SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

129. [1999, 196] *Proposed by Kenneth B. Davenport, 301 Morea Road, Box 491, Frackville, Pennsylvania.*

Let $k \geq 0$ and $i \geq 1$ be integers. Prove that

$$\sum_j \left\langle {k \atop j} \right\rangle \binom{k+i-j}{k+1} = \sum_{m=1}^{i} m^k,$$

where

$$\left\langle {k \atop j} \right\rangle$$

denotes an Eulerian number.

Solution by the proposer and Carl Libis, Antioch College, Yellow Springs, Ohio. Here, an Eulerian number

$$\left\langle {k \atop j} \right\rangle$$

is the number of permutations $\pi_1 \pi_2 \cdots \pi_k$ of $\{1, 2, \ldots, k\}$ that have j ascents, namely, j places where $\pi_i < \pi_{i+1}$. To prove this result we need Worpitsky's identity from R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics,* 2nd ed., Addison-Wesley Publishing Company, Reading, Massachusetts, 1994, p. 269, i.e.

$$m^k = \sum_j \left\langle {k \atop j} \right\rangle \binom{m+j}{k}$$

for integer $k \geq 0$. Using this identity, rearranging terms, changing index variables, and using properties of Eulerian numbers, we have

$$\begin{aligned}\sum_{m=1}^{i} m^k &= \sum_{m=1}^{i}\sum_{j}\left\langle {k \atop j} \right\rangle \binom{m+j}{k} = \sum_{j}\left\langle {k \atop j} \right\rangle \sum_{m=1}^{i}\binom{m+j}{k} \\ &= \sum_{j}\left\langle {k \atop j} \right\rangle \sum_{m=j+1}^{i+j}\binom{m}{k} = \sum_{j}\left\langle {k \atop j} \right\rangle \sum_{m=0}^{i+j}\binom{m}{k} \\ &= \sum_{j}\left\langle {k \atop j} \right\rangle \binom{i+j+1}{k+1} = \sum_{j}\left\langle {k \atop k-1-j} \right\rangle \binom{i+j+1}{k+1} \\ &= \sum_{j}\left\langle {k \atop j} \right\rangle \binom{k+i-j}{k+1}.\end{aligned}$$

130. [1999, 196] *Proposed by Joseph Wiener and William Heller, University of Texas-Pan American, Edinburg, Texas.*

Show that for any $b > 1$, the function

$$f(x) = \big(x^2 + (1-b)\big)e^x + bx$$

has exactly one zero for $x \geq 0$.

Solution I by Chris Farmer (student), Northwest Missouri State University, Maryville, Missouri. $f(x)$ is continuous and differentiable throughout its domain. Furthermore,

$$f(0) = [0^2 + (1-b)]e^0 + b(0) = 1 - b$$

$$f(\sqrt{b-1}) = [(\sqrt{b-1})^2 + (1-b)]e^{\sqrt{b-1}} + b\sqrt{b-1} = b\sqrt{b-1}.$$

Since $b > 1$, $f(0) = 1 - b < 0$ and $f(\sqrt{b-1}) = b\sqrt{b-1} > 0$. Therefore, by the Intermediate Value Theorem, there exists a c such that $0 < c < \sqrt{b-1}$ and $f(c) = 0$.

Consider $f(c)$ and $f(c+a)$, where $a > 0$.

$$f(c) = [c^2 + (1-b)]e^c + bc = 0.$$

$$f(c+a) = [(c+a)^2 + (1-b)]e^{c+a} + b(c+a)$$

$$= [c^2 + 2ac + a^2 + (1-b)]e^{c+a} + bc + ba$$

$$> [c^2 + (1-b) + 2ac + a^2]e^c + bc + ba$$

$$= [c^2 + (1-b)]e^c + bc + (a^2 + 2ac)e^c + ba$$

$$= f(c) + (a^2 + 2ac)e^c + ba$$

$$= (a^2 + 2ac)e^c + ba.$$

Since $a, b, c > 0$, $f(c+a) > 0$. Therefore, $f(x)$ has no zeros greater than c.

Suppose $f(x)$ has another zero d. Then $d \leq c$. Repeating the above argument, with d in place of c, we find that $f(x)$ has no zeros greater than d, so $c \leq d$. It follows that $c = d$.

