A LATTICE FORMULATION FOR ALGEBRAIC AND TRANSCENDENTAL EXTENTIONS IN ABSTRACT ALGEBRAS

By

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MacLane [1][1] has proved that the relatively algebraically closed subfields of any field Ω form a matroid lattice[2], and the transcendence degree is obtained as a lattice-theoretic dimension in this matroid lattice. But he did not consider the lattice-theoretic character of algebraic and transcendental extensions of subfields of Ω. On the other hand, in the lattice of convex sets there are two cases of the adjunctions of a point p to a convex set A, namely (1) p is contained in the least linear set $\bar{A}$ containing A, and (2) p is not contained in $\bar{A}$. In the case (1) the convex set obtained by the adjunction of p to A has the same dimension as A, but in the case (2) by the adjunction the dimension increases. These two cases are similar to a simple algebraic extension and a simple transcendental extension of subfields respectively. Hence we may expect that there exists a lattice theory which contains both the theory of extensions of fields and the theory of extensions of convex sets.

In this paper, I first find a Φ-relatively molecular, upper continuous lattice L such that the lattice of all subalgebras of an abstract algebra with finitary operations and lattice of all subgeometries of an abstract geometry with finitary operations[3] are special cases of such a lattice L, where Φ is the set of principal subalgebras and the set of principal subgeometries, that is points, respectively. Introducing in Φ the dependence relation formulated by van der Waerden [1, p. 204], I investigate the theory of extensions in L, with a view to showing that the theory of extension of fields and the theory of extension of convex sets are the special cases of this general theory.

1) The numbers is square brackets refer to the list of references at the end of this paper.

2) MacLane used "exchange lattice" instead of "matroid lattice", but these two conceptions are equivalent. Cf. Maeda [1] 181.

3) For the abstract geometry with finitary operations, cf. Maeda [1].

§ 1. Φ-Relatively Molecular, Upper Continuous Lattices.

DEFINITION 1·1. Let $a(\neq 0)$ be an element of a lattice L. We call a a *molecular element*[1] of L, when for the directed set $\{a_\delta ; \delta \in D\}$ such that $a_\delta \uparrow a$, some individual a_{δ_0} is equal to a.

Let Φ be a set of molecular elements of L, if $a < b$ implies $a < a \smile h \leqq b$ for some $h \in \Phi$, then L is called *Φ-relatively molecular*. Especially when Φ is a set of points of L, we say that L is *relatively atomic*[2].

REMARK 1·1. An element $a\,(\neq 0)$ of a lattice L is a molecular element of L, if and only if $a = \bigvee(x ; x \in S)$ (where S is a subset of L) implies $a = \bigvee(x ; x \in \nu_0)$ for some finite subset ν_0 of S. For, put $a_\nu = \bigvee(x ; x \in \nu)$ for every finite subset ν of S, then $a_\nu \uparrow a$. Hence there exists ν_0 such that $a = a_{\nu_0} = \bigvee(x ; x \in \nu_0)$. Conversely, if $a_\delta \uparrow a$, then since $a = \bigvee(a_\delta ; \delta \in D)$, we have $a = a_{\delta_1} \smile \cdots \smile a_{\delta_n}$. Therefore $a = a_{\delta_0}$ for δ_0 such that $\delta_1, \dots, \delta_n \leqq \delta_0$.

THEOREM 1·1. *A lattice L is Φ-relatively molecular, if and only if each element $a(\neq 0)$ of L is the join of molecular elements in Φ which are contained in a.*

PROOF. We prove as Maeda [1] p. 88 Lemma 1·1, where molecular elements in Φ must be used instead of points.

