Y \setminus \cup\{V_\lambda : \lambda \in \Lambda_0\} \subset f(X \setminus \cup\{f^{-1}(V_\lambda) : \lambda \in \Lambda_0\}) \in f(\mathcal{H})$ implying thereby that Y is $\nu f(\mathcal{H})$-Lindelöf. □

By taking $\mathcal{H} = \{\phi\}$ in the above theorem, we get the well-known result that μ-Lindelöf is preserved by (μ, ν)-continuous surjections. We also have the following results concerning the pre-images.

Corollary 3.16. *If $f : (X, \mu) \to (Y, \nu, \mathcal{H})$ is a μ-open bijection and $(Y, \nu, \mathcal{H})$ is $\nu\mathcal{H}$- Lindelöf, then $(X, \mu, f^{-1}(\mathcal{H}))$ is $\mu f^{-1}(\mathcal{H})$-Lindelöf.*

Proof. Since $f : (X, \mu) \to (Y, \nu, \mathcal{H})$ is a μ-open bijection, $f^{-1} : (Y, \nu, \mathcal{H}) \to (X, \mu)$ is a (ν, μ)-continuous surjection. Since $(Y, \nu, \mathcal{H})$ is $\nu\mathcal{H}$- Lindelöf, by Theorem 3.15 we obtain $(X, \mu, f^{-1}(\mathcal{H}))$ is $\mu f^{-1}(\mathcal{H})$-Lindelöf. □

4. Sets $\mu\mathcal{H}$-Lindelöf Relative to a Space

Definition 4.1. *Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X. A subset A of X is said to be $\mu\mathcal{H}$-Lindelöf or μ-Lindelöf with respect to $(X, \mu, \mathcal{H})$ if for every μ-covering $\{U_\lambda : \lambda \in \Lambda\}$ of A there exists a countable subset Λ_0 of Λ such that $A \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$.*

Definition 4.2. *A subset A of GTS (X, μ) is said to be ω-μ-open if for each $x \in A$, there exists $U_x \in \mu$ containing x such that $U_x \setminus A$ is a countable set. The complement of an ω-μ-open set is said to be ω-μ-closed. The family of all ω-μ-open sets of (X, μ) is denoted by μ_ω.*

Lemma 4.3. *For any GTS (X, μ), the family μ_ω is GT.*

Proof. It is obvious that $\phi, X \in \mu_\omega$. Let $\{A_\lambda : \lambda \in \Lambda\}$ be any subfamily of μ_ω. Then for each $x \in \bigcup\limits_{\lambda\in\Lambda} A_\lambda$, there exists $\lambda_0 \in \Lambda$ such that $x \in A_{\lambda_0}$. Since $A_{\lambda_0} \in \mu_\omega$, there exists $U_x \in \mu$ containing x such that $U_x \setminus A_{\lambda_0}$ is a countable set. Since $U_x \setminus (\bigcup\limits_{\lambda\in\Lambda} A_\lambda) \subset U_x \setminus A_{\lambda_0}$, $U_x \setminus (\bigcup\limits_{\lambda\in\Lambda} A_\lambda)$ is a countable set. Therefore, $\bigcup\limits_{\lambda\in\Lambda} A_\lambda \in \mu_\omega$. This shows that (X, μ_ω) is GTS. □

Theorem 4.4. *A subset A of a HGTS $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf relative to μ if and only if A is $\mu_\omega\mathcal{H}$-Lindelöf relative to μ_ω.*

Proof. Necessity. Suppose that a subset A of a HGTS $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf relative to μ. Let $\{U_\lambda : \lambda \in \Lambda\}$ be any μ_ω-covering of A. For each $x \in A$ there exists $\lambda(x) \in \Lambda$ such that $x \in U_{\lambda(x)}$. Since $U_{\lambda(x)}$ is μ_ω-open, there exists a μ-open set $V_{\lambda(x)}$ such that $x \in V_{\lambda(x)}$ and $V_{\lambda(x)} \setminus U_{\lambda(x)}$ is countable. The collection $\{V_{\lambda(x)} : x \in A\}$ is a μ-covering of A. Since A is $\mu\mathcal{H}$-Lindelöf relative to μ, there exists a countable subset, $\lambda(x_1), \lambda(x_2), \ldots, \lambda(x_n), \ldots$ such that $A \setminus \cup\left\{V_{\lambda(x_i)} : i \in \mathbb{N}\right\} \in \mathcal{H}$. On the other hand, we have

$$A \setminus \bigcup_{i\in\mathbb{N}} \left\{\left(V_{\lambda(x_i)} \setminus U_{\lambda(x_i)}\right) \cup U_{\lambda(x_i)}\right\} \subset A \setminus \cup\left\{V_{\lambda(x_i)} : i \in \mathbb{N}\right\}$$

and hence,

$$A \setminus [\bigcup_{i\in\mathbb{N}}(V_{\lambda(x_i)} \setminus U_{\lambda(x_i)}) \cap A) \cup (\bigcup_{i\in\mathbb{N}} U_{\lambda(x_i)})] \subset A \setminus \cup\left\{V_{\lambda(x_i)} : i \in \mathbb{N}\right\}.$$

