DEFINITION OF MOMENTUM AND MASS AS AN INVARIANT VECTOR OF THE NEW FUNDAMENTAL GROUP OF TRANSFORMATIONS IN SPECIAL RELATIVITY AND QUANTUM MECHANICS

By

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§ 1. New fundamental group of transformations in special relativity and quantum mechanics.

In special relativity and quantum mechanics, special Lorentz transformations have been taken, as the fundamental group of transformations, to represent the relations between the coordinates of two systems one of which is moving with uniform velocity to the other. In the previous paper [1] we have shown that, for representing the relations between the coordinates of the two systems, a new fundamental group of transformations should be taken in place of special Lorentz transformations. Such a new fundamental group of transformations has been given without writing detailed calculations. In this paper we will describe our theory adding some results.

Now we consider two coordinate systems $K(x, y, z, t)$ and $K'(x', y', z', t')$ and suppose that K' is moving with uniform velocity to K, the $x-$, $y-$, $z-$ components of the velocity of the origin of K' being u^1, u^2, u^3 with respect to K. In order to obtain the relations between the coordinates of K and K', we have placed the following three postulates [1].

I. The velocity of light must be constant ($=c$) for the two systems K and K'.

II. The relations between the coordinates of K and K', should be determined uniquely by the components u^1, u^2, u^3 of the uniform velocity. We denote these relations by

$$x'^{\alpha} = f^{\alpha}(x^1, x^2, x^3, x^4; u^1, u^2, u^3) \qquad (\alpha = 1, \cdots, 4) \qquad , \quad (1.1)$$

where x^1, x^2, x^3, x^4 are used in place of x, y, z, t, and f^{α} are certain functions of x's and u's.

III. The transformations (1.1), u^1, u^2, u^3 being regarded as parameters, must form a 3-parameter continuous transformation group.

From these postulates the general form of the equations (1.1) has been determined in the previous paper [2]. The simplest form of (1.1) has been given by the equations (2.6) in the previous paper [1]. In this paper, for our future purpose, we take $-d^i$ instead of d^i in the previous papers [1, 2], which means no essential difference. Then the equations (2.6) in the previous paper [1], are expressed as follows:

$$\left.\begin{aligned} x'^i &= x^j\left[\delta^i_j - \frac{d^i-u^i/c}{1-(du)/c}\,d_j - d^i\left\{\frac{u_j/c}{\sqrt{1-(uu)/c^2}} - \frac{d_j\sqrt{1-(uu)/c^2}}{1-(du)/c}\right\}\right] \\ &\quad + t\left[d^i\frac{(uu)/c-(du)\{1-\sqrt{1-(uu)/c^2}\}}{\{1-(du)/c\}\sqrt{1-(uu)/c^2}} - \frac{u^i}{1-(du)/c}\right] \\ t' &= [t-(ux)/c^2]/\sqrt{1-(uu)/c^2} \qquad (i,\ j=1,\ 2,\ 3) \end{aligned}\right\} \quad (1.2)$$

In these equations we have made use of the convention, to be used throughout, that when the same index appears twice in a term this term stands for the sum of the terms obtained by giving the index each of its values; thus the first term of the right hand side of (1.2) stands for the sum of 3 terms as j takes the values 1 to 3. And in (1.2), δ^i_j are defined by $\delta^i_j=1$ or 0 as $i=j$ or $i\neq j$, and $d_i(=d^i)$ are any constants satisfying the condition $d_i d^i=1$. Further (du) denotes the inner product of d_i and u^i, namely $(du)=d_i u^i$, and similarly $(uu)=u^i u^i$. The inverse transformation of (1.2) is given by the equations:

$$\left.\begin{aligned} x^i &= x'^i + u^i\frac{t'-(dx')/c}{\sqrt{1-(uu)/c^2}} + d^i\frac{(ux')/c-(dx')\{1-\sqrt{1-(uu)/c^2}\}}{1-(du)/c}, \quad (i=1,\ 2,\ 3) \\ t &= \frac{1}{\sqrt{1-(uu)/c^2}}\left[\frac{\{(du)/c-(uu)/c^2\}\,(dx')/c}{1-(du)/c} + t'\right] \\ &\qquad + [(ux')-(du)(dx')]/c^2[1-(du)/c] \end{aligned}\right\} \quad (1.3)$$

which are obtained by putting $h=k=0$ in the equations (4.1) of the previous paper [2] and by replacing d^i by $-d^i$.