THEOREM 1·2. *In a Φ-relatively molecular, complete lattice L, the following two propositions (α) and (β) are equivalent.*

(α) *L is upper continuous.*[3]

(β) *Let h be an element of Φ and S a subset of Φ. Then $h \leqq \bigvee(k ; k \in S)$ implies $h \leqq k_1 \smile \cdots \smile k_n$ for some $k_i \in S$ $(i = 1, \dots, n)$.*

PROOF. $(\alpha) \rightarrow (\beta)$. Let ν be any finite subset of S, and put

$$a_\nu = \bigvee(k ; k \in \nu), \qquad a = \bigvee(k ; k \in S).$$

Since $a_\nu \uparrow a$, by (α) we have $a_\nu \frown h \uparrow a \frown h$. But $h \leqq a$, we have $a_\nu \frown h \uparrow h$. Since h is a molecular element, there exists $\nu_0 = \{k_1, \dots, k_n\}$ such that $a_{\nu_0} \frown h = h$. That is,

$$h \leqq a_{\nu_0} = k_1 \smile \cdots \smile k_n .$$

$(\beta) \rightarrow (\alpha)$. Using Theorem 1·1 we can prove as Maeda [1] p. 90 Lemma 1·3, where elements of Φ must be used instead of points.

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1) Birkhoff and Frink [1, p. 301] called such an element a $\uparrow$-inaccessible element.

2) Cf. Maeda [1] 88.

3) In a complete lattice L, if $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$, we say that L is an upper continuous lattice.

THEOREM 1·3. *Let $\mathfrak{M}$ be a family of subsets of a set S, which satisfies the following conditions:*

(1°) $S \in \mathfrak{M}$.

(2°) *Let $\mathfrak{M}_0$ be any non-empty subset of $\mathfrak{M}$, then the intersection $\bigwedge(X\,;X \in \mathfrak{M}_0)$ belongs to $\mathfrak{M}$.*

(3°) *Let $\{X_\delta\,;\delta \in D\}$ be any directed subsets of $\mathfrak{M}$, such that $\delta < \delta'$ implies $X_\delta \leqq X_{\delta'}$. Then the set-union $\bigvee(X_\delta\,;\delta \in D)$ belongs to $\mathfrak{M}$.*

Then $\mathfrak{M}$ is a Φ-relatively molecular, upper continuous lattice partially ordered by set-inclusion. If we denote by $\overline{\{p\}}$ the least set in $\mathfrak{M}$ which contains a single element p, then Φ is the set $\{\overline{\{p\}}\,;p \in S\}$ deleting the zero element of $\mathfrak{M}$.

PROOF. (i) By (1°) and (2°), $\mathfrak{M}$ is a complete lattice partially ordered by set-inclusion, where the meet $\bigwedge(X\,;X \in \mathfrak{M}_0)$ is the intersection $\bigwedge(X\,;X \in \mathfrak{M}_0)$. By (3°) the set-union $\bigvee(X_\delta\,;\delta \in D)$ is the join $\bigvee(X_\delta\,;\delta \in D)$, hence $X_\delta \uparrow X$ implies $\bigvee(X_\delta\,;\delta \in D)=X$. By the set-calculation, we have

$$\bigvee(X_\delta \wedge Y\,;\delta \in D) = \bigvee(X_\delta\,;\delta \in D) \wedge Y\,,$$

which means in the complete lattice $\mathfrak{M}$,

$$X_\delta \frown Y \uparrow X \frown Y\,.$$

That is, $\mathfrak{M}$ is upper continuous.[1)]

(ii) By (2°) $\overline{\{p\}} \in \mathfrak{M}$. When $X_\delta \uparrow \overline{\{p\}}$ in $\mathfrak{M}$, since

$$\bigvee(X_\delta\,;\delta \in D) = \overline{\{p\}}\,, \tag{1}$$

we have $p \in \bigvee(X_\delta\,;\delta \in D)$. Therefore $p \in X_{\delta_0}$ for some $\delta_0 \in D$. Then $\overline{\{p\}} \leqq X_{\delta_0}$. But by (1) $\overline{\{p\}} \geqq X_{\delta_0}$, hence we have $\overline{\{p\}} = X_{\delta_0}$. That is $\overline{\{p\}}$ is a molecular element of $\mathfrak{M}$, when $\overline{\{p\}}$ is not the zero element of $\mathfrak{M}$.

