THEORY OF THE SPHERICALLY SYMMETRIC SPACE-TIMES. V. n DIMENSIONAL SPHERICALLY SYMMETRIC SPACE-TIMES

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(Received Sept. 3, 1952)

In this paper we shall give an $n(\geqq 5)$ dimensional generalization of the theory of the spherically symmetric space-times developed by the present writer.[1] Almost all properties of the four dimensional S_0 given in the previous papers also hold for the n dimensional case to within some slight modifications, but as will be seen in due course, the theory concerning the latter is simpler than the four dimensional theory in some respects owing to the condition n-$2\geqq 3$. In this paper, emphasizing the properties which do not hold for the four dimensional S_0, we shall give the main results.

§ 1. Definition of S_0 and identities concerning c.s.

Definition: An $n(\geqq 5)$ dimensional s.s. space-time S_0 is an n dimensional Riemannian space having the following properties:

(I) Its curvature tensor satisfies

$$K_{ijlm}=\overset{1}{\rho}\alpha_{[i}\alpha_{[l}\beta_{j]}\beta_{m]}+\overset{2}{\rho}g_{[i[l}\alpha_{j]}\alpha_{m]}+\overset{3}{\rho}g_{[i[l}\beta_{j]}\beta_{m]}+\overset{4}{\rho}g_{[i[l}g_{j]m]}\,, \tag{F$_1$}$$

where $i, j, \cdots=1, \cdots, n,$ α_i and β_i are mutually orthogonal unit vectors satisfying

$$\nabla_i\alpha_j=\sigma\alpha_i\beta_j+\kappa(g_{ij}-\alpha_i\alpha_j-\beta_i\beta_j)-\bar{\sigma}\beta_i\beta_j\,, \tag{F$_2$}$$

$$\nabla_i\beta_j=\bar{\sigma}\beta_i\alpha_j+\bar{\kappa}(g_{ij}-\alpha_i\alpha_j-\beta_i\beta_j)-\sigma\alpha_i\alpha_j\,, \tag{F$_3$}$$

and $\overset{a}{\rho}$; $\sigma, \bar{\sigma}$; $\kappa, \bar{\kappa}$ are scalars determined from these equations.

(II) One of the five scalars $\overset{1}{\rho}, \cdots, \overset{4}{\rho}$ and K is such that its gradient vector is a linear combination of α_i and β_i.

(III) $$\overset{4}{\rho}+2(\kappa^2+\bar{\kappa}^2)\neq 0\,. \tag{F$_4$}$$

(IV) Moreover for the sake of simplicity and symmetry, we shall assume that the fundamental form is positive definite and $\alpha_s\alpha^s=\beta_s\beta^s=1$.

Corresponding theory for the case of g_{ij} of the type $(--\cdots+)$ and $-\alpha_s\alpha^s=\beta_s\beta^s=1$ is obtained by making some slight modifications. Throughout this paper the similar definitions and the same notations of the c.s., c.v., m-transformation, etc. as those used in the four dimensional case are adopted.

Then we can easily show that various identities similar to those given in §2 of (I) hold. For example,

$$\sigma=\alpha^s\beta^t\nabla_s\alpha_t=-\alpha^s\alpha^t\nabla_s\beta_t,\cdots;(n-2)\kappa-\bar{\sigma}=\nabla_s\alpha^s,\cdots;\alpha^s\nabla_s\alpha^t=\sigma\beta^t,\cdots \tag{1.1}$$

$$\overset{1}{\rho}=4[-K+(n-1)\{(n-2)\overset{1}{\tau}+\overset{2}{\tau}+\overset{3}{\tau}\}]/(n-2)(n-3),\cdots\cdots \tag{1.2}$$

$$\begin{aligned}\overset{1}{\tau}&=\alpha^i\beta^j\alpha^l\beta^m K_{ijlm}=(\overset{1}{\rho}+\overset{2}{\rho}+\overset{3}{\rho}+2\overset{4}{\rho})/4,\\ \overset{2}{\tau}&=\alpha^i\alpha^j K_{ij}=-\{\overset{1}{\rho}+(n-1)\overset{2}{\rho}+\overset{3}{\rho}+2(n-1)\overset{4}{\rho}\},\cdots\cdots\end{aligned} \tag{1.3}$$

$$\alpha^j K_{jl}=\overset{2}{\tau}\alpha_l,\quad \beta^j K_{jl}=\overset{3}{\tau}\beta_l. \tag{1.4}$$

$$4K_{jl}=-\{\overset{2}{\rho}+\overset{3}{\rho}+2(n-1)\overset{4}{\rho}\}g_{jl}-\{\overset{1}{\rho}+(n-2)\overset{2}{\rho}\}\alpha_j\alpha_l-\{\overset{1}{\rho}+(n-2)\overset{3}{\rho}\}\beta_j\beta_l,\cdots \tag{1.5}$$

$$\nabla_k P_{ij}=g_{k[i}(\kappa\beta_{j]}-\bar{\kappa}\alpha_{j]})+F_k P_{ij},\ (-F_i=\kappa\alpha_i+\bar{\kappa}\beta_i). \tag{1.6}$$