For each $\lambda(x_i)$, the set $(V_{\lambda(x_i)} \setminus U_{\lambda(x_i)}) \cap A$ is a countable set and there exists a countable subset $\Lambda_{\lambda(x_i)}$ of Λ such that $(V_{\lambda(x_i)} \setminus U_{\lambda(x_i)}) \cap A \subseteq$

$\cup\{U_\lambda : \lambda \in \Lambda_{\lambda(x_i)}\}$. Therefore, we have

$$\begin{aligned} &A \setminus \left[\left(\bigcup_{i\in\mathbb{N}} \left(\cup\{U_\lambda : \lambda \in \Lambda_{\lambda(x_i)}\}\right)\right) \cup \left(\bigcup_{i\in\mathbb{N}} U_{\lambda(x_i)}\right)\right] \\ &\subset A \setminus \left[\left(\bigcup_{i\in\mathbb{N}} (V_{\lambda(x_i)} \setminus U_{\lambda(x_i)}) \cap A\right) \cup \left(\bigcup_{i\in\mathbb{N}} U_{\lambda(x_i)}\right)\right] \\ &\subset A \setminus \cup\{V_{\lambda(x_i)} : i \in \mathbb{N}\} \in \mathcal{H}. \end{aligned}$$

By the hereditary property of the class $\mathcal{H}$, we have

$$A \setminus \left[\left(\bigcup_{i\in\mathbb{N}} \left(\cup\{U_\lambda : \lambda \in \Gamma_{\lambda(x_i)}\}\right)\right) \cup \left(\bigcup_{i\in\mathbb{N}} U_{\lambda(x_i)}\right)\right] \in \mathcal{H}.$$

Sufficiency. Since $\mu \subset \mu_\omega$ the proof is obvious. □

Corollary 4.5. *A HGTS $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf if and only if the HGTS $(X, \mu_\omega, \mathcal{H})$ is $\mu_\omega\mathcal{H}$-Lindelöf.*

By taking $\mathcal{H} = \{\phi\}$ in the Corollary 4.5, we obtain the following result established in Theorem 4.4.

Theorem 4.6. *A GTS (X, μ) is μ-Lindelöf if and only if (X, μ_ω) is μ_ω-Lindelöf.*

Definition 4.7. [5] *Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X. A subset A of X is said to be $\mu\mathcal{H}$-compact if for every μ-covering $\{U_\lambda : \lambda \in \Lambda\}$ of A there exists a finite subcollection $\{U_\lambda : \lambda \in \Lambda_0\}$ such that $A \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$. X is said to be a $\mu\mathcal{H}$-compact space if X is $\mu\mathcal{H}$-compact as a subset.*

A GTS (X, μ) is said to be μ-Hausdroff [14] for each pair of distinct points x and y in X, there exist μ-open sets U_x and V_y containing x and y, respectively, such that $U \cap V = \phi$.

Lemma 4.8. *$x \notin A^*$ if and only if $(U_x \setminus H) \cap A = \phi$, where $U_x \in \mu(x)$ and $H \in \mathcal{H}$.*

Proof. Let $x \notin A^*$. Then there exist $U_x \in \mu(x)$ such that $A \cap U_x = H \in \mathcal{H}$. It follows that $(U_x \setminus H) \cap A = \phi$. Conversely, suppose that $(U_x \setminus H) \cap A = \phi$ for some $U_x \in \mu(x)$ and $H \in \mathcal{H}$. Then $U_x \cap (X \setminus H) \cap A = (A \cap U_x) \cap (X \setminus H) = \phi$. This implies that $(A \cap U_x) \subset H \in \mathcal{H}$. Hence, $x \notin A^*$. □

Theorem 4.9. *Every $\mu\mathcal{H}$-compact subset of a μ-Hausdorff HGTS $(X, \mu, \mathcal{H})$ is μ^*-closed.*

Proof. Let A be a $\mu\mathcal{H}$-compact subset of a μ-Hausdroff HGTS $(X, \mu, \mathcal{H})$. Let $x \notin A$ then $x \in X \setminus A$. For each $y \in A$, there exist two μ-open sets U_y and V_y containing x and y, respectively, such that $U_y \cap V_y = \phi$. Note that $x \notin c_\mu(V_y)$. Then $\{V_y : y \in A\}$ is a μ-covering of A which is $\mu\mathcal{H}$- compact.