Let $X < Y$ in $\mathfrak{M}$, there exists an element $p \in S$ such that $p \notin X$, $p \in Y$. Since $\overline{\{p\}} \nleqq X$ and $\overline{\{p\}} \leqq Y$, we have $X < X \smile \overline{\{p\}} \leqq Y$. That is, if we put $\Phi = \{\overline{\{p\}}\,;p \in \mathfrak{M}\}$ deleting the zero element of $\mathfrak{M}$, $\mathfrak{M}$ is Φ-relatively molecular.

REMARK 1·2. By Theorem 1·3, the lattice $L(A)$ of all subalgebras of an abstract algebra A with finitary operations is a Φ-relatively molecular, upper continuous lattice, where Φ is the set of principal subalgebras of A.

Maeda [1, p. 87] showed that an abstract lattice L is isomorphic with the lattice of subgeometries of a suitable abstract geometry G with finitary operations, if and only if it is a relatively atomic, upper continuous lattice.

1) Cf. Birkhoff and Frink [1] 301, and Birkhoff [1] 64, Ex. 3 (*b*).

In this case G is the set of points of L.

REMARK 1·3. We give here some notes on matroid lattices. A lattice L is called *semi-modular* if it satisfies

(ξ') If a and b cover c, and $a\neq b$, then $a\smile b$ covers a and b.

And a relatively atomic, upper continuous, semi-modular lattice is called a *matroid lattice* which is equivalent to an *exchange lattice* defined by MacLane [1, p. 456] as a relatively atomic, upper continuous lattice which satisfies the following exchange axiom:

(η') If p, q are points, and $a< a\smile q\leq a\smile p$, then $a\smile p = a\smile q$.[1)]

The sublattice $L(a, b)$[2)] of a matroid lattice L is also a matroid lattice, and points of $L(a, b)$ are all the elements of L of the form $a\smile p$, where p is a point of L such that $a< a\smile p\leq b$.[3)]

In a complete lattice L, a set N of elements is called an *independent system* if and only if $\bigvee(x\,;x\in N_1)\frown\bigvee(x\,;x\in N_2)=0$ for any disjoint subsets N_1, N_2 of N. If L is upper continuous, N is an independent system if and only if every finite subset of N is an independent system.[4)] When L is a matroid lattice, a finite set of points $\{p_1, \dots, p_n\}$ is independent, if and only if

$$(p_1\smile \cdots \smile p_i)\frown p_{i+1} = 0 \quad \text{for} \quad i = 1, \dots, n-1.^{5)}$$

For an element a of a matroid lattice L, the independent system N of points, such that $a=\bigvee(p\,;p\in N)$ is called a *basis* of a. If N_1, N_2 are two bases of a, then N_1 and N_2 have the same cardinal number.[6)]

THEOREM 1·4. *In a matroid lattice L, if P is a set of points such that $a=\bigvee(p\,;p\in P)$, then there exists a subset N of P such that N is the basis of a.*

PROOF. Let $\mathfrak{M}$ be the family of all independent systems composed of the points contained in P. $\mathfrak{M}$ is partially ordered by set-inclusion. Let $\mathfrak{M}_0$ be a directed subfamily of $\mathfrak{M}$, then by Remark 1·3, the set-union $\bigcup(X\,;X\in\mathfrak{M}_0)$ is an independent system and it is an upper bound of $\mathfrak{M}_0$ in $\mathfrak{M}$. Hence by Zorn's lemma there exists a maximal set N in $\mathfrak{M}$. If q is a point of P with $q\not\leq\bigvee(p\,;p\in N)$, then by Maeda [2, p. 179, Lemma 6] the set obtained by adjoining q to N is an independent system. This contradicts