Corresponding to the equations above, we can obtain the formula for sum of velocities. Namely from (1.3), putting $dx^i/dt=w^i$ and $dx'^i/dt'=v^i$, we have

$$w^i = \frac{\begin{bmatrix} v^i\{1-(du)/c\} + d^i\left[(uv)/c-(dv)\{1-\sqrt{1-(uu)/c^2}\}\right] \\ +u^i\{1-(dv)/c\}\{1-(du)/c\}/\sqrt{1-(uu)/c^2} \end{bmatrix}}{\frac{\{(du)/c-(uu)/c^2\}(dv)/c+1-(du)/c}{\sqrt{1-(uu)/c^2}} + \frac{(uv)-(du)(dv)}{c^2}} \tag{1.4}$$

which represents the composition law of velocities u^i and v^i.

Taking (1.2) or (1.3) as the fundamental group of transformations, it will be considered newly to define physical quantities (e.g. momentum, mass, energy) and wave equations in quantum mechanics. In the next section a part of this problem will be considered. (The problem to define wave equations will be treated in our future paper.)

§ 2. Definition of momentum and mass as an invariant vector of (1.3) and (1.4).

In special relativity, *momentum* and *mass* are defined such that the equations of motion are invariant under special Lorentz transformations. Corresponding to the above, as the first step, we will try to define newly momentum and mass as an invariant vector of the fundamental group of transformations (1.3). For this purpose we consider a particle moving in the system K', the $x'-$, $y'-$, $z'-$ components of the velocity of the particle being v^1, v^2, v^3. Here we assume that momentum and mass constitute a vector $M^\alpha(v)$ ($\alpha=1, \cdots, 4$) in 4-dimensional space-time, where $M^\alpha(v)$ are functions of v^1, v^2, v^3 and do not contain x''s. By transforming the coordinates from K' to K, the vector $M^\alpha(v)$ is transformed to $\frac{\partial x^\alpha}{\partial x'^\beta} M^\beta(v)$ ($\alpha, \beta=1, \cdots, 4$), and the components v^i of the velocity of the particle are transformed to w^i given by (1.4). Hence, in order that the vector $M^\alpha(v)$ may be expressed in the same form for two systems K and K', it must be that

$$M^\alpha(w) = \frac{\partial x^\alpha}{\partial x'^\beta} M^\beta(v) \qquad (\alpha, \beta = 1, \cdots, 4) \tag{2.1}$$

where $M^\alpha(w)$ denotes the function obtained by replacing v^i by w^i in the expression $M^\alpha(v)$ We call the vector $M^\alpha(v)$ which satisfies the equations (2.1), *invariant vector* of the transformations (1.3) and (1.4).

Now assuming that momentum and mass constitute an invariant vector of (1.3) and (1.4), we will find such an invariant vector $M^\alpha(v)$. Denoting the time component of the vector by M, (*i.e.* $\boldsymbol{M}^4=\boldsymbol{M}$) the solution of (2.1) is given by: (see § 3)

$$\left.\begin{aligned} M^h &= \frac{m}{\sqrt{1-(vv)/c^2}}v^h + \frac{n\sqrt{1-(vv)/c^2}+(lv)/c}{1-(dv)/c}cd^h + cl^h \\ M &= \frac{m}{\sqrt{1-(vv)/c^2}} + \frac{n\sqrt{1-(vv)/c^2}+(lv)/c}{1-(dv)/c} \quad (h=1,\ 2,\ 3) \end{aligned}\right\} \tag{2.2}$$

where m and n are arbitrary constants and l^h are any constants satisfying the condition $(ld)=0$. The vector defined by (2.2), gives new definition of *momentum* and *mass* as the invariant vector of the fundamental group of transformations (1.3) and (1.4). It may be noticeable that new constants are introduced in our theory. Physical interpretation of (2.2) will be done afterwards. (Also see [**N. B**] at the end of this paper.)

