ON AXIOM OF BETWEENNESS (Continued)

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Chapter III

On open-betweenness in partially ordered set

When $x>y>z$ or $x<y<z$ for three elements x, y aud z in a partially ordered set, this relation is called open-betweenness $[xyz]$.

Now, let A be a set and R be some collection of $\alpha\equiv\{abc\}\ (a, b, c \neq)$ which is an element of triple product space $A\times A\times A$. In this chapter, we shall consider the conditions in order that R may be the set of open-betweenness. In other words, we shall require the complete system of conditions for open-betweenness. We use the following notation: $|acb|^{\times}$ means that $\{acb\}$ is not contained in R.

In this chapter, we shall have the following results:

Theorem. *The following system* $\mathfrak{O}.\mathfrak{B}$ *is the complete system of conditions for open-betweenness.*

$$\mathfrak{O}.\mathfrak{B}\begin{cases}\text{OB3.} & |abc|\to|cba|.\\ \text{OB4.} & |abc|\to|acb|^{\times}.\\ \text{OB6.} & \|a_1, a_2, b_1\|\cdot\|a_2, a_3, b_2\|\cdots\cdots\|a_{2n+1}, a_1, b_{2n+1}\|\to|a_i a_{i+1} a_{i+2}|.\\ & \textit{for at least one } i\ (1\leqq i\leqq 2n+1);\ n=1, 2, \dots;\ a_{2n+1+i}=a_i\\ & \textit{and } a_i, a_{i+1}, b_i \neq\ (i=1, 2, \dots, 2n+1).\\ \text{OB7.} & |abc|\cdot\|b, d, e\|\to|abd|\ \text{or}\ |dbc|.\end{cases}$$

Proof. The necessity of the conditions OB3, OB4, OB6 and OB7 is clear from (9·3) in chapter I. And by the following examples these conditions are independent of each other.

Example	Set	R	
1	a, b, c	$\{abc\}$	only OB3 is not satisfied
2	a, b, c	$\{abc\}\ \{acb\}\ \{cba\}\ \{bca\}$	OB4 „
3	a, b, c	$\{abp\}\ \{bcq\}\ \{car\},\{pba\}\ \{qcb\}\ \{rac\}$	OB6 „
	p, q, r	$\{cbp\}\ \{bar\}\ \{acq\},\{pbc\}\ \{rab\}\ \{qca\}$	
		$\{par\}\ \{rcq\}\ \{qbp\},\{rap\}\ \{qcr\}\ \{pbq\}$	
		$\{qar\}\ \{rbp\}\ \{pcq\},\{raq\}\ \{pbr\}\ \{qcp\}$	
4	a, b, c d, e	$\{abc\}\ \{bde\}\ \{cba\}\ \{edb\}$	OB7 „

And by the same way as that of §2 in chapter I, we see that the condition OB6 is independent as to index n.

Note. From the following example we see that this system $\mathfrak{O}.\mathfrak{B}$ does not decompose in two subsets.

Example. Set$\equiv\{a_1, a_2, \dots, a_n, \dots\}$

$\{a_1a_2a_3\}$ $\{a_1a_2a_4\}$ ...... $\{a_1a_2a_n\}$ ......, $\{a_3a_2a_1\}$ $\{a_4a_2a_1\}$ ...... $\{a_na_2a_1\}$ ......,

$\{a_1a_3a_4\}$ $\{a_1a_3a_5\}$ ...... $\{a_1a_3a_n\}$ ......, $\{a_4a_3a_1\}$ $\{a_5a_3a_1\}$ ...... $\{a_na_3a_1\}$ ......,

....

....

$\{a_1a_na_{n+1}\}$ $\{a_1a_na_{n+2}\}$ .... , $\{a_{n+1}a_na_1\}$ $\{a_{n+2}a_na_1\}$ .... ,

....

....

$\{a_2a_3a_4\}$ $\{a_2a_3a_5\}$ ...... $\{a_2a_3a_n\}$ ......, $\{a_4a_3a_2\}$ $\{a_5a_3a_2\}$ ...... $\{a_na_3a_2\}$, ...

....

....

Next, we shall prove that the system of conditions $\mathfrak{O}.\mathfrak{B}$ is sufficient for open-betweenness. In this proof, we shall use the following notations.

$C \Rrightarrow D$: If there exists C, we add D to C.

$R(\mathfrak{O}.\mathfrak{B})$: R which is closed concerning the system $\mathfrak{O}.\mathfrak{B}$.