Therefore, $f(x)$ has exactly one zero.

Also solved by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and the proposers.

131. [1999, 197] *Proposed by Kenneth B. Davenport, 301 Morea Road, Box 491, Frackville, Pennsylvania.*

Show that if

$$A = \sum_{n=0}^{\infty}\left(\frac{1}{9n+1} - \frac{1}{9n+4}\right), \qquad B = \sum_{n=0}^{\infty}\left(\frac{1}{9n+5} - \frac{1}{9n+8}\right),$$

$$C = \sum_{n=0}^{\infty}\left(\frac{1}{9n+2} - \frac{1}{9n+5}\right), \qquad D = \sum_{n=0}^{\infty}\left(\frac{1}{9n+4} - \frac{1}{9n+7}\right),$$

then $A + B = (C+D)\alpha$, where $\alpha = 2\cos(\pi/9)$.

Solution by the proposer. Begin by noting that

$$A + B = \int_0^1 \frac{1+x^4}{1+x^3+x^6}\,dx$$

and

$$C + D = \int_0^1 \frac{x+x^3}{1+x^3+x^6}\,dx. \tag{1}$$

Using the *Tables of Indefinite Integrals* by G. Petit Bois, Dover Publications, Inc., New York, 1961, p. 105, we have the formula

$$\int \frac{x^m}{a+bx^n+cx^{2n}}\,dx = \frac{1}{ncq^{2n-m-1}\sin\epsilon}\cdot$$
$$\sum_{k=0}^{n-1}\Bigg[-\sin(n-m-1)\epsilon_k\cdot\frac{1}{2}\cdot\ln(x^2-2qx\cos\epsilon_k+q^2)$$
$$+\cos(n-m-1)\epsilon_k\tan^{-1}\left[\frac{x\sin\epsilon_k}{q-x\cos\epsilon_k}\right]\Bigg], \tag{2}$$

where

$$q = \left(\frac{a}{c}\right)^{\frac{1}{2n}}, \quad \cos\epsilon = \frac{-b}{2\sqrt{ac}}, \quad \epsilon_k = \frac{2k\pi+\epsilon}{n},$$

and

$$b^2-4ac<0, \quad m<2n.$$

Since $a=b=c=1$, and $n=3$, (2) may be simplified and noting $\epsilon = 2\pi/3$,

$$\frac{2\sqrt{3}}{9}\sum_{k=0}^{2}\left[-\sin(2-m)\epsilon_k\cdot\frac{1}{2}\ln(2-2\cos\epsilon_k)+\cos(2-m)\epsilon_k\tan^{-1}\left[\frac{\sin\epsilon_k}{1-\cos\epsilon_k}\right]\right]$$

and we must have

$$\epsilon_0 = \frac{2\pi}{9}, \quad \epsilon_1 = \frac{8\pi}{9}, \quad \text{and} \quad \epsilon_2 = \frac{14\pi}{9}. \tag{3}$$

Taking the A sum with $m = 0$ we have

$$A = \frac{2\sqrt{3}}{9} \sum_{k=0}^{2} \left[-\sin(2\epsilon_k) \cdot \frac{1}{2} \ln(2 - 2\cos\epsilon_k) + \cos(2\epsilon_k) \tan^{-1} \left[\frac{\sin\epsilon_k}{1 - \cos\epsilon_k} \right] \right] \tag{4}$$

$$= \frac{2\sqrt{3}}{9} \left[-\sin\frac{4\pi}{9} \cdot \frac{1}{2} \cdot \ln\left(2 - 2\cos\frac{2\pi}{9}\right) - \sin\frac{16\pi}{9} \cdot \frac{1}{2} \ln\left(2 - 2\cos\frac{8\pi}{9}\right) \right.$$

$$\left. - \sin\frac{28\pi}{9} \cdot \frac{1}{2} \ln\left(2 - 2\cos\frac{14\pi}{9}\right) + \cos\frac{4\pi}{9} \cdot \frac{7\pi}{18} + \cos\frac{8 \cdot 2\pi}{9} \cdot \frac{\pi}{18} - \cos\frac{28\pi}{9} \cdot \frac{5\pi}{18} \right]$$