If we add the condition that the vector defined by (2.2) is moreover invariant under the group of rotations in x, y, z space (which is equivalent to the condition that the vector $M^\alpha(v)$ is invariant under the general Lorentz transformation group), we can show that the condition is equivalent to $n=0$ and $l^i=0$ $(i=1, 2, 3)$. Hence, under the condition of spherical symmetry, (2.2) becomes

$$M^h = \frac{m}{\sqrt{1-(vv)/c^2}}v^h, \quad M = \frac{m}{\sqrt{1-(vv)/c^2}},$$

which coincides with the ordinary definition of momentum and mass.

From the definition of mass (2.2), energy E is defined by the same way as in special relativity, viz. $E=Mc^2$. Then we have the relation between energy and momentum as follows: $E^2/c^2-(MM)=c^2\{m^2+2mn-(ll)\}$. Further if we put $M^h=\underset{(1)}{M^h}+\underset{(2)}{M^h}+cl^h$, $M=\underset{(1)}{M}+\underset{(2)}{M}$,

where
$$\underset{(1)}{M^h} = \frac{m}{\sqrt{1-(vv)/c^2}}v^h, \quad \underset{(2)}{M^h} = \frac{n\sqrt{1-(vv)/c^2}+(lv)/c}{1-(dv)/c}cd^h$$
$$\underset{(1)}{M} = \frac{m}{\sqrt{1-(vv)/c^2}}, \quad \underset{(2)}{M} = \frac{n\sqrt{1-(vv)/c^2}+(lv)/c}{1-(dv)/c}.$$

then $\underset{(1)}{M^h}$, $\underset{(1)}{M}$ and $\underset{(2)}{M^h}$, $\underset{(2)}{M}$ constitute a vector in 4-dimensional space time respectively, and the following relations hold:

$$c^2\underset{(1)}{M^2}-(\underset{(1)(1)}{MM}) = c^2m^2, \quad c^2\underset{(2)}{M^2}-(\underset{(2)(2)}{MM}) = 0.$$

§ 3. Invariant vector of the transformations (1.3) and (1.4).

In this section we will show that the solution of (2.1) is given by (2.2). For this purpose we will consider the infinitesimal transformations of (1.3) and (1.4), regarding u^1, u^2, u^3 as parameters. From (1.3), differentiating x^i and t by u^j and putting $u^1=u^2=u^3=0$ in the resulting equations, we have:

$$\left[\frac{\partial x^i}{\partial u^j}\right]_{u=0}=\delta^i_j\left\{t'-\frac{(dx')}{c}\right\}+\frac{d^ix'^j}{c},\quad \left[\frac{\partial t}{\partial u^j}\right]_{u=0}=\frac{x'^j}{c^2}\quad (i,j=1,2,3) \tag{3.1}$$

By the similar manner, from (1.4) we have:

$$\left[\frac{\partial w^i}{\partial u^j}\right]_{u=0}=\delta^i_j\left\{1-\frac{(dv)}{c}\right\}+\frac{d^iv^j}{c}-\frac{v^iv^j}{c^2}. \tag{3.2}$$

From (3.1) and (3.2), the infinitesimal transformations for x''s, t' and v's by an infinitesimal change δ_T of parameter u^j, are given by:

$$\left.\begin{aligned} x^i&=x'^i+\left[\delta^i_j\left\{t'-\frac{(dx')}{c}\right\}+\frac{d^ix'^j}{c}\right]\delta_T,\quad t=t'+\frac{x'^j}{c^2}\delta_T\\ w^i&=v^i+\left[\delta^i_j\left\{1-\frac{(dv)}{c}\right\}+\frac{d^iv^j}{c}-\frac{v^iv^j}{c^2}\right]\delta_T. \end{aligned}\right\} \tag{3.3}$$