$R(\mathfrak{B}'')$: R which is closed concerning the system $\mathfrak{B}''$.

$R(2.1)$: the set which is obtained by operating $\| a,b,c \| \Rrightarrow |abb|, |aab|$ and $|aaa|$ to (all elements of) R.

$R(\mathfrak{O}.\mathfrak{B}; 2.1)$: $(R(\mathfrak{O}.\mathfrak{B}))(2.1)$

Now, we consider the collection $R(\mathfrak{O}.\mathfrak{B})$. If we construct $R(\mathfrak{O}.\mathfrak{B}; 2.1)$ by operating $\| a, b, c \| \Rrightarrow |abb|, |aab|$ and $|aaa|$ to (all elements of) $R(\mathfrak{O}.\mathfrak{B})$, then we have $R(\mathfrak{O}.\mathfrak{B}; 2.1) = R(\mathfrak{O}.\mathfrak{B}; 2.1; \mathfrak{B}'')$, that is, $R(\mathfrak{O}.\mathfrak{B}; 2.1)$ is closed concerning the system of conditions $\mathfrak{B}''$.

For; since it is clear that $R(\mathfrak{O}.\mathfrak{B}; 2.1)$ satisfies the conditions B1, B2$''$ and B3, we shall prove that it satisfies the conditions B4$''$, B6$''$ and B7$''$.

B4$''$: From the construction of $R(\mathfrak{O}.\mathfrak{B}; 2.1)$, $\{xab\} \in R(\mathfrak{O}.\mathfrak{B}; 2.1) \rightarrow x \neq b$ and $\{xba\} \in R(\mathfrak{O}.\mathfrak{B}; 2.1) \rightarrow x \neq a$. And when $x \neq a$ and b, from OB4 $|xab|$ and $|xba|$ are not compatible. Therefore, $R(\mathfrak{O}.\mathfrak{B}; 2.1)$ satisfies the condition B4$''$.

From the construction of $R(\mathfrak{O}.\mathfrak{B}; 2.1)$, if $\| a, b, b \|$ exists in $R(\mathfrak{O}.\mathfrak{B}; 2.1)$, then there exists $\| a, b, c \|$ (for some c) in $R(\mathfrak{O}.\mathfrak{B}; 2.1)$. ....(1)

B6$''$: We prove this by decomposing two cases:

Case 1. $x_i \neq a_i$ and a_{i+1} $(i=1, 2, \dots, 2n+1)$:

$$\| a_1, a_2, x_1 \| \cdot \| a_2, a_3, x_2 \| \cdots\cdots \| a_{2n+1}, a_1, x_{2n+1} \| \xrightarrow{\text{OB6}} |a_i a_{i+1} a_{i+2}|.$$

Case 2. $x_i = a_i$ (for some i) ⓞⓡ $x_i = a_{i+1}$:

$$\| a_i, a_{i+1}, x_i \| \xrightarrow{(1)} \| a_i, a_{i+1}, b_i \| \text{ where } a_i, a_{i+1}, b_i \neq . \quad \cdots\cdots\cdots (2)$$

So, $\| a_1, a_2, x_1 \| \cdot \| a_2, a_3, x_2 \| \cdots\cdots \| a_{2n+1}, a_1, x_{2n+1} \| \xrightarrow{(2)} \| a_1, a_2, y_1 \| \cdot \| a_2, a_3, y_2 \|$ $\cdots\cdots \| a_{2n+1}, a_1, y_{2n+1} \|$ (where $a_i, a_{t+1}, y_i \neq$) $\xrightarrow{\text{OB6}} |a_i a_{i+1} a_{i+1}|$. Therefore, $R(\mathfrak{D}.\mathfrak{B}; 2.1)$ satisfies the condition B6″.

B7″: Case 1. $x \neq b$ and d: $|abc| \cdot \| b, d, x \| \xrightarrow{\text{OB7}} |abd|$ ⓞⓡ $|dbc|$.

Case 2. $x=b$ or $x=d$: $|abc| \; \| b, d, x \| \xrightarrow{(1)} |abc| \cdot \| b, d, e \| \xrightarrow{\text{OB7}} |abd|$ ⓞⓡ $|dbc|$.

Therefore, $R(\mathfrak{D}.\mathfrak{B}; 2.1)$ satisfies the condition B7″.

So, we have $R(\mathfrak{D}.\mathfrak{B}; 2.1) = R(\mathfrak{D}.\mathfrak{B}; 2.1; \mathfrak{B}'')$.

