ANOTHER PROOF OF THE CHANGE OF VARIABLE FORMULA FOR d-DIMENSIONAL INTEGRALS

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The Volume 105, Number 7 issue of *The American Mathematical Monthly* published a new proof of the change of the variable formula for d-dimensional integrals

$$\int_{\mathbb{R}^d} f(x)\,d\lambda(x) = |\det(A)| \int_{\mathbb{R}^d} f(Ax)\,d\lambda(x) \tag{1}$$

with an invertible matrix A, based on group theoretical arguments [1]. In this note we provide another proof of (1) to illustrate an application of elementary measure theory and the singular value decomposition.

We use the same notation as in [1]. The measure $\lambda \circ A^{-1}$, which is defined by $(\lambda \circ A^{-1})(B) = \lambda(A^{-1}(B))$ for all Borel sets B, is equivalent to (i.e., absolutely continuous with respect to each other) the Lebesgue measure λ for any $A \in GL(d, \mathbb{R})$.

Proposition 1.

$$\int_{\mathbb{R}^d} f(Ax)\,d\lambda(x) = \int_{\mathbb{R}^d} f(x)\,d(\lambda \circ A^{-1})(x). \tag{2}$$

Proof. Let $f = 1_B$, where 1_B is the characteristic function of B. Then

$$\int_{\mathbb{R}^d} 1_B(Ax)\,d\lambda(x) = \lambda(A^{-1}(B)) = \int_{\mathbb{R}^d} 1_B(x)\,d(\lambda \circ A^{-1})(x),$$

i.e., (2) is true for all characteristic functions, which implies that (2) is satisfied by all simple functions. Since f is the limit of a sequence of simple functions [3], using a limiting process we see that (2) is valid for all integrable functions f.

Proposition 2. If $A \in GL(d, \mathbb{R})$ is orthogonal, then $(\lambda \circ A^{-1})(B) = \lambda(B)$ for all Borel sets B.

Proof. Every orthogonal matrix is a product of several rotations and reflections which do not change the Lebesgue measure of a Borel set.

Proposition 3. If $A \in GL(d,\mathbb{R})$ is diagonal, then (1) is true.

Proof. Let $A = \mathrm{diag}(a_1, \ldots, a_d)$. Then it is obvious that

$$(\lambda \circ A^{-1})(B) = \left| \prod_{i=1}^{d} a_i \right|^{-1} \lambda(B) = |\det(A)|^{-1} \lambda(B).$$

Hence, (1) follows from Proposition 1.

Now, by the singular value decomposition theorem [2], $A = U^T D V$, where U and V are orthogonal matrices and D is an invertible diagonal matrix. Thus, using the above propositions,

$$\int_{\mathbb{R}^d} f(Ax)\, d\lambda(x) = \int_{\mathbb{R}^d} (f \circ U^T \circ D)(Vx)\, d\lambda(x) = \int_{\mathbb{R}^d} (f \circ U^T)(Dx)\, d\lambda(x)$$

$$= |\det(D)|^{-1} \int_{\mathbb{R}^d} f(U^T x)\, d\lambda(x) = |\det(A)|^{-1} \int_{\mathbb{R}^d} f(x)\, d\lambda(x).$$

Therefore, (1) is proved.

References

1. P. Dierolf and V. Schmidt, "A Proof of the Change of Variable Formula for d-Dimensional Integrals," *The American Mathematical Monthly,* 105 (1998), 654–656.

2. R. Horn and C. Johnson, *Matrix Analysis,* Cambridge University Press, New York, 1985.

3. H. L. Royden, *Real Analysis,* Macmillan, New York, 1968.

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