Substituting (3.3) into (2.1) and expanding the resulting equations in power series of δ_T, we have:

$$\left.\begin{aligned} &\left[\delta^i_j\left\{1-\frac{(dv)}{c}\right\}+\frac{d^iv^j}{c}-\frac{v^iv^j}{c^2}\right]\frac{\partial}{\partial v}M^h=\delta^h_j\left\{M-\frac{(dM)}{c}\right\}+\frac{d^hM^j}{c}\\ &\left[\delta^i_j\left\{1-\frac{(dv)}{c}\right\}+\frac{d^iv^j}{c}-\frac{v^iv^j}{c^2}\right]\frac{\partial}{\partial v}M=\frac{M^j}{c^2}\quad (h,i,j=1,2,3) \end{aligned}\right\} \tag{3.4}$$

(The first and the second equations of the above, are deduced from the equations corresponding to the cases $\alpha=h$ (h=1, 2, 3) and $\alpha=4$ in (2.1)). In order to simplify the left hand member of (3.4) we introduce new variables $\bar{v}^k$ in place of v^k by the following relations:

$$\bar{v}^k=[v^k-cd^k]/\sqrt{1-(vv)/c^2}\qquad (k=1,2,3) \tag{3.5}$$

By the new variables, the operators of the left hand side of (3.4) are expressed as follows:

$$\left[\delta^i_j\left\{1-\frac{(dv)}{c}\right\}+\frac{d^iv^j}{c}-\frac{v^iv^j}{c^2}\right]\frac{\partial}{\partial v^i}=-\delta^k_j\frac{(d\bar{v})}{c}\frac{\partial}{\partial\bar{v}^k},$$

since $$\left[\delta^i_j\left\{1-\frac{(dv)}{c}\right\}+\frac{d^iv^j}{c}-\frac{v^iv^j}{c^2}\right]\frac{\partial\bar{v}^k}{\partial v^i}=-\delta^k_j\frac{(d\bar{v})}{c}. \tag{3.6}$$

Therefore, by the new variables $\bar{v}^k$, the equations (3.4) are expressed as:

$$\left.\begin{aligned} &-\frac{(d\bar{v})}{c}\frac{\partial}{\partial \bar{v}^j}M^h = \delta^h_j\left\{M-\frac{(dM)}{c}\right\}+\frac{d^h M^j}{c} \\ &-c\,(d\bar{v})\,\partial M/\partial \bar{v}^j = M^j \qquad (h,\, j=1,\, 2,\, 3) \end{aligned}\right\} \tag{3.7}$$

Solving these equations, we have:

$$\left.\begin{aligned} &M^h = m\bar{v}^h - \frac{cd^h}{(d\bar{v})}\left[\frac{mc}{2}\left\{1+\frac{(\bar{v}\bar{v})}{c^2}\right\}+nc+(l\bar{v})\right]+cl^h \\ &M = -\frac{1}{(d\bar{v})}\left[\frac{mc}{2}\left\{1+\frac{(\bar{v}\bar{v})}{c^2}\right\}+nc+(l\bar{v})\right] \qquad (h=1,\, 2,\, 3) \end{aligned}\right\} \tag{3.8}$$

where m and n are arbitrary constants and l^h are any constants satisfying the condition $(ld)=0$. From the above, expressing $\bar{v}^k$ in terms of v^k by (3.5), we have (2.2) which is the solution of (2.1).

Appendix

§ 4. Determination of the operators of the infinitesimal transformations which generate the fundamental group of transformations satisfying the postulates I, II and III.