From the above result and (9·4) in Chap. I, we can introduce the partial order in R^* such that $R(2.1)$ is the set of betweenness and so that R is the set of open-betweenness.[1] Hence, the system $\mathfrak{D}.\mathfrak{B}$ is the sufficient conditions for open-betweenness.

Thus, we have proved the Theorem.

Remark: For the open-betweenness of each special case with which we have dealt in Chap. II, we have the complete system of conditions by replacing OB6 [1], ..., OB6 [4] and OB6 [5] respectively for OB6 in the system $\mathfrak{D}.\mathfrak{B}$: that is,

1. For the case where A has inner radius r:
   OB6 [1]: $\| a_1, a_2, b_1 \| \cdot \| a_2, a_3, b_2 \| \cdots\cdots \| a_{2n+1}, a_1, b_{2n+1} \| \rightarrow |a_i a_{i+1} a_{i+2}|$.
   for at least one i $(1 \leqq i \leqq 2n+1)$, $n=1, 2, \dots, r$ and $a_{2n+1+i} = a_i$.
2. For the case where A has a center:
   OB6 [2] $\| a, b, c \| \rightarrow \| a_0, a, b \|$ for some one fixed element a_0.
3. For the case where A has one extreme element:
   OB6 [3]: $\| a, b, c \| \rightarrow |a_0 ab|$ or $|a_0 ba|$ for some one fixed element a_0.
4. For the case where A has two extreme elements:
   OB6 [4]: $\| a, b, c \| \rightarrow |a_0 a a_\infty|$ for some two fixed elements a_0 and a_∞.
5. For the case where A is simply ordered:
   OB6 [5]: $\| a, b, c \| \cdot \| d, e, f \| \rightarrow \| a_0, a, d \|$ for some one fixed element a_0

Chapter IV

On betweenness in quasi-partially ordered set

In a quasi-partially ordered set[2], (i.e. the set in which a binary relation $x \geqq y$ is defined, which satisfies P1 and P3), when $x \geqq y \geqq z$ or $x \leqq y \leqq z$

1) Cf. §4 in Chap. I Page 187: meaning of S^*).
2) Cf. G. Birkhoff; Lattice Theory (1948) P. 4.

for three elements x, y and z, this relation is called closed-Q-betweenness $\langle xyz \rangle$.

In this chapter, we shall investigate the conditions in order that R, which is some collecton of elements of triple product space $A \times A \times A$, may be the set of closed-Q-betweenness of set A.

§1. Necessary conditions for closed-Q-betweenness.

In this section, we shall require the conditions which closed-Q-betweenness must satisfy. About this, we shall have the following result:

Theorem 1. *The following conditions must be held for closed-Q-betweenness in a quasi-partially ordered set A.*

B1. $\langle aaa \rangle$ *for all* a $(a \in A)$.

B2. $\langle abx \rangle \to \langle aab \rangle$.

B3. $\langle axb \rangle \to \langle bxa \rangle$.

QB5. $\langle axb \rangle \cdot \langle xby \rangle$ $x \not\equiv b \to \langle aby \rangle$.

QB6. $\langle a_1 a_2 x_1 \rangle \cdot \langle a_2 a_3 x_2 \rangle \cdots \langle a_{2n+1} a_1 x_{2n+1} \rangle \to \langle a_i a_{i+1} a_{i+2} \rangle$.
for at least one i $(1 \leqq i \leqq 2n+1)$, $n=1, 2, \dots$, $a_{2n+1+i}=a_i$ *and* $a_i \not\equiv a_{i+1}$ $(i=1, 2, \dots, 2n+1)$.

QB7$_1$. $\langle abc \rangle \cdot \langle bdx \rangle$ $a \not\equiv b$, $b \not\equiv c$, $b \not\equiv d \to \langle abd \rangle$ or $\langle dbc \rangle$.

B7$_2$. $\langle axb \rangle$ $b \equiv c \to \langle abc \rangle$.

where $p \equiv q$ *means that* $\langle xpq \rangle$ *and* $\langle xqp \rangle$ *for some* x. ......(1.1)

Proof. By the similar way to that in the proof of the necessity of the conditions B1, B2, B3, B5 and B7 in §1 of Chap. I, we see that the conditions B1, B2, B3, QB5 and QB7$_1$ are necessary for closed-Q-betweenness. So, we prove the necessity of the conditions B6 and B7$_2$.

B6: We prove by decomposing two cases.