$$B = \frac{2\sqrt{3}}{9} \sum_{k=0}^{2} \left[-\sin(-2\epsilon_k) \cdot \frac{1}{2} \ln(2 - 2\cos\epsilon_k) + \cos(-2\epsilon_k) \tan^{-1} \left[\frac{\sin\epsilon_k}{1 - \cos\epsilon_k} \right] \right] \tag{5}$$

for $m = 4$; but since B differs only in the sign of the logarithmic part of its sum, then $A + B$ will cancel the logarithmic parts of the summation and so we must have

$$A + B = \frac{2\sqrt{3}}{9} \cdot 2 \left[\frac{7\pi}{18} \cos\frac{4\pi}{9} + \frac{\pi}{18} \cos\frac{16\pi}{9} - \frac{5\pi}{18} \cos\frac{28\pi}{9} \right]. \tag{6}$$

Now taking the C sum with $m = 1$ we have

$$C = \frac{2\sqrt{3}}{9} \sum_{k=0}^{2} -\sin(\epsilon_k) \cdot \frac{1}{2} \ln(2 - 2\cos\epsilon_k) + \cos(\epsilon_k) \tan^{-1} \left[\frac{\sin\epsilon_k}{1 - \cos\epsilon_k} \right] \tag{7}$$

$$= \frac{2\sqrt{3}}{9} \left[-\sin\frac{2\pi}{9} \cdot \frac{1}{2} \cdot \ln\left(2 - 2\cos\frac{2\pi}{9}\right) - \sin\frac{8\pi}{9} \cdot \frac{1}{2} \ln\left(2 - 2\cos\frac{8\pi}{9}\right) \right.$$

$$\left. - \sin\frac{14\pi}{9} \cdot \frac{1}{2} \ln\left(2 - 2\cos\frac{14\pi}{9}\right) + \frac{7\pi}{18} \cdot \cos\frac{2\pi}{9} + \frac{\pi}{18} \cdot \cos\frac{8\pi}{9} - \frac{5\pi}{18} \cos\frac{14\pi}{9} \right].$$

And the D sum with $m = 3$ yields

$$D = \frac{2\sqrt{3}}{9} \sum_{k=0}^{2} \left[-\sin(-\epsilon_k) \cdot \frac{1}{2} \ln(2 - 2\cos\epsilon_k) + \cos(-\epsilon_k) \tan^{-1} \left[\frac{\sin\epsilon_k}{1 - \cos\epsilon_k} \right] \right]. \quad (8)$$

As in (5), D differs only in the sign of the logarithmic part of its sum, and so $C + D$ will cancel the logarithmic parts of the summation and you will have

$$C + D = \frac{2\sqrt{3}}{9} \cdot 2 \left[\frac{7\pi}{18} \cos\frac{2\pi}{9} + \frac{\pi}{18} \cos\frac{8\pi}{9} - \frac{5\pi}{18} \cos\frac{14\pi}{9} \right]. \quad (9)$$

It now remains to show that

$$A + B = (C + D)\alpha.$$

(6) may be simplified to

$$\frac{2\sqrt{3}\pi}{18} \left[5\cos\frac{\pi}{9} + 7\cos\frac{4\pi}{9} - \cos\frac{7\pi}{9} \right] \quad (10)$$

and now simplifying (9) and multiplying through by α we have

$$(C + D)\alpha = \frac{2\sqrt{3}\pi}{81} \left[14\cos\frac{\pi}{9} \cos\frac{2\pi}{9} + 2\cos\frac{\pi}{9} \cos\frac{8\pi}{9} - 10\cos\frac{\pi}{9} \cos\frac{14\pi}{9} \right] \quad (11)$$

$$= -7\left(\cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} \right) + \cos\frac{7\pi}{9} - 1 + 5\left(\cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} \right)$$

$$= \frac{7}{2} - 7\cos\frac{8\pi}{9} + \cos\frac{7\pi}{9} - 1 + 5\cos\frac{4\pi}{9} - \frac{5}{2} = 7\cos\frac{\pi}{9} + 5\cos\frac{4\pi}{9} + \cos\frac{7\pi}{9}.$$

This simplification was reached after using product and half-angle formulas from the *Handbook of Mathematical Functions*, edited by M. Abramowitz and Irene Stegun, Dover Publications, 9th ed. 1970, 4.3.32 and 4.3.36, p. 72, 73.