1) Cf. Maeda [2] 181, Theorem 3.
2) $L(a, b)$ consists of all elements $x\in L$ such that $a\leq x\leq b$.
3) MacLane [1] 458, Theorem 4.
4) Maeda [2] 178, Lemma 4.
5) Sasaki and Fujiwara [1] 184, Lemma 2.
6) MacLane [1] 458, Theorem 6.

the maximality of N. Hence $q \leqq \bigvee(p\,;\,p \in N)$ for every $q \in P$. Since $N \leqq P$, we have

$$a = \bigvee(q\,;\,q \in P) = \bigvee(p\,;\,p \in N).$$

§ 2. Dependence Relation in Φ-Relatively Molecular, Upper Continuous Lattices.

In this section, L is a Φ-relatively molecular, upper continuous lattice.

DEFINITION 2·1. For an element h of Φ and a non-empty subset S of Φ, we shall define a relation "*h depends on S*" with the following five properties: 1)

(1°) If $h \leqq \bigvee(k\,;\,k \in S)$, then h depends on S.

(2°) If h depends on S and $S \leqq T$, then h depends on T.

(3°) If h depends on S, then h depends on some finite subset S_0 of S.

(4°) If h depends on $S_0 = \{k_1, \dots, k_n\}$ but not on any proper subset of S_0, then k_n depends on $\{k_1, \dots, k_{n-1}, h\}$.

(5°) If h depends on S and if every element of S depends on T, then h depends on T.

Let S and T be two subsets of Φ. If every element of S depends on T, we say that S *depends on* T. If S depends on T and T depends on S we say that S and T are *equivalent* with respect to this dependence relation.2)

If every element h of S does not depend on the set obtained by S deleting h, we say that S is *irreducible* with respect to this dependence relation.3)

REMARK 2·1. By (2°) and (3°), a set S is irreducible, if and only if every finite subset of S is irreducible.

REMARK 2·2. If an element h of Φ depends on an element k of Φ, then k depends on h. For, since h does not depend on the proper subset of the single element k, by (4°) k depends on h.

DEFINITION 2·2. For $a \in L$ and $S \leqq \Phi$, put

$$S(a) = \{k\,;\,k \leqq a,\ k \in \Phi\}, \quad a(S) = \bigvee(k\,;\,k \in S).$$

DEFINITION 2·3. For $h \in \Phi$ and $0 < M \leqq \Phi$, we say that *h depends on*

1) Our postulates are related to the properties of algebraic dependence as formulated by van der Waerden [1, p. 204], where instead of (1°), the following postulate (1′) is used.

(1′) Each h depends on the set of the single element h.

2) Cf. van der Waerden [1] 205.

3) Cf. van der Waerden [1] 205.

M with respect to a if and only if h depends on $S(a)\cup M$.

LEMMA 2·1. *The dependence relation relative to a defined in Definition 2·3 have the following properties*:

(1_a) *If* $h\leq a\smile\bigvee(k\,;\,k\in M)$, *then h depends on M with respect to a.*

(2_a) *If h depends on M with respect to a and* $M\leq N$, *then h depends on N with respect to a.*

(3_a) *If h depends on M with respect to a, then h depends on some finite subset* M_0 *of M with respect to a.*

(4_a) *If h depends on* $M_0=\{k_1,\dots,k_n\}$ *with respect to a, but not on any proper subset of* M_0, *then k depends on* $\{k_1,\dots,k_{n-1},h\}$ *with respect to a.*

(5_a) *If h depends on M with respect to a and if every element of M depends on N with respect to a, then h depends on N with respect to a.*

PROOF. (1_a) By Theorem 1·1, $a=\bigvee(k\,;\,k\in S(a))$. Hence if $h\leq a\smile\bigvee(k\,;\,k\in M)$, then $h\leq\bigvee(k\,;\,k\in S(a)\cup M)$, and h depends on M with respect to a.

(2_a) and (3_a) are evident.