The condition $(2\nabla_{[k}\nabla_{l]}\alpha_j=K_{kijl}\alpha^i,\ 2\nabla_{[k}\nabla_{l]}\beta_j=K_{kijl}\beta^i)$ is equivalent to that $\sigma, \bar{\sigma};\ \kappa, \bar{\kappa}$ are s.s. and

$$\begin{aligned}&\overset{1}{\tau}+\sigma^2+\bar{\sigma}^2=\alpha^s\bar{\sigma}_s+\beta^s\sigma_s,\ \bar{\kappa}(\bar{\sigma}+\kappa)+\beta^s\kappa_s=0,\ \kappa(\sigma+\bar{\kappa})+\alpha^s\bar{\kappa}_s=0,\\ &\overset{2}{\rho}+2\overset{4}{\rho}=-4(-\sigma\bar{\kappa}+\kappa^2+\alpha^s\kappa_s),\ \overset{3}{\rho}+2\overset{4}{\rho}=-4(-\bar{\sigma}\kappa+\bar{\kappa}^2+\beta^s\bar{\kappa}_s).\end{aligned} \tag{1.7}$$

Bianchi's identity is equivalent to

$$\begin{cases}\overset{1}{\rho}_i=(\alpha^s\overset{1}{\rho}_s)\alpha_i+(\beta^s\overset{1}{\rho}_s)\beta_i,\ \overset{2}{\rho}_i=(\alpha^s\overset{2}{\rho}_s)\alpha_i+\{\overset{1}{\rho}\bar{\kappa}+\overset{2}{\rho}\sigma-\overset{3}{\rho}(\bar{\kappa}+\sigma)\}\beta_i,\\ \overset{3}{\rho}_i=\{\overset{1}{\rho}\kappa+\overset{3}{\rho}\bar{\sigma}-\overset{2}{\rho}(\kappa+\bar{\sigma})\}\alpha_i+(\beta^s\overset{3}{\rho}_s)\beta_i,\ \overset{4}{\rho}_i=\overset{2}{\rho}\kappa\alpha_i+\overset{3}{\rho}\bar{\kappa}\beta_i.\end{cases} \tag{1.8}$$

Theorems [2.1] and [2.2] in (I) hold as they are.

§2. Standard form for c.s. and s.s. coordinate system

As in the four dimensional case we can prove the following theorem:

Theorem [2.1] *For any c.s. of an S_0 there exists a coordinate system in which the fundamental form is given by*

$$ds^2=A(r,t)dr^2+B(r,t)dl^2+C(r,t)dt^2,\quad (A,B,C>0), \tag{2.1}$$

where

$$dl^2=(d\theta^2)^2+\sin^2\theta^2[(d\theta^3)^2+\sin^2\theta^3\{(d\theta^4)^2+\cdots+\sin^2\theta^{n-2}(d\theta^{n-1})^2\cdots\}], \tag{2.2}$$

and $$\gamma\equiv K_{1a}{}^{an}=0,\quad (a=2,\cdots,n-1;\ \textit{not summed for } a), \tag{2.3}$$

hold, and the c.s. takes the form

$$\overset{1}{\rho}=4(\xi+\eta-\alpha-\beta),\ \overset{2}{\rho}=4(\alpha-\eta),\ \overset{3}{\rho}=4(\beta-\eta),\ \overset{4}{\rho}=2\eta, \tag{2.4}$$

$$\begin{cases} \alpha_i = \sqrt{A}\,\delta_i^1, \ \beta_i = \sqrt{C}\,\delta_i^n, \quad (x^1 \equiv r,\ x^a \equiv \theta^a,\ x^n \equiv t), \\ \sigma = -\dot{A}/2A\sqrt{C}, \ \bar{\sigma} = -C'/2C\sqrt{A}; \ \kappa = B'/2B\sqrt{A}, \ \bar{\kappa} = \dot{B}/2B\sqrt{C}, \end{cases} \tag{2.5}$$

$$where \quad \alpha \equiv K_{1a}{}^{\cdot\cdot 1a},\ \beta \equiv K_{an}{}^{\cdot\cdot an},\ \xi \equiv K_{1n}{}^{\cdot\cdot 1n},\ \eta \equiv K_{ab}{}^{\cdot\cdot ab}, \ \begin{pmatrix} a, b = 2, \cdots, n-1;\ not \\ summed\ for\ a\ and\ b \end{pmatrix}. \tag{2.6}$$

This coordinate system is standard for the c. s. dl^2 in (2.1) and (2.2) defines an $(n-2)$ dimensional space of constant curvature whose scalar curvature is equal to 1. By a transformation $(r, t) \to (\bar{r}, \bar{t})$ satisfying $(\partial r/\partial \bar{r})(\partial r/\partial \bar{t})A + (\partial t/\partial \bar{r})(\partial t/\partial \bar{t})C = 0$, we have a s. s. coordinate system, in which (2.3) is not necessarily satisfied though ds^2 takes the form (2.1). In general, however, any s. s. fundamental form can be brought into the form in which (2.3) holds by a suitable transformation of (r, t) and such a coordinate system is the standard one for g_{ij}. Hence in a standard coordinate system for g_{ij}, one c. s. is given by (2.4) and (2.5).