Case 1. $a_i \not\equiv a_{i+1}$ $(i=1, 2, \dots, 2n+1)$: In this case, by the similar way to that in §1 Chap. I, we see that the condition B6 is necessary.

Case 2. $a_i \equiv a_{i+1}$ for some i: $a_i \equiv a_{i+1} \xrightarrow{(1.1)} a_i \geqq a_{i+1}$ and $a_i \leqq a_{i+1}$. And $\langle a_{i+1} a_{i+2} x_{i+1} \rangle \to a_{i+1} \geqq a_{i+2}$ or $a_{i+1} \leqq a_{i+2}$. So, in this case, we have $\langle a_i a_{i+2} a_{i+2} \rangle$.

B7$_2$: $\langle axb \rangle \to a \geqq b$ or $a \leqq b$, and $b \equiv c \to b \geqq c$ and $b \leqq c$.
So, $\langle axb \rangle \cdot b \equiv c \to a \geqq b \geqq c$ or $a \leqq b \leqq c$. Therefore, we have $\langle abc \rangle$.

Thus, we have Theorem 1.

§2. Independency of conditions.

In this section, we shall investigate the independency of the conditions

B1, ... , $B7_2$ in the following system $\mathfrak{Q}.\mathfrak{B}$ by the same way as that in §2 Chap. I.

$$\mathfrak{Q}.\mathfrak{B}\left\{\begin{array}{rl}
\text{B1.} & |aaa| \text{ for all } a\ (a\in A).^{1)} \\
\text{B2.} & |abx|\to|aab|. \\
\text{B3.} & |axb|\to|bxa|. \\
\text{QB5.} & |axb|\cdot|xby|\ x\not\equiv b\to|aby|. \\
\text{B6.} & |a_1a_2x_1|\cdot|a_2a_3x_2|\cdots\cdots|a_{2n+1}a_1x_{2n+1}|\to|a_ia_{i+1}a_{i+2}|. \\
 & \text{for at lest one } i\ (1\leq i\leq 2n+1),\ n=1,2,\dots,\ a_{2n+1+i}=a_i \text{ and} \\
 & a_i\neq a_{i+1}\ (i=1,2,\dots,2n+1). \\
\text{QB7}_1. & |abc|\cdot|bdx|\ c\not\equiv b,\ b\not\equiv c,\ b\not\equiv d\to|abd| \text{ or } |dbc|. \\
\text{B7}_2. & |axb|\cdot b\equiv c\to|abc|. \\
 & \text{where } p\equiv q \text{ means that } |xpq| \text{ and } |xqp| \text{ for some } x.
\end{array}\right. \qquad \cdots\cdots(2.1)$$

We shall have the following result:

Theorem 2. *The conditions B1, B2, B3, QB5, B6, QB5, B6, $QB7_1$ and $B7_2$ are independent of each other.*

Proof. By the examples 1, 2, 3, 5, 6 and 7 in Chap. I, each of the conditions B1, B2, B3, QB5, B6 and $QB7_1$ is independent of the others and $B7_2$. For the independence of $B7_2$, we have the following examples.

Example 1. $a\equiv b$:

$$\{aab\}\ \{baa\}\ \{abb\}\ \{bba\}\ \{aaa\}\ \{bbb\}\ \{aba\} \qquad \cdots\cdots\cdots(2.2)$$

This example satisfies the conditions B1, B2, B3, QB5, B6 and $QB7_1$, but not $B7_2$. For, $\{baa\}\ \{aab\}\ \{aba\}\Longrightarrow\{baa\}\ a\equiv b \overset{B7_2}{\Longrightarrow}\{bab\}$, but $\{bab\}$ does not exist in (2.2).

Example 2. $a\not\equiv b$:

$$\{abb\}\ \{aab\}\ \{bba\}\ \{baa\}\ \{bcc\}\ \{bbc\}\ \{ccb\}\ \{cbb\}$$
$$\{bcb\}\ \{cbc\}\ \{aaa\}\ \{bbb\}\ \{ccc\} \qquad \cdots\cdots\cdots(2.3)$$

This example satisfies B1, B2, B3, QB5, B6 and $QB7_1$, but not $B7_2$. For, $\{aab\}\ \{bbc\}\ \{bcb\}\Longrightarrow\{aab\}\cdot b\equiv c \overset{B7_2}{\Longrightarrow}\{abc\}$, but $\{abc\}$ does not exist in (2.3).

Thus, we have Theorem 2.

§ 3. Lemmas concerning $R(\mathfrak{Q}.\mathfrak{B})$.