(4_a) Let h depend on $M=\{k_1,\dots,k_n\}$ with respect to a, but not on any proper subset of M_0. By Definition 2·1 (3°), there exists a finite subset U_0 of $S(a)\cup M_0$ such that h depends on U_0. Let U_0 be the least subset which has the above property, then U_0 contains all elements of M_0. Hence if we put $M_1=U_0-M_0$ then $M_1\leq S(a)$, and h depends on $M_1\cup M_0$ but not on any proper subset $M_1\cup M_0$. Therefore by Definition 2·1 (4°) k_n depends on $M_1\cup\{k_1,\dots,k_{n-1},h\}$. Consequently k_n depends on $S(a)\cup\{k_1,\dots,k_{n-1},h\}$, and ($4_a$) holds.

(5_a) is evident.

DEFINITION 2·4. As Definition 2·1, we define *equivalency* and *irreducibleness with respect to a.*

DEFINITION 2·5. When S is a subset of Φ, $\bar{S}$ means the set of all elements of Φ which depends on S. If $S=\bar{S}$, we say that S is a *closed set* with respect to this dependence relation.

By Definition 2·1 (5°), we have $\bar{\bar{S}}=\bar{S}$. Hence $\bar{S}$ is a closed set.

Let $\mathfrak{L}(\Phi)$ mean the set of all closed subsets of Φ.

THEOREM 2·1. $\mathfrak{L}(\Phi)$ *is a matroid lattice, partially ordered by set-inclusion, where* $\overline{\{h\}}$ $(h\in\Phi)$ *is a point or a zero point in* $\mathfrak{L}(\Phi)$.[1]

1) This theorem is essentially given by MacLane [1, p. 457 Thereom 1] without proof. $\{h\}$ means the set with a single element h.

PROOF. (i) It is evident that $\Phi \in \mathfrak{L}(\Phi)$. Let $\{M_\alpha ; \alpha \in I\}$ be any non-empty subset of $\mathfrak{L}(\Phi)$, and put $T = \bigwedge(M_\alpha ; \alpha \in I)$. If h depends on T, by Definition 2·1 (2°), h depends on M_α. Since M_α is closed, we have $h \in M_\alpha$ for every $\alpha \in I$. Therefore $h \in T$. That is, $T \in \mathfrak{L}(\Phi)$.

Next let $\{M_\delta ; \delta \in D\}$ be a directed subsystem of $\mathfrak{L}(\Phi)$, such that $\delta < \delta'$ implies $M_\delta \leqq M_{\delta'}$. Put $S = \bigvee(M_\delta ; \delta \in D)$. If h depends on S, by Definition 2·1 (3°), h depends on a finite subset $\{h_1, \dots, h_n\}$ of S. Then there exists δ_0 such that $h_i \in M_{\delta_0}$ for $i = 1, \dots, n$. Hence $h \in M_{\delta_0}$. Consequently $h \in S$ and $S \in \mathfrak{L}(\Phi)$.

Therefore by Theorem 1·3, $\mathfrak{L}(\Phi)$ is a Φ -relatively molecular, upper continuous lattice, where $\Phi = \{\overline{\{h\}} ; h \in \Phi\}$[1] deleting the zero element of $\mathfrak{L}(\Phi)$.

(ii) If there exists $S \in \mathfrak{L}(\Phi)$ such that $0 < S \leqq \overline{\{h\}}$, then for any $k \in S$ we have $\overline{\{k\}} \leqq S \leqq \overline{\{h\}}$. Hence k depends on h. Therefore by Remark 2·2, h depends on k, and we have $\overline{\{h\}} = \overline{\{k\}}$. Consequently $S = \overline{\{h\}}$. That is $\overline{\{h\}}$ is a point in $\mathfrak{L}(\Phi)$.

Therefore $\mathfrak{L}(\Phi)$ is a relatively atomic, upper continuous lattice.