The non-vanishing components of the Christoffel symbol and the curvature tensor in a s. s. coordinate system are given by

$$\begin{aligned}
&\{{}^{1}_{11}\} = A'/2A,\ \{{}^{n}_{11}\} = -\dot{A}/2C,\ \{{}^{1}_{1n}\} = \dot{A}/2A,\ \{{}^{n}_{1n}\} = C'/2C,\ \{{}^{1}_{nn}\} = -C'/2A,\ \{{}^{n}_{nn}\} = \dot{C}/2C; \\
&\{{}^{a}_{1a}\} = B'/2B,\ \{{}^{a}_{na}\} = \dot{B}/2B;\ \{{}^{3}_{23}\} = \cdots = \{{}^{n-1}_{2, n-1}\} = \cot\theta^2; \\
&\{{}^{4}_{34}\} = \cdots = \{{}^{n-1}_{3, n-1}\} = \cot\theta^3, \cdots, \{{}^{n-1}_{n-2, n-1}\} = \cot\theta^{n-2}; \\
&\{{}^{1}_{22}\} = -B'/2A,\ \{{}^{1}_{33}\} = -(B'/2A)\sin^2\theta^2, \cdots, \{{}^{1}_{n-1, n-1}\} = -(B'/2A)\sin^2\theta^2 \cdots \sin^2\theta^{n-2}; \\
&\{{}^{n}_{22}\} = -\dot{B}/2C,\ \{{}^{n}_{33}\} = -(\dot{B}/2C)\sin^2\theta^2, \cdots, \{{}^{n}_{n-1, n-1}\} = -(\dot{B}/2C)\sin^2\theta^2 \cdots \sin^2\theta^{n-2}; \\
&\{{}^{2}_{33}\} = -\sin\theta^2\cos\theta^2;\ \{{}^{2}_{44}\} = -\sin\theta^2\cos\theta^2\sin^2\theta^3,\ \{{}^{3}_{44}\} = -\sin\theta^3\cos\theta^3; \qquad (2.7) \\
&\{{}^{2}_{55}\} = -\sin\theta^2\cos\theta^2\sin^2\theta^3\sin^2\theta^4,\ \{{}^{3}_{55}\} = -\sin\theta^3\cos\theta^3\sin^2\theta^4,\ \{{}^{4}_{55}\} = -\sin\theta^4\cos\theta^4; \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\{{}^{2}_{n-1, n-1}\} = -\sin\theta^2\cos\theta^2\sin^2\theta^3 \cdots \sin^2\theta^{n-2},\ \{{}^{3}_{n-1, n-1}\} = -\sin\theta^3\cos\theta^3\sin^2\theta^4 \cdots\cdots \\
&\sin^2\theta^{n-2}, \cdots, \{{}^{n-3}_{n-1, n-1}\} = -\sin\theta^{n-3}\cos\theta^{n-3}\sin^2\theta^{n-2},\ \{{}^{n-2}_{n-1, n-1}\} = -\sin\theta^{n-2}\cos\theta^{n-2},
\end{aligned}$$

$$\begin{aligned}
\alpha &= -(1/4AB)\{2B'' - B'^2/B - A'B'/A + \dot{A}\dot{B}/C\}, \\
\beta &= -(1/4BC)\{2\ddot{B} - \dot{B}^2/B - \dot{B}\dot{C}/C + B'C'/A\}, \\
\gamma &= (1/4BC)\{2\dot{B}' - \dot{B}B'/B - \dot{A}B'/A - \dot{B}C'/C\}, \\
\eta &= (1/B^2)\{B - B'^2/4A - \dot{B}^2/4C\}, \\
\xi &= -(1/4AC)\{2(\ddot{A} + C'') - A'C'/A - \dot{A}\dot{C}/C - \dot{A}^2/A - C'^2/C\},
\end{aligned} \tag{2.8}$$

where $a, b = 2, \cdots, n-1$ and not summed for a and b.

In the standard coordinate system for the c. s. it holds that

$$4\alpha = \overset{2}{\rho} + 2\overset{4}{\rho},\ 4\beta = \overset{3}{\rho} + 2\overset{4}{\rho},\ \gamma = 0,\ 4\xi = 4\overset{1}{\tau} = \overset{1}{\rho} + \overset{2}{\rho} + \overset{3}{\rho} + 2\overset{4}{\rho},\ 2\eta = \overset{4}{\rho}, \tag{2.9}$$

and
$$F_i = -(\kappa\alpha_i + \bar{\kappa}\beta_i) = \nabla_i F = (-B'/2B, \cdots, 0, -\dot{B}/2B). \tag{2.10}$$

We normalize F by $\overset{4}{\rho} = 2(e^{2F} - \kappa^2 - \bar{\kappa}^2)$. (2.11)

Hence in the above coordinate system we have $B = e^{-2F}$. (2.12)

Furthermore we can prove the following identities concerning F:

$$\alpha^i \beta^j \nabla_i \nabla_j F = \alpha^j \beta^i \nabla_i \nabla_j F = \kappa\bar{\kappa} = -(\sigma\kappa + \alpha^s \bar{\kappa}_s) = -(\bar{\sigma}\bar{\kappa} + \beta^s \kappa_s), \text{ etc.} \quad (2.13)$$

§ 3. Freedom of c. s. in an S_0

In the same way as in the four dimensional case, we have the following theorems concerning $\varepsilon-$, $i-$, and ω-transformations:

Theorem [3. 1] *Let* $[K]$: $(\alpha_i, \beta_i, \overset{a}{\rho}, \cdots)$ *be a c. s. of an* S_0. *Then* (α^*_i, β^*_i) *given by*

$$(i)\ \ \alpha^*_i = \varepsilon\alpha_i,\ \ \beta^*_i = \bar{\varepsilon}\beta_i, \qquad (ii)\ \ \alpha^*_i = \beta_i,\ \ \beta^*_i = \alpha_i, \quad (3.1)$$

where $\varepsilon^2 = \bar{\varepsilon}^2 = 1$, *are again c. v. of the* S_0 *and the remaining members of the c. s. corresponding to (i) and (ii) are given by (i)* $[K_1]$: $\overset{a}{\rho}{}^* = \overset{a}{\rho}$, $\sigma^* = \bar{\varepsilon}\sigma$, $\bar{\sigma}^* = \varepsilon\bar{\sigma}$, $\kappa^* = \varepsilon\kappa$, $\bar{\kappa}^* = \bar{\varepsilon}\bar{\kappa}$, $(F^* = F)$, *and (ii)* $[K_2]$: $\overset{1}{\rho}{}^* = \overset{1}{\rho}$, $\overset{2}{\rho}{}^* = \overset{3}{\rho}$, $\overset{3}{\rho}{}^* = \overset{2}{\rho}$, $\overset{4}{\rho}{}^* = \overset{4}{\rho}$, $\sigma^* = \bar{\sigma}$, $\bar{\sigma}^* = \sigma$, $\kappa^* = \bar{\kappa}$, $\bar{\kappa}^* = \kappa$, $(F^* = F)$, *respectively.*

Theorem [3. 2] *A necessary and sufficient condition that*

$$\alpha^*_i = \alpha_i \cos\omega - \beta_i \sin\omega,\ \ \beta^*_i = \alpha_i \sin\omega + \beta_i \cos\omega,\ \ (\omega \neq m\pi/2,\ m \text{ is any integer}), \quad (3.2)$$

be again c. v. of the S_0 *is given by* $\overset{2}{\rho} = \overset{3}{\rho}$ *and* $\nabla_i \omega = \omega_i = (\alpha^s \omega_s)\alpha_i + (\beta^s \omega_s)\beta_i$, *and* $\overset{a}{\rho}{}^*$, σ^*, $\cdots$ *corresponding to* (α^*_i, β^*_i) *are given by*

$$\begin{aligned} &\overset{a}{\rho}{}^* = \overset{a}{\rho},\ (F^* = F);\ \kappa^* = \kappa\cos\omega - \bar{\kappa}\sin\omega,\ \bar{\kappa}^* = \kappa\sin\omega + \bar{\kappa}\cos\omega; \\ &\sigma^* = (\sigma - \alpha^s\omega_s)\cos\omega + (\bar{\sigma} + \beta^s\omega_s)\sin\omega,\ \bar{\sigma}^* = -(\sigma - \alpha^s\omega_s)\sin\omega + (\bar{\sigma} + \beta^s\omega_s)\cos\omega. \end{aligned} \quad (3.3)$$

Both α^i and β^i give the principal directions of K_{ij} and corresponding invariants are $\overset{2}{\tau}$ and $\overset{3}{\tau}$ respectively. In the standard coordinate system for a c. s., non-vanishing components of $K_i^{\cdot j}$ are

$$K_1^{\cdot 1} = -\{\xi + (n-2)\alpha\},\ K_a^{\cdot a} = -\{(\alpha + \beta) + (n-3)\eta\},\ K_n^{\cdot n} = -\{\xi + (n-2)\beta\}, \quad (3.4)$$

where $a = 2, \cdots, n-1$ and not summed for a. Hence the n principal invariants of K_{ij} are

$$\begin{cases} \overset{1}{\nu} = -\{\overset{1}{\rho} + \overset{3}{\rho} + (n-1)(\overset{2}{\rho} + 2\overset{4}{\rho})\}/4,\ \ \overset{3}{\nu} = \cdots = \overset{n-1}{\nu} = -\{\overset{2}{\rho} + \overset{3}{\rho} + 2(n-1)\overset{4}{\rho}\}/4, \\ \overset{n}{\nu} = -\{\overset{1}{\rho} + \overset{2}{\rho} + (n-1)(\overset{3}{\rho} + 2\overset{4}{\rho})\}/4. \end{cases} \quad (3.5)$$

Hence, in general, there exists an $(n-2)$-ple invariants. Next,

$$\overset{1}{\nu} = \overset{n}{\nu},\ \ \overset{1}{\nu} = \overset{a}{\nu},\ \ \overset{n}{\nu} = \overset{a}{\nu},\ \ (a = 2, \cdots, n-1), \quad (3.6)$$

are equivalent to

$$\overset{2}{\rho} = \overset{3}{\rho}\,, \quad \overset{1}{\rho} + (n-2)\overset{2}{\rho} = 0\,, \quad \overset{1}{\rho} + (n-2)\overset{3}{\rho} = 0\,, \tag{3.7}$$

respectively and in the standard coordinate system these become

$$\alpha = \beta\,, \quad \xi - \beta + (n-3)(\alpha - \eta) = 0\,, \quad \xi - \alpha + (n-3)(\beta - \eta) = 0\,. \tag{3.8}$$

Then we can prove the following to within ε- and i-transformations:
Case I. When ν's are of the form $(a, \cdots, a, b, c)$, $(a, b, c \neq)$, c.s. is determined uniquely.
Case II. When ν's are of the form $(a, \cdots, a, b, b)$, $(a \neq b)$, c.s. is determined uniquely to within an ω-transformation. Since $n-2>2$, the numbers of a and b can not be equal. So there is no need of the special treatment of the case in which B and ξ are constant, different from the four dimensional case.
Case III. When ν's are of the form $(a, \cdots, a, b)$, $(a \neq b)$. When and only when ds^2 is reducible to

$$ds^2 = e^{2g(t)}(1 - r^2/4R^2)^{-2}(dr^2 + r^2 dl^2) + dt^2\,, \quad (R^2 \text{ is a constant} \neq 0), \tag{3.9}$$