In the following sections, we shall prove that the system of conditions $\mathfrak{Q}.\mathfrak{B}$ is sufficient for closed-Q-betweeneness of set A.

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1) See P. 183 in §9 Chap. I meaning of $|xyz|$ and meaning of $A\to B$).

When R, which is some collection of elements of triple product space $A\times A\times A$, satisfies the system $\mathfrak{Q}.\mathfrak{B}$, R is said to be closed concerning the system $\mathfrak{Q}.\mathfrak{B}$ and represented by $R(\mathfrak{Q}.\mathfrak{B})$.

In this section, we shall investigate the properties of $R(\mathfrak{Q}.\mathfrak{B})$.

Lemma 3.1. $x\equiv y,\ y\equiv z \rightarrow x\equiv z$

Proof. $x\equiv y \xrightarrow{(2.1)} |axy|$ for some $a \xRightarrow{B3} |yxa| \xRightarrow{B2} |yyx| \xRightarrow{B3} |xyy|$;
$|xyy|\cdot y\equiv z \xRightarrow{B7_2} |xyz|$.
$|xyz|\cdot z\equiv y \xRightarrow{B7_2} |xzy| \xRightarrow{B3} |yzx|$. And $|xyz| \xRightarrow{B3} |zyx|$,
$|zyx|\cdot x\equiv y \xRightarrow{B7_2} |zxy| \xRightarrow{B3} |yxz|$.

So, $|yzx|\cdot|yxz| \xRightarrow{(2.1)} x\equiv z$.

Lemma 3.2. $|byd|\cdot b\equiv d \rightarrow y\equiv b$.

Proof. $|byd| \xrightarrow{B2} |bby| \xRightarrow{B3} |ybb|$; $|ybb|\cdot b\equiv d \xRightarrow{B7_2} |ybd|$.
$|byd|\cdot|ybd| \xRightarrow{B3} |dyb|\cdot|dby| \xRightarrow{(2.1)} y\equiv b$.

Lemma 3.3. $|acb|\cdot b\equiv d \rightarrow |acd|$.

Proof. $|acb| \xrightarrow{B3} |bca| \xRightarrow{B2} |bbc| \xRightarrow{B3} |cbb|$; $|cbb|\cdot b\equiv d \xRightarrow{B7_2} |cbd|$.
We proceed by decomposing two cases.

Case 1. $b\not\equiv c$: $|acb|\cdot|cbd| \xRightarrow{B3} |dbc|\cdot|bca| \xRightarrow{QB5} |dca| \xRightarrow{B3} |acd|$.

Case 2. $b\equiv c$: $c\equiv b,\ b\equiv d \xrightarrow{\text{lemma 3.1}} c\equiv d$.
$|acb| \xrightarrow{B2} |aac|$; $|aac|\cdot c\equiv d \xRightarrow{B7_2} |acd|$.

So, we have that $|acb|\cdot b\equiv d \rightarrow |acd|$.

Lemma 3.4. $|abc|\ b\equiv d \rightarrow |adc|$.

Proof. $|abc| \xrightarrow{B2} |aab|$; $|aab|\ b\equiv d \xRightarrow{B7_2} |abd|$; $|abd|\cdot d\equiv b \xRightarrow{B7_2} |adb|$. $\cdots$(3.1)
Similarly, by interchanging a and c, we have that $|abc|\cdot b\equiv d \xRightarrow{B3} |cba|\cdot b\equiv d$ $\Longrightarrow |cbd|$ and $|cdb|$. $\cdots$(3.2)
We prove by decomposing three cases.

Case 1. $a\equiv b$: $a\equiv b,\ b\equiv d \xrightarrow{\text{lemma 3.1}} a\equiv d$. From the above and (3.2),
$|cbd|\cdot d\equiv a \xRightarrow{B7_2} |cda| \xRightarrow{B3} |adc|$.

Case 2. $a\not\equiv b,\ b\equiv c$: $c\equiv b,\ b\equiv d \xrightarrow{\text{lemma3.1}} c\equiv d$.
$|abc|\ c\equiv d \xrightarrow{B7_2} |acd|$; $|acd|\cdot d\equiv c \xRightarrow{B7_2} |adc|$.