(iii) In $\mathfrak{L}(\Phi)$, let $S < S \smile \overline{\{h\}} \leqq S \smile \overline{\{k\}}$. Then h depends on $S \smile \overline{\{k\}}$. Since $S \smile \overline{\{k\}} \leqq \overline{S \bigvee \{k\}}$, h depends on $S \bigvee \{k\}$. Therefore by Definition 2·1 (3°), there exists a finite subset $\{g_1, \dots, g_m\}$ of S such that h depends of $\{g_1, \dots, g_m, k\}$. In this case we can take $\{g_1, \dots, g_m\}$ such that h does not depend on proper subsets of $\{g_1, \dots, g_m, k\}$. Hence by Definition 2·1 (4°), k depends on $\{g_1, \dots, g_m, h\}$, that is $k \in S \smile \overline{\{h\}}$. Consequently $\overline{\{k\}} \leqq S \smile \overline{\{h\}}$, and $S \smile \overline{\{h\}} = S \smile \overline{\{k\}}$. Thus $\mathfrak{L}(\Phi)$ satisfies (η') in Remark 1·3, and $\mathfrak{L}(\Phi)$ is a matroid lattice.

DEFINITION 2·6. For $a \in L$, if $h \in \Phi$ depends of $S(a)$, we say that h *depends on* a.[2] We call a a *closed element* of L,[3] if the relation h depends on a always implies $h \leqq a$. Denote by $\bar{L}$ the set of all closed elements of L.

LEMMA 2·2. *Let S be a subset of Φ. If $h \in \Phi$ depends on $a(S)$, then h depends on S.*

PROOF. Since $a(S) = \bigvee(k ; k \in S)$, h depends on $T = \{k ; k \leqq \bigvee(k ; k \in S)\}$. By Definition 2·1 (1°), $k \in T$ depends on S, and by Definition 2·1 (5°) h depends on S.

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1) In Theorem 1·3, $\overline{\{h\}}$ means the least set in $\mathfrak{L}(\Phi)$, which contains a single element h. But this is equivalent to the closed set $\overline{\{h\}}$ in Definition 2·5.

2) If $h \leqq a$, since $h \leqq \bigvee(k ; k \in S(a))$, by Definition 2·1 (1°) h depends on $S(a)$, that is, h depends on a.

3) The notion "closed element" is introduced by Prenowitz [1] 670.

THEOREM 2·2. *Between $\mathfrak{L}(\Phi)$ and $\overline{L}$, there exists a one-to-one correspondence such that*

$$S \rightarrow a(S), \quad a \rightarrow S(a),$$

and $\overline{L}$ is a matroid lattice by the same partially ordering as L.

PROOF. (i) Let $S \in \mathfrak{L}(\Phi)$. If h depends on $a(S)$, by Lemma 2·2, h depends on S. Since S is a closed set, we have $h \in S$, that is $h \leq a(S)$. Therefore $a(S)$ is a closed element.

(ii) Let $a \in \overline{L}$. If h depends on $S(a)$, by Definition 2·6, h depends on a. Since a is a closed element, we have $h \leq a$, that is, $h \in S(a)$. Therefore $S(a)$ is a closed set.

(iii) Let $S \in \mathfrak{L}(\Phi)$. If $h \leq a(S)$, since $h \leq \bigvee(k; k \in S)$, by Definition 2·1 (1°), h depends on S, that is, $h \in S$. Conversely if $h \in S$, then $h \leq a(S)$. Therefore, $h \leq a(S)$ if and only if $h \in S$, and

$$S = S(a(S)).$$

By Theorem 1·1 we have

$$a = \bigvee(k; k \leq a) = a(S(a)).$$

Therefore by $S \rightarrow a(S)$, $a \rightarrow S(a)$, there exists a one-to-one correspondence between $\mathfrak{L}(\Phi)$ and $\overline{L}$, preserving the partially ordering. Hence $\overline{L}$ is lattice-isomorphic to $\mathfrak{L}(\Phi)$, and by Theorem 2·1, $\overline{L}$ is a matroid lattice.