where $\ddot{g} + e^{-2g}/R^2 \neq 0$,[2)] the S_0 admits m-transformation and in other cases c.s. is determined uniquely. In the S_0 defined by (3.9) one of the following relations holds for all c.s.: (i) $\overset{1}{\rho} = \overset{2}{\rho} = \bar{\sigma} = \sigma + \bar{\kappa} = \alpha^s \overset{a}{\rho}_s = 0$, ($\alpha_i$ is determined uniquely), (ii) $\overset{1}{\rho} = \overset{3}{\rho} = \sigma = \bar{\sigma} + \kappa = \beta^s \overset{a}{\rho}_s = 0$, ($\beta_i$ is determined uniquely).
Case IV. When ν's are of the form $(a, \cdots, a)$. When the S_0 is of constant curvature c.s. is determined to within ω- and m-transformations and its ds^2 is reducible to

$$ds^2 = (1 + k^2 r^2)^{-1} dr^2 + r^2 dl^2 + (1 + k^2 r^2) dt^2\,, \quad \begin{pmatrix} k^2 \text{ is any const.} \\ \text{including } 0 \end{pmatrix}. \tag{3.10}$$

When the S_0 is not of constant curvature we can easily prove that c.s. is determined to within an ω-transformation, different from the four dimensional case.

Hence we have determined the freedom of the c.s. in all S_0's. If we use c.s. in place of ν's we have the following table to within ε- and i-transformations.

In all cases ρ's are determined uniquely and it is also to be noticed that *ρ's are invariant under any m-transformation* as is seen from the fact that they are determined by K_{ijlm} in the standard coordinate system for the c.s.

$$
S\left\{\begin{array}{ll}
\begin{array}{c} S_b \\ (\overset{2}{\rho}\neq\overset{3}{\rho}) \end{array} & \left\{\begin{array}{lll}
\overset{1}{\rho}+(n-2)\overset{2}{\rho}\neq 0,\ \ \overset{1}{\rho}+(n-2)\overset{3}{\rho}\neq 0 & \text{unique} & (\text{Case I}) \\
\overset{1}{\rho}+(n-2)\overset{2}{\rho}= 0,\ \ \overset{1}{\rho}+(n-2)\overset{3}{\rho}\neq 0 & & (\text{Case III}) \\
\quad\left\{\begin{array}{l} \overset{1}{\rho}=\overset{2}{\rho}=\bar{\sigma}=\sigma+\bar{\kappa}=\alpha^s\overset{a}{\rho}_s=0 \ \text{for all c.s.} \quad (m) \\ \qquad (\alpha_i \text{ is determined uniquely}) \\ \text{other } S_0\text{'s} \end{array}\right. & \begin{array}{l} \\ \\ \text{unique} \end{array} & \\
\overset{1}{\rho}+(n-2)\overset{2}{\rho}\neq 0,\ \ \overset{1}{\rho}+(n-2)\overset{3}{\rho}= 0 & & (\text{Case III}) \\
\quad\left\{\begin{array}{l} \overset{1}{\rho}=\overset{3}{\rho}=\sigma=\bar{\sigma}+\kappa=\beta^s\overset{a}{\rho}_s=0 \ \text{for all c.s.} \quad (m) \\ \qquad (\beta_i \text{ is determined uniquely}) \\ \text{other } S_0\text{'s} \end{array}\right. & \begin{array}{l} \\ \\ \text{unique} \end{array} &
\end{array}\right. \\
\begin{array}{c} S_a \\ (\overset{2}{\rho}=\overset{3}{\rho}) \end{array} & \left\{\begin{array}{lll}
\overset{1}{\rho}+(n-2)\overset{2}{\rho}=\overset{1}{\rho}+(n-2)\overset{3}{\rho}\neq 0 & (\omega) & (\text{Case II}) \\
\quad ,, \quad = \quad ,, \quad =0 \left\{\begin{array}{l} \overset{1}{\rho}\neq 0 \\ \overset{1}{\rho}=0 \end{array}\right. & \begin{array}{l} (\omega) \\ (\omega, m) \end{array} & \begin{array}{l} (\text{Case IV}) \\ (\text{Case IV}) \end{array} \\
\qquad\qquad\qquad (\text{const. curvature}) & &
\end{array}\right.
\end{array}\right.
$$

§ 4. Miscellaneous theorems

As regards the formulae for c. s. in any s. s. coordinate system, theorem characterizing S_{I} and S_{II}, theorem concerning s. s. coordinate system of an S_a, etc., we can obtain similar results as those given in § 5 of (**I**) by making some slight modifications. So we shall omit them here and shall add two theorems concerning an S_0.[3]

Theorem [4.1] *In an S_0, ${}_nC_2$ principal invariants of $K_{AB}\equiv K_{ijlm}$ are all real. They are composed of two $(n-2)$-ple roots, one ${}_{n-2}C_2$-ple root and one simple root in the general case.*

Proof. The theorem is evident from the fact that in any s. s. coordinate system the equation $|K_A^{\cdot B}-\nu\delta_A^B|=0$ becomes

$$(\nu-\xi)(\nu-\eta)^p\{\nu^2-(\alpha+\beta)\nu+(\alpha\beta-\gamma^2A/C)\}^{n-2}=0\,. \quad (p={}_{n-2}C_2)\,. \tag{4.1}$$

Theorem [4.2] *A necessary and sufficient condition for an S_0 to be an S_{II} is that $2(n-2)$ of the principal invariants of K_{AB} is equal to zero and another ${}_{n-2}C_2$ are equal to a non-vanishing constant. (The remaining one is arbitrary).*[3]

Proof. It is evident that the condition is necessary. Conversely if the condition is satisfied we have η=const.$\neq0$ and $\alpha+\beta=\alpha\beta-\gamma^2A/C=0$ in any s. s. coordinate system, from which we can easily prove that this S_0 can not be an S_{I}.

