ANOTHER ELEMENTARY PROOF OF THE CONVERGENCE-DIVERGENCE OF p-SERIES

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Recently Khan [2] gave a simple proof of the convergence-divergence of the p-series $\sum_{n=1}^{\infty} 1/n^p$. The divergence of this series for $p \leq 1$ was shown by contradiction while the convergence of the series for $p > 1$ was established by the boundedness of the monotonic partial sums [2]. Here, we give a more direct and very elementary proof of the same by using only the sum of a geometric series. Moreover, a telescoping method is used to find sums of some interesting series.

We use the following simple fact:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1. \tag{1}$$

We consider integers j from 2^m to $2^{m+1} - 1$ ($m = 0, 1, 2, \dots$), and note that the number of terms are $2^{m+1} - 1 - (2^m - 1) = 2^{m+1} - 2^m = 2^m$. Then, for any p we write the p-series as

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{m=0}^{\infty} \sum_{j=2^m}^{2^{m+1}-1} \frac{1}{j^p}. \tag{2}$$

If $p > 1$, it then follows from (1) and (2) that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{m=0}^{\infty} \frac{2^m}{2^{mp}} = \sum_{m=0}^{\infty} \frac{1}{2^{m(p-1)}} = \frac{2^{p-1}}{2^{p-1} - 1},$$

and the series converges. To show the divergence for $p \leq 1$, we consider $p = 1$ first. It is clear from (2) that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{m=0}^{\infty} \sum_{j=2^m}^{2^{m+1}-1} \frac{1}{j} \geq \sum_{m=0}^{\infty} \frac{2^m}{2^{m+1}} = \infty.$$

If $p \leq 1$, then $\sum_{n=1}^{\infty} 1/n^p \geq \sum_{n=1}^{\infty} 1/n = \infty$, and the series diverges. Hence, the convergence-divergence of the p-series has been established for every value of p.

The preceding observation can be further used to determine the convergence/divergence of some other series $\sum_{n=k}^{\infty} f(n)$ $(k \geq 1)$, where $f(x)$ is a decreasing positive function. For example, we have

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} = \sum_{m=1}^{\infty} \sum_{j=2^m}^{2^{m+1}-1} \frac{1}{j(\ln j)^p},$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \leq \sum_{m=1}^{\infty} \frac{2^m}{2^m(m\ln 2)^p} = \sum_{m=1}^{\infty} \frac{1}{c\; m^p}, \quad c = (\ln 2)^p,$$

and the series converges if $p > 1$ by the p-series. Moreover,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \geq \sum_{m=1}^{\infty} \frac{2^m}{2^{m+1} \ln 2^{m+1}} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{(m+1)\ln 2} = +\infty,$$

and thus, $\sum_{n=2}^{\infty} 1/(n(\ln n)^p)$ diverges for $p \leq 1$. Perhaps, even more interesting is the series $\sum_{n=2}^{\infty} 1/((\ln n)^p)$ for any $p > 0$. Clearly,

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} = \sum_{m=1}^{\infty} \sum_{j=2^m}^{2^{m+1}-1} \frac{1}{(\ln j)^p} \geq \sum_{m=1}^{\infty} \frac{2^m}{(\ln 2^{m+1})^p} = \sum_{m=1}^{\infty} \frac{2^m}{(m+1)^p(\ln 2)^p}.$$

Since $\lim_{m\to\infty} 2^m/(m^p) = \infty$ $(m - p\ln m = O(m)$ and $2^{m-p\ln m} \to \infty)$, the series $\sum_{n=2}^{\infty} 1/((\ln n)^p)$ diverges for every fixed p.

It is interesting to note that the telescoping method can be used to find the sum of the series $\sum_{n=1}^{\infty} n/(2^n)$ and $\sum_{n=1}^{\infty} n^2/(2^n)$, etc. However, as suggested by a colleague we consider a more general case, namely, $\sum_{n=1}^{\infty} n^k r^n$, $|r| < 1$. First we observe that the series converges absolutely. To see this let $r \neq 0$ and note that $|r|^n = \exp(-\alpha n)$, where $\alpha = -\ln|r| > 0$. Since $\exp(\alpha n) > (\alpha^{k+2} n^{k+2})/((k+2)!)$, hence, $n^k|r|^n = n^k \exp(-\alpha n) < \lambda_k/(n^2)$, for all $n \geq 1$, where $\lambda_k = ((k+2)!)/(\alpha^{k+2})$. Thus, $\sum_{n=1}^{\infty} n^k|r|^n \leq \lambda_k \sum_{n=1}^{\infty} 1/(n^2) < \infty$ by the p-series. Therefore, letting

$a_n = n^k r^n$, $\lim_{n\to\infty} n^k r^n = \lim_{n\to\infty} a_n = 0$. It then follows that $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges, and $\sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1$. Now set $S(k) = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^k r^n$. Clearly, $S(0) = r/(1-r)$, and applying the binomial theorem we have

$$a_n - a_{n+1} = n^k r^n - (n+1)^k r^{n+1} = (1-r)n^k r^n - \sum_{j=0}^{k-1}\binom{k}{j} n^j r^{n+1}.$$

Now summing over n and using the telescoping sum we obtain

$$r = a_1 = \sum_{n=1}^{\infty}(a_n - a_{n+1}) = \sum_{n=1}^{\infty}(1-r)n^k r^n - \sum_{n=1}^{\infty}\sum_{j=0}^{k-1} r\binom{k}{j} n^j r^n$$

$$= (1-r)\sum_{n=1}^{\infty} n^k r^n - r\sum_{j=0}^{k-1}\binom{k}{j}\sum_{n=1}^{\infty} n^j r^n = (1-r)S(k) - r\sum_{j=0}^{k-1}\binom{k}{j} S(j).$$

Hence, the preceding equation gives

$$S(k) = \frac{r}{1-r}\left(1 + \sum_{j=0}^{k-1}\binom{k}{j} S(j)\right) = S(0)\left(1 + \sum_{j=0}^{k-1}\binom{k}{j} S(j)\right).$$

In particular, for $r = 1/2$, using the recurrence we obtain

$$S(0) = \sum_{n=1}^{\infty}\frac{1}{2^n} = 1, \qquad S(1) = \sum_{n=1}^{\infty}\frac{n}{2^n} = 2,$$

$$S(2) = \sum_{n=1}^{\infty}\frac{n^2}{2^n} = 6, \qquad S(3) = \sum_{n=1}^{\infty}\frac{n^3}{2^n} = 26, \quad \text{etc.}$$

It is tempting to find a closed formula for $S(k)$, but unfortunately it appears that such a formula is intractable and cannot be obtained even for the special case $r = 1/2$.

References

1. T. Cohen and W. J. Knight, "Convergence and Divergence of $\sum_{n=1}^{\infty} 1/n^p$," *Mathematics Magazine,* 52 (1979), 178.

2. R. A. Khan, "Convergence-Divergence of p-Series," *College Mathematics Journal,* 32 (2001), 206–207.

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