REMARK 2·3. For $M_\alpha \in \mathfrak{L}(\Phi)$ $(\alpha \in I)$, put $T = \bigwedge(M_\alpha; \alpha \in I)$. Then by Theorem 2·1 proof (i), we have $T \in \mathfrak{L}(\Phi)$. That is, T is the meet of $\{M_\alpha; \alpha \in I\}$ in $\mathfrak{L}(\Phi)$. Therefore by Theorem 2·2, $a(T)$ is the meet of $\{a(M_\alpha); \alpha \in I\}$ in $\overline{L}$.

Since $T \leq M_\alpha$, we have $a(T) \leq a(M_\alpha)$ for every $\alpha \in I$. Hence $a(T) \leq \bigwedge(a(M_\alpha); \alpha \in I)$ in L. If $h \leq a(M_\alpha)$ for every $\alpha \in I$, then by Definition 2·1 (1°) h depends on M_α. M_α being a closed set, $h \in M_\alpha$ for every $\alpha \in I$. Hence $h \in T$, and $h \leq a(T)$. Consequently

$$a(T) = \bigwedge(a(M_\alpha); \alpha \in I)$$

in L. Therefore the meets of $\{a(M_\alpha); \alpha \in I\}$ in L and $\overline{L}$ coincide. Therefore in $\overline{L}$ we may use the same symbol $\frown$, $\bigwedge$ as in L.

But the join in $\overline{L}$ of $\{a(M_\alpha); \alpha \in I\}$ differs from that in L. Hence in $\overline{L}$ we use $\vee$, $\bigvee$ for joins.

DEFINITION 2·7. For $a \in L$, $\bar{a}$ means the least closed element which contains a.

LEMMA 2·3. *For any $a \in L$, we have*

$$a(\overline{S(a)}) = \bar{a}.$$

PROOF. It is evident that $S(a)\leqq S(\bar{a})$. By Theorem 2·2, $S(\bar{a})$ is a closed set, hence we have $\overline{S(a)}\leqq S(\bar{a})$. Since by Theorem 1·1, $a(S(\bar{a}))=\bar{a}$, we have $a(\overline{S(a)})\leqq\bar{a}$. By Theorem 2·2, $a(\overline{S(a)})$ is a closed set and

$$a=a(S(a))\leqq a(\overline{S(a)}).$$

Therefore $\bar{a}\leqq a(\overline{S(a)})$. Thus we have $a(\overline{S(a)})=\bar{a}$.

REMARK 2·4. For $h\in\Phi$, we have $S(h)=\{h\}$. Hence by Lemma 2·3 we have $a(\overline{\{h\}})=\bar{h}$. By Theorem 2·1 $\overline{\{h\}}$ being a point or a zero element in $\mathfrak{L}(\Phi)$, by Theorem 2·2 $\bar{h}$ is a point or a zero element in $\bar{L}$.

LEMMA 2·4. (i) *$h\in\Phi$ depends on $S\leqq\Phi$, if and only if $h\leqq a(\bar{S})$.*

(ii) *$h\in\Phi$ depends on $a\in L$, if and only if $h\leqq\bar{a}$.*

PROOF. (i) If h depends on S, then $h\in\bar{S}$, that is $h\leqq a(\bar{S})$. Conversely if $h\leqq a(\bar{S})$, then h depends on $\bar{S}$ by Definition 2·1 (1°). Therefore $h\in\bar{S}$ and h depends on S.

(ii) By Definition 2·6, "h depends on a" is equivalent to "h depends on $S(a)$". Hence by (i) this is equivalent to "$h\leqq a(\overline{S(a)})$." Since by Lemma 2·3 $a(\overline{S(a)})=\bar{a}$, h depends on a if and only if $h\leqq\bar{a}$.