§ 5. Orthogonal ennuple intrinsic to an S_0

This problem was not treated in the four dimensional case. In any

s. s. coordinate system, $(n-2)$ unit vectors $\overset{a}{\gamma}_i$, $(a=2, \cdots, n-1)$, defined by

$$\overset{2}{\gamma}_i=(0, \sqrt{B}, 0, \cdots, 0), \quad \overset{3}{\gamma}_i=(0, 0, \sqrt{B}\sin\theta^2, 0, \cdots, 0), \cdots,$$
$$\overset{n-1}{\gamma}_i=(0, \cdots, 0, \sqrt{B}\sin\theta^2\cdots\sin\theta^{n-2}, 0), \tag{5.1}$$

form an orthogonal ennuple together with α_i and β_i. These $\overset{a}{\gamma}_i$ satisfy

$$\nabla_i\overset{a}{\gamma}_j=\overset{a}{\gamma}_i(F_j-\overset{2}{\mu}\overset{2}{\gamma}_j-\cdots-\overset{a-1}{\mu}\,\overset{a-1}{\gamma}_j)+\overset{a}{\mu}(\overset{a+1}{\gamma}_i\,\overset{a+1}{\gamma}_j+\cdots+\overset{n-1}{\gamma}_i\,\overset{n-1}{\gamma}_j), \quad (a=2, \cdots, n-1), \tag{F$_5$}$$

where

$$\overset{2}{\mu}=\cot\theta^2/\sqrt{B}, \quad \overset{3}{\mu}=\cot\theta^3/\sqrt{B}\sin\theta^2, \cdots, \overset{n-2}{\mu}=\cot\theta^{n-2}/\sqrt{B}\sin\theta^2\cdots\sin\theta^{n-3}, \tag{5.2}$$

and

$$\overset{1}{\mu}=\overset{n-1}{\mu}=\overset{n}{\mu}=0. \tag{5.3}$$

Then $\overset{a}{\mu}$ satisfies

$$\nabla_i\overset{b}{\mu}=\overset{b}{\mu}(F_i-\overset{2}{\mu}\overset{2}{\gamma}_i-\cdots-\overset{b-1}{\mu}\,\overset{b-1}{\gamma}_i)-\{(\overset{2}{\mu})^2+\cdots+(\overset{b}{\mu})^2+e^{2F}\}\overset{b}{\gamma}_i, \quad (b=2, \cdots, n-2). \tag{F$_6$}$$

Conversely we shall assume that $(n-2)$ unit vectors $\overset{a}{\gamma}_i$ constitute an orthogonal ennuple together with α_i and β_i, and satisfy (F$_5$) where $\overset{a}{\mu}$, $(a=1, \cdots, n-1)$, are determined from (F$_5$) and (5.3). Moreover we shall assume that these μ's satisfy (F$_6$). Then we can prove:

Theorem [5.1] *In an S_0, $\overset{a}{\gamma}_i$ and $\overset{a}{\mu}$ are determined uniquely to within ε- and m-transformations where ε-transformation is defined by*

$$\overset{a}{\gamma}{}^*_i=\overset{a}{\varepsilon}\,\overset{a}{\gamma}_i, \quad \overset{a}{\mu}{}^*=\overset{a}{\varepsilon}\,\overset{a}{\mu}, \quad ((\overset{a}{\varepsilon})^2=1; \textit{ not summed for } a). \tag{5.4}$$

Proof. The proof concerning (5.4) is evident. We shall prove the remaining part by showing that in any s. s. coordinate system $\overset{a}{\gamma}_i$ and $\overset{a}{\mu}$ are transformable into the form of (5.1) and (5.2) to within an ε-transformation by a suitable transformation which keeps ds^2 invariant. First take any s. s. coordinate system. Then from (F$_5$) and (F$_6$), we have

$$\sqrt{\{e^{2F}+(\overset{2}{\mu})^2+\cdots+(\overset{a-1}{\mu})^2\}}\,\overset{a}{\gamma}_j=\nabla_j T_a, \quad (a=2, \cdots, n-1). \tag{5.5}$$

If we transform the coordinate system by

$$\bar{r}=r, \quad \bar{\theta}^a=T_a, \quad \bar{t}=t, \quad (a=2, \cdots, n-1), \tag{5.6}$$

then the forms of α_i, β_i, $\overset{a}{\rho}$, $\cdots$, F, g_{1}, g_{1n} $(=0)$ and g_{nn} are kept invariant and in the new coordinate system, we have

$$g_{ij}=0, \quad (i\neq j), \quad \overset{a}{\gamma}_i\sqrt{\{e^{2F}+(\overset{2}{\mu})^2+\cdots+(\overset{a-1}{\mu})^2\}}=\delta^a_i, \quad \overset{a}{\gamma}_a=\sqrt{g_{aa}}, \tag{5.7}$$

to within an ε-transformation. From (F$_6$), we have

$$\nabla_i X_b = -(1+X_b^2)\delta_i^b \text{ where } X_b \equiv \overset{b}{\mu}/\sqrt{\{e^{2F}+\cdots+(\overset{a-1}{\mu})^2\}},\ (b=2,\cdots,n-2), \tag{5.8}$$

from which we get $X_b=\cot(\theta^b+\text{const.})$. By the transformation $\theta^b \to \bar{\theta}^b = \theta^b + \text{const.}$ we obtain (5.2). Then using (F_5) and (5.7) we can show that ds^2 is of the same form as its original one and $\overset{a}{\gamma}_i$ take the form (5.1). Q.E.D.