Therefore, there exists a finite subset Λ_0 of A such that $A \backslash \cup\{V_y : y \in \Lambda_0\} \in \mathcal{H}$. Now $x \notin c_\mu(V_y)$ for each $y \in A$ implies $x \notin \bigcup_{y\in\Lambda_0} c_\mu(V_y) = c_\mu(\bigcup_{y\in\Lambda_0} V_y)$. Let $U = X \setminus c_\mu(\bigcup_{y\in\Lambda_0} V_y)$ and let $H = A \setminus c_\mu(\bigcup_{y\in\Lambda_0} V_y) \subset A \setminus \bigcup_{y\in\Lambda_0} V_y = H_1$, where $H_1 \in \mathcal{H}$. Since $U \in \mu_x$ and $H \in \mathcal{H}$, by Theorem 2.7 $U \setminus H$ is a μ^*-open set containing x and $(U \setminus H) \cap A = \emptyset$. This implies that $x \notin A^*$ (by Lemma 4.8). Hence, $A^* \subset A$, so A is μ^*-closed. □

Theorem 4.10. *Let (X, μ) be a GTS and $\mathcal{H}$ be an ideal on X, then a finite union of sets which are $\mu\mathcal{H}$- Lindelöf relative to a space $(X, \mu, \mathcal{H})$ is a $\mu\mathcal{H}$-Lindelöf relative to X.*

Proof. Let A_1 and A_2 be two subsets which are $\mu\mathcal{H}$-Lindelöf relative to X and let $A = A_1 \cup A_2$. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a μ-covering of A. Hence, $\{U_\lambda : \lambda \in \Lambda\}$ is a μ-covering of A_1 and A_2. Since A_1 and A_2 are $\mu\mathcal{H}$-Lindelöf relative to X, there exist finite subfamily $\{H_1, H_2\} \subset \mathcal{H}$ and countable subsets Λ_0 and Λ_1 of Λ such that $A_1 \setminus \cup\{U_{\lambda_i} : \lambda_i \in \Lambda_0\} = H_1$ and $A_2 \setminus \cup\{U_{\lambda_k} : \lambda_k \in \Lambda_1\} = H_2$. Now we have

$$\begin{aligned} A &= A_1 \cup A_2 \\ &\subset (\cup\{U_{\lambda_i} : \lambda_i \in \Lambda_0\}) \cup (\cup\{U_{\lambda_k} : \lambda_k \in \Lambda_1\}) \cup (H_1 \cup H_2). \end{aligned}$$

This implies $(A_1 \cup A_2) \setminus (\cup\{U_{\lambda_i} : \lambda_i \in \Lambda_0\}) \cup (\cup\{U_{\lambda_k} : \lambda_k \in \Lambda_1\}) \subset H_1 \cup H_2 \in \mathcal{H}$. Therefore, $A = A_1 \cup A_2$ is $\mu\mathcal{H}$- Lindelöf relative to X. This proves that the union of two $\mu\mathcal{H}$-Lindelöf sets is $\mu\mathcal{H}$-Lindelöf. For finite unions, the proof proceeds by induction on the number of sets. □

Remark 4.11. *If the class $\mathcal{H}$ is not an ideal then the union of finite subsets which are $\mu\mathcal{H}$-Lindelöf relative to X is not $\mu\mathcal{H}$-Lindelöf relative to X.*

Let $(X, \mu, \mathcal{H})$ be a HGTS and let $A \subseteq X$, $A \neq \phi$. We denote by $\mathcal{H}_A$ the collection $\{H \cap A : H \in \mathcal{H}\}$ and by (A, μ_A) the subspace of (X, μ) on A. It is clear that the collection μ_A is a generalized topology on A and the collection $\mathcal{H}_A$ is a hereditary class of subsets in A. Then we have the following theorem.

Theorem 4.12. *Let $(X, \mu, \mathcal{H})$ be a $\mu\mathcal{H}$-Lindelöf HGTS and A be a μ-closed subset of X. Then $(A, \mu_A, \mathcal{H}_A)$ is $\mu_A\mathcal{H}_A$-Lindelöf.*

Proof. Let $\{U_\lambda \cap A : U_\lambda \in \mu, \lambda \in \Lambda\}$ be a μ_A-covering of A. Then $\{U_\lambda : U_\lambda \in \mu, \lambda \in \Lambda\} \cup \{X \setminus A\}$ is a μ-covering of X and hence, there exists a countable subset Λ_0 of Λ such that $X \setminus [\cup\{U_\lambda : \lambda \in \Lambda_0\} \cup (X \setminus A)] = H \in \mathcal{H}$.