In the previous paper [2] we have shown that the fundamental group of transformations satisfying the postulates I, II and III of § 1 is generated by the infinitesimal transformations with the following operators of the form:

$$P_i = Q_i + c_{ij}R_j + c_i Q \qquad (i,\, j=1,\, 2,\, 3) \tag{4.1}$$

which are the equations (2.2) of the previous paper [2]. Here Q_i, R_i and Q are the operators defined by:

$$\left.\begin{aligned} &Q_i = x^i\partial/c\partial t + ct\partial/\partial x^i && (i,\, j,\, k=1,\, 2,\, 3) \\ &R_i = \varepsilon_{ijk}x^j\partial/\partial x^k, \quad Q = x^\alpha\partial/\partial x^\alpha, && (\alpha=1, \cdots, 4) \end{aligned}\right\} \tag{4.2}$$

where ε_{ijk} are such that

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } i,\, j,\, k \text{ is an even permutation of } 1,\, 2,\, 3, \\ -1 \text{ if } i,\, j,\, k \text{ is an odd permutation of } 1,\, 2,\, 3, \\ 0 \text{ in any other cases,} \end{cases}$$

and c_{ij} and c_i are constants determined from the condition that the commutators of P_i and P_j are expressed by linear combinations of P_k $(k=1, 2, 3)$ with constant coefficients, i.e.

$$[P_i P_j] = d_{ijk}P_k\,. \tag{4.3}$$

Without describing calculation, we have stated that the constants c_{ij}, c_i and d_{ijk} which satisfy the condition (4.3), are given by the equations (2.3) of the previous paper [2], namely:

$$\left.\begin{aligned} &c_{ij}=\varepsilon_{ijk}d^k+hd_id_j, \quad c_i=kd_i \qquad (i,j,k=1,2,3) \\ &d_{ijk}=hd^l(d_j\varepsilon_{ilk}-d_i\varepsilon_{ljk})+d_i\delta_{jk}-d_j\delta_{ik} \end{aligned}\right\} \tag{4.4}$$

In this section, by writing calculation, we will show that the equations (4.4) are deduced from the condition (4.3). Substituting (4.1) into (4.3) and using the following relations:

$$\left.\begin{aligned} &[Q_iQ_j]=\varepsilon_{ijk}R_k, \quad [R_iR_j]=-\varepsilon_{ijk}R_k \qquad (i,j,k=1,2,3) \\ &[Q_iR_j]=-\varepsilon_{ijk}Q_k, \quad [QQ_i]=0, \quad [QR_i]=0, \end{aligned}\right\} \tag{4.5}$$

we can express (4.3) as follows:

$$\begin{aligned} &\varepsilon_{ijk}R_k+c_{jl}\varepsilon_{ilk}Q_k-c_{il}\varepsilon_{ljk}Q_k-c_{il}c_{jm}\varepsilon_{lmk}R_k \\ &\qquad =d_{ijk}(Q_k+c_{kl}R_l+c_kQ). \qquad (i,j,k=1,2,3) \end{aligned}$$

From the above, since the operators Q, R_k, and Q_k (k=1, 2, 3) are linearly independent, we have:

$$d_{ijk}c_k=0, \qquad (h,i,j,k,l=1,2,3) \tag{4.6}$$

$$\varepsilon_{ijk}-c_{il}c_{jm}\varepsilon_{lmk}=d_{ijh}c_{hk}, \tag{4.7}$$

$$c_{jl}\varepsilon_{ilk}-c_{il}\varepsilon_{ljk}=d_{ijk}. \tag{4.8}$$

Substituting (4.8) into (4.7) we have the equations for c_{ij}:

$$\varepsilon_{ijk}-c_{il}c_{jm}\varepsilon_{lmk}=(c_{jl}\varepsilon_{ilh}-c_{il}\varepsilon_{ljh})c_{hk}. \tag{4.9}$$

In order to solve the equations (4.9) we put

$$c_{ij}=g_{ij}+f_{ij}, \tag{4.10}$$

were g_{ij} and f_{ij} represent the symmetric and antisymmetric parts of c_{ij}. Substituting (4.10) into (4.9), we have:

$$\left.\begin{aligned} &\varepsilon_{ijk}-(g_{il}g_{jm}+g_{il}f_{jm}+f_{il}g_{jm}+f_{il}f_{jm})\varepsilon_{lmk} \\ &\quad =\varepsilon_{ilh}(g_{jl}g_{hk}+g_{jl}f_{hk}+f_{jl}g_{hk}+f_{jl}f_{hk}) \\ &\quad -\varepsilon_{ljh}(g_{il}g_{hk}+g_{il}f_{hk}+f_{il}g_{hk}+f_{il}f_{hk}). \end{aligned}\right\} \tag{4.11}$$