Case 3. $a\not\equiv b,\ b\not\equiv c$: From (3.2), $|cdb| \xRightarrow{B2} |ccd| \xRightarrow{B3} |dcc|$. $\cdots\cdots$(3.3)
And $|abc| \xRightarrow{B3} |cba| \xRightarrow{B2} |ccb| \xRightarrow{B3} |bcc|$; $|abc|\cdot|bcc|\cdot b\not\equiv c \xRightarrow{QB5} |acc| \xRightarrow{B2} |aac|$ $\xRightarrow{B3} |caa|$. $\cdots\cdots\cdots$(3.4)
From (3.1), (3.3) and (3.4), we have $|adb|$, $|dcc|$ and $|caa|$. So, by B6 there exists at least one of $|adc|$, $|dca|$ and $|cad|$ in $R(\mathfrak{Q}.\mathfrak{B})$. $\cdots\cdots\cdots$(3.5)
If there exists $|dca|$, $|dca| \xRightarrow{B3} |acd|$; $|acd|\ d\equiv b \xRightarrow{\text{lemma 3.3}} |acb|$. $|abc|\cdot|acb| \xRightarrow{(2.1)}$ $b\equiv c$. This contradicts the assumption $b\not\equiv c$, so $|dca|$ may not occur. And

similarly by using $|dac|$, $b\equiv d$, $|cba|$ and $b\not\equiv a$, we can conclude that $|cad|$ i.e. $|dac|$ may not occur. So, in (3. 5) only $|adc|$ exists in $R(\mathfrak{Q}.\mathfrak{B})$. Therefore, we have $|abc|\cdot b\equiv d\rightarrow|adc|$.

Lemma 3. 5. $|xyz|\cdot x\equiv x_1,\ y\equiv y_1,\ z\equiv z_1\rightarrow|x_1y_1z_1|$.

Proof. $|xyz|\cdot y\equiv y_1\xrightarrow{\text{lemma 3.4}}|xy_1z|$. $|xy_1z|\cdot z\equiv z_1\xRightarrow{\text{lemma 3.3}}|xy_1z_1|\xRightarrow{\text{B3}}|z_1y_1x|$. $|z_1y_1x|\cdot x\equiv x_1\xRightarrow{\text{lemma 3.3}}|z_1y_1x_1|\xRightarrow{\text{B3}}|x_1y_1z_1|$.

§ 4. Sufficient conditions for closed-Q-betweenness.

In this section, we shall prove that the system of conditions $\mathfrak{Q}.\mathfrak{B}$ is sufficient for Q-betweenness of set A.

When $|xab|$ and $|xba|$ for some x, we define $a\equiv b$. Then this relation σ: $x\equiv y$ is an equivalence relation in set A. For;

1) From B1, $|xxx|$ for all x. So we have $x\equiv x$.

2) $x\equiv y\xrightarrow{\text{def.}}y\equiv x$.

3) $x\equiv y,\ y\equiv z\xrightarrow{\text{lemma 3.1}}x\equiv z$.

Now, we classify the set A by this equivalence relation, and represent the classes $X_\alpha^*, X_\beta^*, \dots\dots$. And we define $A(\sigma)$ and $R(\sigma)$ as follows:

$$A(\sigma)\equiv\{X_\sigma\}\ ;\ R(\sigma)\equiv\{|X_\alpha X_\beta X_\gamma|\,|x\in X_\alpha^*,\ y\in X_\beta^*,\ z\in X_\gamma^*, |xyz|\in R\}.^{1)}$$

From Lemma 3. 5, for any three elements x_1, y_1 and z_1 such that

$$x_1\in X_\alpha^*,\ y_1\in X_\beta^*,\ z_1\in X_\gamma^*,\ |X_\alpha X_\beta X_\gamma|\in R(\sigma)\rightarrow|x_1y_1z_1|\in R(\mathfrak{Q}.\mathfrak{B}). \quad\cdots\cdots\cdots(4.1)$$

If we define $A(\sigma)$ and $R(\sigma)$ as above, then from the system $\mathfrak{Q}.\mathfrak{B}$, $A(\sigma)$ and $R(\sigma)$ satisfy the system $\mathfrak{B}'$ (Cf. § 9 Chap. I). So, we can introduce the partial order in $A(\sigma)\equiv\{X_\sigma\}$ such that $R(\sigma)$ is the set of closed betweenness. $\cdots\cdots\cdots(4.2)$

Now, by the similar process to that of § 7 in Chap. I, we shall prove the sufficiency of the system $\mathfrak{Q}.\mathfrak{B}$.

[1] We define the following binary relation in set A:

(1) $x>y$ if and only if $X_\alpha>X_\beta\ (\alpha\neq\beta)$, $x\in X_\alpha^*$, $y\in X_\beta^*\ (\alpha\neq\beta)$.