We now rewrite this last expression as

$$5\cos\frac{\pi}{9} + \left(2\cos\frac{\pi}{9}\right) + 7\cos\frac{4\pi}{9} - \left(2\cos\frac{4\pi}{9}\right) - \cos\frac{7\pi}{9} + \left(2\cos\frac{7\pi}{9}\right). \qquad (12)$$

It remains to show that

$$\cos\frac{\pi}{9} - \cos\frac{4\pi}{9} + \cos\frac{7\pi}{9} = 0. \qquad (13)$$

But

$$\begin{aligned}
&\cos\frac{\pi}{9} - \cos\frac{4\pi}{9} + \cos\frac{7\pi}{9} \\
&= \cos\frac{\pi}{9} - \cos\frac{\pi}{9}\cos\frac{3\pi}{9} + \sin\frac{\pi}{9}\sin\frac{3\pi}{9} + \cos\frac{\pi}{9}\cos\frac{6\pi}{9} - \sin\frac{\pi}{9}\sin\frac{6\pi}{9} \\
&= \cos\frac{\pi}{9} - \frac{1}{2}\cos\frac{\pi}{9} + \frac{\sqrt{3}}{2}\sin\frac{\pi}{9} - \frac{1}{2}\cos\frac{\pi}{9} - \frac{\sqrt{3}}{2}\sin\frac{\pi}{9} = 0.
\end{aligned}$$

This completes the proof.

132. [1999,197] *Proposed by Don Redmond, Southern Illinois University, Carbondale, Illinois.*

Let F_n denote the nth Fibonacci number. That is, $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. In 1883 Cesaro showed that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n} \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3n}.$$

Prove the following generalization of Cesaro's result.

Let r and s be roots of the quadratic equation

$$x^2 - ax - b = 0. \qquad (1)$$

Define the two sequences $\{P_n\}$ and $\{Q_n\}$ by

$$Q_n = \frac{r^n - s^n}{r - s} \quad \text{and} \quad P_n = cr^n + ds^n,$$

where c and d are constants. If $j \geq 2$, then

$$\sum_{k=0}^{n} \binom{n}{k} (bQ_{j-1})^{n-k} Q_j^k P_k = P_{jn}.$$

Solution by the proposer. We begin with a lemma.

Lemma. If x satisfies (1), then, for $n \geq 1$,

$$x^n = Q_n x + bQ_{n-1}.$$

Proof. It is clear that $Q_0 = 0$, $Q_1 = 1$, $Q_2 = a$ and that, for $n \geq 1$,

$$Q_{n+1} = aQ_n + bQ_{n-1}.$$

We proceed by induction on n.

For $n = 1$ and 2 we have

$$x = Q_1 x + bQ_0 \quad \text{and} \quad x^2 = ax + b = Q_2 x + bQ_1,$$

so that the result is true in these cases.

If we assume that the result is true for $n = m \geq 1$, that is,

$$x^m = Q_m x + bQ_{m-1},$$

then, for $n = m + 1$, we have

$$\begin{aligned} x^{m+1} &= x \cdot x^m = x(Q_m x + bQ_{m-1}) = x^2 Q_m + xbQ_{m-1} \\ &= (ax + b)Q_m + xbQ_{m-1} = x(aQ_m + bQ_{m-1}) + bQ_m \\ &= xQ_{m+1} + bQ_m \end{aligned}$$

which is the result for $n = m + 1$ and the lemma follows.

We now prove the main result. We have

$$\sum_{k=0}^{n} \binom{n}{k} (bQ_{j-1})^{n-k} Q_j^k P_k = \sum_{k=0}^{n} \binom{n}{k} (bQ_{j-1})^{n-k} Q_j^k (cr^k + ds^k)$$

$$= c \sum_{k=0}^{n} \binom{n}{k} (bQ_{j-1})^{n-k} Q_j^k r^k + d \sum_{k=0}^{n} \binom{n}{k} (bQ_{j-1})^{n-k} Q_j^k s^k$$

$$= c(Q_j r + bQ_{j-1})^n + d(Q_j s + bQ_{j-1})^n = cr^{jn} + ds^{jn} = P_{jn},$$

by the lemma. The result follows.

Also solved by José Luis Díaz, Universidad Politécnica de Cataluña, Terrassa, Spain and Kenneth B. Davenport, 301 Morea Road, Box 491, Frackville, Pennsylvania.