Further, by putting

$$f_{ij}=\varepsilon_{ijk}d_k \qquad (i,j,k=1,2,3), \tag{4.12}$$

we express f_{ij} in terms of d_k (k=1, 2, 3) and substitute (4.12) into (4.11). Then, after some calculation, using the relations:

$$\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl},$$

we have:

$$\left.\begin{array}{l}\varepsilon_{ijk}-g_{il}g_{jm}\varepsilon_{klm}-\varepsilon_{ijp}d_pd_k\\ \quad+g_{jl}g_{km}\varepsilon_{ilm}-\varepsilon_{jkp}d_pd_i\\ \quad-g_{il}g_{km}\varepsilon_{jlm}-\varepsilon_{kip}d_pd_j\end{array}\right\}=2g_{jk}d_i-2g_{ik}d_j \tag{4.13}$$

From (4.13), taking the symmetric and antisymmetric parts with respect to j and k, we have:

$$-g_{il}g_{jm}\varepsilon_{klm}-g_{il}g_{km}\varepsilon_{jlm}=2g_{jk}d_i-g_{ki}d_j-g_{ij}d_k \tag{4.14}$$

and

$$\left.\begin{array}{l}\varepsilon_{ijk}+g_{jl}g_{km}\varepsilon_{ilm}\\ -\varepsilon_{ijp}d_pd_k-\varepsilon_{jkp}d_pd_i-\varepsilon_{kip}d_pd_j\end{array}\right\}=g_{ij}d_k-g_{ik}d_j\,. \tag{4.15}$$

In the equations (4.14), if we put $j=k$ and sum up for $j=1, 2, 3$, it follows that

$$g_{ij}d_j-Gd_i=0\,, \tag{4.16}$$

where we put $G=g_{jj}$. Hence, if we multiply (4.15) by d_k and sum up for $k=1, 2, 3$, we have:

$$\varepsilon_{ijk}d_k+Gg_{jl}d_m\varepsilon_{ilm}-\varepsilon_{ijp}d_p(d_kd_k)=g_{ij}(d_kd_k)-Gd_id_j\,. \tag{4.17}$$

From the above, taking the antisymmetric part with respect to i and j, we have

$$\varepsilon_{ijk}d_k-\varepsilon_{ijp}d_p(d_kd_k)=0\,, \tag{4.18}$$

since the antisymmetric part of the second term in the left hand side of the equation (4.17) is reduced to zero as follows:

$$g_{jl}d_m\varepsilon_{ilm}-g_{il}d_m\varepsilon_{jlm}=\varepsilon_{ijk}(g_{km}d_m-d_kg_{ll})=0\,.$$

Equations (4.18) are rewritten as

$$\varepsilon_{ijp}d_p(1-d_kd_k)=0\,.$$

Hence it must be that

$$d_kd_k=1\,, \tag{4.19}$$

unless $d_p=0$ $(p=1, 2, 3)$. But, when $d_p=0$ $(p=1, 2, 3)$, (4.14) and (4.15) become

$$g_{il}g_{jm}\varepsilon_{klm}+g_{il}g_{km}\varepsilon_{jlm}=0$$

and

$$\varepsilon_{ijk}+g_{jl}g_{km}\varepsilon_{ilm}=0$$

respectively, and the solutions of these equations become $g_{ij}=\pm i\delta_{ij}$, which are imaginary. Therefore, if we confine ourselves to real transformations, we must have $d_k d_k=1$. Then the equations (4.15) are reduced to

$$g_{jl} g_{km} \varepsilon_{ilm} = g_{ij} d_k - g_{ik} d_j , \tag{4.20}$$

since the remaining terms of (4.15) vanish as follows:

$$\varepsilon_{ijk} - \varepsilon_{ijp} d_p d_k - \varepsilon_{jkp} d_p d_i - \varepsilon_{kip} d_p d_j = \varepsilon_{ijk} - \varepsilon_{ijk} d_l d_l = 0$$