Now, we have

$$\begin{aligned} A \cap H &= A \cap (X \setminus [\cup\{U_\lambda : \lambda \in \Lambda_0\} \cup (X \setminus A)]) \\ &= A \cap (X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\}) \cap A \\ &= A \cap (X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\}) = A \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \\ &= A \setminus (A \cap [\cup\{U_\lambda : \lambda \in \Lambda_0\}]) = A \setminus \cup\{U_\lambda \cap A : \lambda \in \Lambda_0\} \end{aligned}$$

Therefore, we have $A \setminus \cup\{U_\lambda \cap A : \lambda \in \Lambda_0\} = A \cap H \in \mathcal{H}_A$. This shows that A is a $\mu\mathcal{H}_{\mathcal{A}}$-Lindelöf set. □

The well-known result that a μ-closed subspace of a μ-Lindelöf space is μ-Lindelöf which is a special case by taking $\mathcal{H} = \{\phi\}$.

Theorem 4.13. *Let $(X, \mu, \mathcal{H})$ be a HGTS and $A \subseteq X$. If for each μ-open set U containing A there is a $\mu_B\mathcal{H}_{\mathcal{B}}$-Lindelöf set B with $A \subset B \subset U$, then A is $\mu_A\mathcal{H}_{\mathcal{A}}$-Lindelöf.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a μ_A-covering of A, where $U_\lambda = V_\lambda \cap A$ such that $V_\lambda \in \mu$. By the given condition, there exists a $\mu_B\mathcal{H}_B$-Lindelöf set B with $A \subset B \subset \cup V_\lambda$. Then $\{V_\lambda \cap B : \lambda \in \Lambda\}$ is a μ_B-covering of B. By assumption B is $\mu_B\mathcal{H}_B$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $B \setminus \cup\{V_\lambda \cap B : \lambda \in \Lambda_0\} \in \mathcal{H}_B$. Let $B \setminus \cup\{V_\lambda \cap B : \lambda \in \Lambda_0\} = H \cap B$, where $H \cap B \in \mathcal{H}_B$ and $H \in \mathcal{H}$. Since $B = \cup\{V_\lambda \cap B : \lambda \in \Lambda_0\} \cup (H \cap B)$. Then $A \cap B = A \cap (\cup\{V_\lambda \cap B : \lambda \in \Lambda_0\} \cup (H \cap B)) = \cup\{V_\lambda \cap B \cap A : \lambda \in \Lambda_0\} \cup (H \cap B \cap A)$. This implies $A = \cup\{V_\lambda \cap A : \lambda \in \Lambda_0\} \cup (H \cap A)$. It follows that $A \setminus \cup\{V_\lambda \cap A : \lambda \in \Lambda_0\} \subseteq H \cap A \in \mathcal{H}_A$. Therefore, A is $\mathcal{H}_A$-Lindelöf. □

Theorem 4.14. *Every μg-closed subset of a $\mu\mathcal{H}$-Lindelöf space is $\mu\mathcal{H}$-Lindelöf relative to X.*

Proof. Let A be any μg-closed of $(X, \mu, \mathcal{H})$ and $\{U_\lambda : \lambda \in \Lambda\}$ be any cover of A by μ-open sets in X. Since A is μg-closed, $A \subset \cup U_\lambda$ implies $c_\mu(A) \subset \cup U_\lambda$. Then the family $\{U_\lambda : \lambda \in \Lambda\} \cup \{X \setminus c_\mu(A)\}$ is a μ-covering of X and hence, there exists a countable subset Λ_0 of Λ such that $X \setminus [\cup\{U_\lambda : \lambda \in \Lambda_0\} \cup (X \setminus c_\mu(A))] \in \mathcal{H}$. Now, we have

$$\begin{aligned} X \setminus [\cup\{U_\lambda : \lambda \in \Lambda_0\} \cup (X \setminus c_\mu(A))] &= [X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\}] \cap c_\mu(A) \\ &\supset [X \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\}] \cap A \\ &= A \setminus [\cup\{U_\lambda : \lambda \in \Lambda_0\}]. \end{aligned}$$

Therefore, we have $A \setminus [\cup\{U_\lambda : \lambda \in \Lambda_0\}] \in \mathcal{H}$. Thus, A is $\mu\mathcal{H}$-Lindelöf relative to X. □

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