For the orthogonal ennuple formed by α_i, β_i and $\overset{a}{\gamma}_i$ above stated the following relations hold:

$$\begin{aligned}
&\nabla_s \overset{b}{\gamma}{}^s=(n-b-1)\overset{b}{\mu},\ \nabla_s \overset{n-1}{\gamma}{}^s=0,\ \alpha^s\nabla_s\overset{a}{\gamma}_i=\beta^s\nabla_s\overset{a}{\gamma}_i=0,\ (a=2,\cdots,n-1,b=2,\cdots,n-2),\\
&\overset{b}{\gamma}{}^s\nabla_s\overset{a}{\gamma}_i=0,\ (b<a);\ =\overset{a}{\mu}\overset{b}{\gamma}_i,\ (b>a);\ =F_i-\overset{2}{\mu}\overset{2}{\gamma}_i-\cdots-\overset{a-1}{\mu}\overset{a-1}{\gamma}_i,\ (b=a),\\
&\overset{2}{\mu}=-\overset{2}{\gamma}{}^s\overset{3}{\gamma}{}^t\nabla_t\overset{3}{\gamma}_s=-\overset{2}{\gamma}{}^s\overset{4}{\gamma}{}^t\nabla_t\overset{4}{\gamma}_s=\cdots=\overset{3}{\gamma}{}^s\overset{3}{\gamma}{}^t\nabla_s\overset{2}{\gamma}_t=\cdots,\quad \text{etc.}\\
&\alpha^s\nabla_i\overset{a}{\gamma}_s=-\kappa\overset{a}{\gamma}_i,\ \beta^s\nabla_i\overset{a}{\gamma}_s=-\bar{\kappa}\overset{a}{\gamma}_i,\ \alpha^s\overset{a}{\gamma}{}^t\nabla_s\overset{a}{\gamma}_t=-\kappa,\cdots,\alpha^s\overset{a}{\mu}_s=\overset{a}{\mu}\alpha^sF_s=-\kappa\overset{a}{\mu},\ \text{etc.},\\
&\overset{b}{\gamma}{}^s\overset{a}{\mu}_s=0,\ (b>a);\ =-\overset{a}{\mu}\overset{b}{\mu},\ (b<a);\ =-\{e^{2F}+\cdots+(\overset{a}{\mu})^2\},\ (b=a).
\end{aligned} \tag{5.9}$$

We can easily obtain the corresponding results in the four dimensional case. If we put $\alpha_i=\overset{1}{\lambda}_i$, $\beta_i=\overset{n}{\lambda}_i$, $\overset{a}{\gamma}_i=\overset{a}{\lambda}_i$, the non-vanishing components of the coefficients of rotation $\gamma_{ijk}(=-\gamma_{jik})$ for this orthogonal ennuple are given by

$$\gamma_{n11}=\sigma,\ \gamma_{1nn}=\bar{\sigma},\ \gamma_{a1a}=-\kappa,\ \gamma_{ana}=-\bar{\kappa};\ \gamma_{aba}=0,\ (b>a);\ =-\overset{b}{\mu},\ (b<a). \tag{5.10}$$

from which we can obtain the geometric interpretation of $\sigma, \bar{\sigma}, \kappa, \bar{\kappa}$ and μ's.

§ 6. Group of motions

We shall denote the group of rotations in an $(n-1)$ dimensional euclidean space by G_q, $(q=(n-1)(n-2)/2)$. Then in the coordinate system of

$$dm^2=(dz^1)^2+\cdots+(dz^{n-1})^2, \tag{6.1}$$

its symbols are given by $z^a\partial_b - z^b\partial_a$, $(a\neq b;\ a, b=1,\cdots,n-1)$. If we transform (6.1) into the polar coordinate form by the transformation:

$$\begin{aligned}
z^1=r\cos\theta^2,\ z^2=r\sin\theta^2\cos\theta^3,\cdots,\ z^{n-2}&=r\sin\theta^2\sin\theta^3\cdots\sin\theta^{n-2}\cos\theta^{n-1},\\
z^{n-1}&=r\sin\theta^2\sin\theta^3\cdots\sin\theta^{n-2}\sin\theta^{n-1},
\end{aligned} \tag{6.2}$$

we have $dm^2=dr^2+r^2dl^2$, where dl^2 is given by (2.2). Then we have

$$\begin{aligned}
z^2\partial_1-z^1\partial_2 &= -\cos\theta^3\partial/\partial\theta^2+\cot\theta^2\sin\theta^3\partial/\partial\theta^3,\\
z^3\partial_1-z^1\partial_3 &= -\sin\theta^3\cos\theta^4\partial/\partial\theta^2-\cot\theta^2\cos\theta^3\cos\theta^4\partial/\partial\theta^3\\
&\qquad +\cot\theta^2\operatorname{cosec}\theta^3\sin\theta^4\partial/\partial\theta^4,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
z^{n-1}\partial_{n-2}-z^{n-2}\partial_{n-1} &= -\partial/\partial\theta^{n-1}.
\end{aligned} \tag{6.3}$$