Moreover, we can easily see that the equations (4.14) are satisfied identically by (4.20). So that the equations (4.20) are the only equations to be solved. In order to solve the equations (4.20), we put

$$H_{ij} = g_{ij} - G d_i d_j \tag{4.21}$$

and express the equations (4.20) in terms of H_{ij}. Then

$$H_{jl} H_{km} \varepsilon_{ilm} + G(H_{jl} d_k d_m + H_{km} d_j d_l)\, \varepsilon_{ilm} = H_{ij} d_k - H_{ik} d_j \tag{4.22}$$

Multiplying the above by d_k, summing up for $k=1, 2, 3$, and using the relations $H_{km} d_k=0$, we have

$$G H_{jl} d_m \varepsilon_{ilm} = H_{ij} , \tag{4.23}$$

by which (4.22) become

$$H_{jl} H_{km} \varepsilon_{ilm} = 0 .$$

The equations (4.23) are expressed as

$$H_{jl} (G d_m \varepsilon_{ilm} - \delta_{il}) = 0 .$$

Calculating the determinant made by the coefficients of H_{jl} in the above, we have

$$| G d_m \varepsilon_{ilm} - \delta_{il} | = -(1+G),$$

which does not vanish as long as G is real. Hence it must be that $H_{jl}=0$. So we have the solutions of (4.20), as given by $g_{ij}=G d_i d_j$. Here, G may be any constant, which we denot by h. Then by (4.10) and (4.12), the general solutions of (4.9) are given by

$$c_{ij} = h d_i d_j + \varepsilon_{ijk} d_k \tag{4.24}$$

where d_k ($k=1, 2, 3$) are any constants subject to the condition $d_k d_k=1$.

From (4.8), using (4.24), d_{ijk} are determined by the equations

$$d_{ijk} = h d_l (d_j \varepsilon_{ilk} - d_i \varepsilon_{ljk}) + d_i \delta_{jk} - d_j \delta_{ik} . \tag{4.25}$$

Finally, we will solve the equations (4.6). By (4.25), (4.6) become

$$h d_l (d_j \varepsilon_{ilk} c_k - d_i \varepsilon_{ljk} c_k) + d_i c_j - d_j c_i = 0 \tag{4.26}$$

Multiplying the above by d_j, summing up for $j=1, 2, 3$, and using the relations (4.19), it follows that

$$(hd_l\varepsilon_{lik}+d_id_k-\delta_{ik})\,c_k=0, \tag{4.27}$$

from which we have

$$c_i=kd_i, \qquad (i=1, 2, 3) \tag{4.28}$$

where k is an arbitrary constant. By (4.28) the equations (4.26) are satisfied identically. So that the general solutions of (4.26) are given by (4.28).

References

T. Shibata, [1] Some properties of Lorantz transformations, this journal vol. **16**, No. 2 (1952), 285-290.

[2] Fundamental group of transformations in special relativity and quantum mechanics, ibid., vol. **16**, No. 1 (1952), 61—66.

[N. B] The equations (2.2) of this paper which give the definition of momentum and mass, become simpler when we take the case where the properties under consideration are axially symmetric. Namely, if we suppose that the vector of momentum and mass defined by the equations (2.2) is invariant under the rotations about the axis whose direction cosines are d^1, d^2, d^3, we have $l^h=0$ $(h=1, 2, 3,)$. Hence in this case, momentum and mass are defined by

$$M^h=\frac{m}{\sqrt{1-(vv)/c^2}}v^h+\frac{n\sqrt{1-(vv)/c^2}}{1-(dv)c}cd^h \qquad (h=1, 2, 3)$$

$$M=\frac{m}{\sqrt{1-(vv)/c^2}}+\frac{n\sqrt{1-(vv)/c^2}}{1-(dv)/c}$$

m and n being arbitrary constants. (See the next paper).