(2) $x\geqq y$ if $x, y\in X_\alpha^*$.

[2] The binary relation which is defined by [1] is a quasi-order.

For; From the definition (2) in [1], it is clear that this binary relation satisfies P1, so we prove that the binary relation satisfies P3.

Suppose that $x\in X_\alpha^*$, $y\in X_\beta^*$, $z\in X_\gamma^*$ and $x\geqq y$, $y\geqq z$.

$$x\geqq y\xrightarrow{\text{def.}[1]}X_\alpha\geqq X_\beta,\ y\geqq z\xrightarrow{\text{def.}[1]}X_\beta\geqq X_\gamma.$$
$$X_\alpha\geqq X_\beta,\ X_\beta\geqq X_\gamma\xRightarrow{(4.2)}X_\alpha\geqq X_\gamma\xRightarrow{\text{def.}[1]}x\geqq z.$$

1) We use the natation X_α in the case when we consider a class X_α^* as one element.

So, this binary relation satisfies P3. Therefore it is a quasi-order.

[3] $x \geq y \geq z$ or $x \leq y \leq z \rightarrow |xyz|$.

For; Suppose that $x \in X_\alpha^*$, $y \in X_\beta^*$, $z \in X_\gamma^*$. $x \geq y \geq z \xrightarrow{\text{def.}(1)} X_\alpha \geq X_\beta \geq X_\gamma \xRightarrow{(4.2)} |X_\alpha X_\beta X_\gamma| \xRightarrow{(4.1)} |xyz|$.

In the case where $x \leq y \leq z$, we have $|xyz|$ by the same way as above.

[4] $|xyz| \rightarrow x \geq y \geq z$ or $x \leq y \leq z$.

For; Suppose $x \in X_\alpha^*$, $y \in X_\beta^*$, $z \in X_\gamma^*$. $|xyz| \xrightarrow{\text{def. B}(\sigma)} |X_\alpha X_\beta X_\gamma| \xRightarrow{(4.2)} X_\alpha \geq X_\beta \geq X_\gamma$ or $X_\alpha \leq X_\beta \leq X_\gamma \xRightarrow{\text{def.}(1)} x \geq y \geq z$ or $x \leq y \leq z$.

From the above results, we have the following Theorem:

Theorem 3. *The system of conditions $\mathfrak{Q}.\mathfrak{B}$ is sufficient for closed-Q-betweenness of set A.*

From the Theorem 1, 2, and 3, we have:

Theorem 4. *The system of conditions $\mathfrak{Q}.\mathfrak{B}$ is the complete system of conditions (necessary, sufficient, and mutually independent) for closed-Q-betweenness of set A.*

Remark 1. From the same consideration as that in §9 Chap. I, the complete system of conditions for closed-Q-betweenness of some set is as follows:

$$\mathfrak{Q}'.\mathfrak{B}\left\{\begin{array}{ll}
\text{QB}(1.2)'. & |xyz| \rightarrow |xxy|. \\
\text{B3.} & |axb| \rightarrow |bxa|. \\
\text{QB5.} & |axb| \cdot |xby| \cdot x \not\equiv b \rightarrow |aby|. \\
\text{B6.} & |a_1 a_2 x_1| \cdot |a_2 a_3 x_2| \cdots\cdots |a_{2n+1} a_1 a_{2n+1}| \rightarrow |a_i a_{i+1} a_{i+2}|. \\
 & \text{for at least one } i \ (1 \leq i \leq 2n+1),\ n=1,2,\dots,\ a_{2n+1+i}=a_i \\
 & \text{and } a_i \neq a_{i+1}\ (i=1,2,\dots,2n+1). \\
\text{QB7}_1. & |abc| \cdot |bdx|\ a \not\equiv b,\ b \not\equiv c,\ b \not\equiv d \rightarrow |abd| \text{ or } |dbc|. \\
\text{B7}_2. & |axb| \cdot b \equiv c \rightarrow |abc|. \\
 & \text{where } p \equiv q \text{ means that } |xpq| \text{ and } |xqp| \text{ for some } x.
\end{array}\right.$$

Remark 2. When the set A is one of the five special cases of quasi-partially ordered set (Cf. Chap. II): (1) the case being bounded; (2) having a center; (3) having one extreme element; (4) having two extreme elements; (5) being simply ordered, we have the complete system of conditions for closed-Q-betweenness of set A by replacing B6[1], B6[2], B6[3], B6[4] and B6[5] respectively for B6 in the system $\mathfrak{Q}.\mathfrak{B}$.