As in the case of four dimension, by solving the Killing's equation we have

Theorem [6.1] *If we classify S_0's from the standpoint of the group of motions we obtain the same scheme as the one given in* § 4 *of* (**II**). *The numbers of the parameters of the respective groups are given by*

$$S_{\mathrm{I}}:\begin{cases}[A],[B]\cdots\cdots n(n+1)/2\,;\ [C]\cdots\cdots(n^2-n+2)/2\,;\ [D],[E]\cdots\cdots n(n-1)/2\,;\\ [F_1],[F_2]\cdots\cdots(n^2-3n+4)/2\,;\ \ [G],[H]\cdots\cdots q(=(n-1)(n-2)/2).\end{cases}$$

$$S_{\mathrm{II}}:\ [I]\cdots\cdots(n^2-3n+8)/2\,;\ \ [J_1],[J_2]\cdots\cdots(n^2-3n+4)/2\,;\ \ [K]\cdots\cdots q.$$

We can easily express this classification in an invariant form by using c.s. of each space-time and also can obtain the concrete forms of the symbols for the motions of each group but we shall omit them here.

§ 7. Class

As in the case of four dimension, by solving

$$K_{ijlm}=e(b_{il}b_{jm}-b_{im}b_{jl}),\ \ \nabla_k b_{ij}-\nabla_j b_{ik}=0,\ \ (e=\pm 1), \tag{7.1}$$

we can prove the following theorem:

Theorem [7.1] *The class of an S_0 is* 2 *or* 1 *or* 0. *And a necessary and sufficient condition that an S_0 be class one is given by*

$$\overset{4}{\rho}\neq 0,\ \ 2\overset{1}{\rho}\overset{4}{\rho}=\overset{2}{\rho}\overset{3}{\rho}\,. \tag{7.2}$$

Specially when the S_0 is conformally flat, (7.2) is reduced to ($\overset{4}{\rho}\neq 0$, $\overset{2}{\rho}\overset{3}{\rho}=0$). In any s.s. coordinate system (7.2) becomes

$$\eta\neq 0,\ \ \xi\eta=\alpha\beta-\gamma^2 C/A\,. \tag{7.3}$$

Further when the S_0 is an S_{I} and the coordinate system is such that $B=r^2$, (7.3) becomes

$$A\neq 1,\ \ 2(\ddot{A}+C'')-(\dot{A}^2+A'C')/(A-1)-(C'^2+\dot{A}\dot{C})/C=0\,. \tag{7.4}$$

When (7.2) is satisfied b_{ij} is determined uniquely to within a sign different from the four dimensional case where, in some special cases, b_{ij} has some arbitrariness other than a sign owing to the fact that the rank of $[b_{ij}]$ can be less than three.

By using the theorem in § 4 we can express (7.2) in terms of the principal invariants Ω, $\bar{\Omega}$, Ω_1 and Ω_2 of K_{AB}, where both Ω and $\bar{\Omega}$ are $(n-2)$-ple roots and Ω and Ω_2 are ${}_{n-2}C_2$-ple and simple roots respectively. In fact (7.2) is equivalent to

$$\Omega_1 \neq 0, \quad \Omega\bar{\Omega} = \Omega_1\Omega_2. \tag{7.5}$$

Furthermore, concerning the concrete form of b_{ij}, that of z^α, etc. we can make similar consideration as in the four dimensional case.

§ 8. Spherically symmetric conformal transformation

In this section we shall give some results concerning s. s. conformal transformation. As is easily seen all the results given in § 1, …, § 6 of (**IV**) hold also in the n dimensional case to within some slight modifications. For instance, as regards the theorem [2. 1] of (**IV**), (2. 10) holds as it is, and in place of (2. 11) we have

$$\overset{4}{\rho}{}^* = e^{-2v}\left\{\overset{4}{\rho} - \frac{4}{n-2}(g^{st} - \alpha^s\alpha^t - \beta^s\beta^t)v_{st} - 2p\right\} \qquad (p = v_i v^i),$$
$$= e^{-2v}\left\{\overset{4}{\rho} - \frac{4}{n-2}\Box v - \frac{2(n-4)}{n-2}p + \frac{4}{n-2}(\alpha^s\alpha^t + \beta^s\beta^t)v_{st}\right\}. \tag{8.1}$$

But with respect to § 7 of (**IV**), the circumstances are somewhat different. Now we shall give the main differences. When $n \geq 5$, in the proof of [7. 2] of (**IV**), we can show that c. v. of the type $(P_{1n}=0)$ can not satisfy (F_1). Hence the main theorem [7. 2] of (**IV**) must be replaced by the following better one:

Theorem [8. 1] *If an S_0 is not conformally flat, a necessary and sufficient condition that the space-time $S_0^{'}$ obtained by a conformal transformation from the S_0 be s. s. again is that the conformal transformation be s. s.*

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Notes

1) (**I**) H. Takeno, Journ. Math. Soc. Japan, **3**(1951), 317; (**II**) ——, this Journal, **16** (1952), 67; (**III**) ——, this Journal, **16** (1952), 291; (**IV**) ——, this Journal, **16** (1952), 299. The same notations as in these papers are used throughout the present paper. See also H. Takeno, Prog. Theor. Phys. 8 (1952), 317.

2) This condition is equivalent to $\overset{2}{\rho} \neq \overset{3}{\rho}$, since $\alpha = \xi = -(\dot{g}^2 + \ddot{g})$ and $\beta = \eta = -\dot{g}^2 + e^{-2g}/R^2$ in the standard coordinate system for the c. s.

3) See also H. Takeno, this Journal, **12** (1942), 125. (Japanese).