§ 5. Open-Q-betweenness in quasi-partially ordered set.

When $x > y > z$ or $x < y < z$ for three elements x, y and z in quasi-partially ordered set, this relation is called open-Q-betweenness $\langle xyz \rangle_0$.

In this section, we shall investigate the conditions in order that R, which is some collection of elements of triple product space $A\times A\times A$, may be the set of open-Q-betweenness.

About this, we shall have the following Theorem:

Theorem 5. *The system of conditions* $\mathfrak{O}.\mathfrak{Q}.\mathfrak{B}$:

$$\mathfrak{O}.\mathfrak{Q}.\mathfrak{B}\begin{cases} \text{B3}''. & |abc|\to|cba|. \\ \text{OB6}''. & \|a_1,a_2,b_1\|\cdot\|a_2,a_3,b_2\|\cdots\cdots\|a_{2n+1},a_1,b_{2n+1}\|\to|a_ia_{i+1}a_{i+2}|. \\ & \text{for at least one } i\ (1\leq i\leq 2n+1),\ n=1,2,\dots,\ a_{2n+1+l}=a_l. \\ \text{OB7}_1''. & |abc|\cdot\|b,d,e\|\ a\not\equiv b,\ b\not\equiv c,\ l\not\equiv d\to|abd|\ \text{or}\ |dbc|. \\ \text{OB7}_2''. & \|a,d,b\|\cdot b\equiv c\to|abc|. \\ & \text{where } p\equiv q \text{ means that } |rpq| \text{ and } |rqp| \text{ for some } r. \end{cases}$$

is the complete system of conditions of open-Q-betweenness.

Proof. From the Theorem 1 and 2, it is clear that the conditions are necessary and mutually independent. So, we shall prove the sufficiency of these conditions.

Similarly to the proof of the sufficiency of the system $\mathfrak{Q}.\mathfrak{B}$ (Cf. Chap. III), we construct $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)$ by operating $\|a,b,c\|\Rrightarrow|aab|$, $|abb|$ and $|aaa|$ to $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B})$. Then we have $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)=R(\mathfrak{Q}.\mathfrak{B})$.

For; From the construction of $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)$, it is clear that $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)$ satisfies the conditions B1, B2 and B3.

QB5: From Lemma 3.2 and $|axb|\cdot x\not\equiv b$, we have $a\not\equiv b$. So, we prove this by decomposing two cases:

Case 1. $b\not\equiv y$: $|xby|\cdot|axb|\xrightarrow{\text{OB7}_1''}|aby|$ or $|xba|$.

If $|xba|$, $|axb|\cdot|abx|\xrightarrow{(2.1)}x\equiv b$. This contradicts the assumption, so $|xba|$ may not occur. Hence, in this case, we have $|axb|\cdot|xby|\ x\not\equiv b\to|aby|$.

Case 2. $b\equiv y$:

(i) $x\neq a$ and b: $|axb|\cdot b\equiv y\xrightarrow{\text{OB7}_2''}|aby|$.

(ii) $x=a$ or b: $|axb|\to\|a,b,c\|$ for some c; $\|a,b,c\|\cdot b\equiv y\xrightarrow{\text{OB7}_2''}|aby|$.

Therefore, R satisfies QB5.

By the same way as that of obtaining B6 and B7_1 in the proof of the sufficiency of the system $\mathfrak{Q}.\mathfrak{B}$ in Chap. III, we see that $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)$ satisfies B6 and QB7_1. And from $\text{OB7}_2''$, we easily see that $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)$ satisfies B7_2.

Hence $R(\mathfrak{O}.\mathfrak{Q}.\mathfrak{B};2.1)=R(\mathfrak{Q}.\mathfrak{B})$.

From the above results and Theorem 4, we can introduce a quasi-order in R^* such that R is the set of open-Q-betweenness. Therefore, the system of conditions $\mathfrak{O}.\mathfrak{Q}.\mathfrak{B}$ is the complete conditions for-Q-betweenness.

Thus, we have proved Theorem 5.

Remork 3. When set A is one of the five special cases of quasi-partially ordered set (Cf. Remark 2), we have the complete system of conditions by replacing OB6 [1], OB6 [2], OB6 [3], OB6 [4] and OB6 [5] respectively for B6 in the system $\mathfrak{O}.\mathfrak{Q}.\mathfrak{B}$.

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