LEMMA 2·5. *For $a_\alpha\in L\,(\alpha\in I)$, we have in $\bar{L}$*

$$\overline{\bigvee(a_\alpha\,;\,\alpha\in I)}=\mathsf{V}(\bar{a}_\alpha\,;\,\alpha\in I). \tag{1}$$

PROOF. Since $\bar{a}_\alpha\leqq\overline{\bigvee(a_\alpha\,;\,\alpha\in I)}$, we have $\mathsf{V}(\bar{a}_\alpha\,;\,\alpha\in I)\leqq\overline{\bigvee(a_\alpha\,;\,\alpha\in I)}$. Since $a_\alpha\leqq\bar{a}_\alpha\leqq\mathsf{V}(\bar{a}_\alpha\,;\,\alpha\in I)$ in L, we have $\bigvee(a_\alpha\,;\,\alpha\in I)\leqq\mathsf{V}(\bar{a}_\alpha\,;\,\alpha\in I)$. Since $\mathsf{V}(\bar{a}_\alpha\,;\,\alpha\in I)$ is a closed element we have $\overline{\bigvee(a_\alpha\,;\,\alpha\in I)}\leqq\mathsf{V}(\bar{a}_\alpha\,;\,\alpha\in I)$. Thus we have (1).

LEMMA 2·6. *$h\in\Phi$ depends on $M\leqq\Phi$ with respect to a, if and only if h depends on M with respect to $\bar{a}$.*

PROOF. (i) Necessary. Since $S(a)\leqq S(\bar{a})$, by Definition 2·3, it is evident.

(ii) Sufficient. If h depends on M with respect to $\bar{a}$, then h depends on $S(\bar{a})\bigcup M$. If $g\in S(\bar{a})$, then $g\leqq\bar{a}$. And by Lemma 2·4 (ii) g depends on a, that is, g depends on $S(a)$. Hence by Definition 2·1 (2°) and (5°), h depends on $S(a)\bigcup M$, that is, h depends on M with respect to a.

LEMMA 2·7. *The following three propositions (α), (β) and (γ) are equivalent.*

(α) *h depends on M with respect to a.*

(β) *h depends on $a\smile\bigvee(k\,;\,k\in M)$.*

(γ) $\bar{h}\leqq\bar{a}\vee\mathsf{V}(\bar{k}\,;\,k\in M)$.

PROOF. $(\alpha)\rightarrow(\beta)$. Since $S(a)\bigcup M\leqq S(a\smile\bigvee(k\,;\,k\in M))$, by Definition 2·3 and Definition 2·6, $(\alpha)\rightarrow(\beta)$ is evident.

$\beta)\rightarrow(\alpha)$. Since

$$a \smile \bigvee(k\,;\,k\in M)=\bigvee(k\,;\,k\in S(a))\smile\bigvee(k\,;\,k\in M)=\bigvee(k\,;\,k\in S(a)\cup M),$$

by Lemma 2·2, $(\beta)\rightarrow(\alpha)$ is evident.

$(\beta)\rightleftarrows(\gamma)$. By Lemma 2·4 (ii) and Lemma 2·5, (β) is equivalent to

$$h\leqq\bar{a}\vee\bigvee(k\,;\,\bar{k}\in M),$$

which is equivalent to (γ).

THEOREM 2·3. *Two subsets M, N of Φ are equivalent with respect to a, if and only if in $\bar{L}$*

$$\bar{a}\vee\bigvee(\bar{h}\,;\,h\in M)=\bar{a}\vee\bigvee(\bar{k}\,;\,k\in N). \tag{1}$$

PROOF. (i) Necessary. Since $h\in M$ depends on N with respect to a, by Lemma 2·7 we have

$$\bar{h}\leqq\bar{a}\vee\bigvee(\bar{k}\,;\,k\in N).$$

Hence $$\bar{a}\vee\bigvee(\bar{h}\,;\,h\in M)\leqq\bar{a}\vee\bigvee(\bar{k}\,;\,k\in N).$$

Interchanging M and N, we have (1).

(ii) Sufficient. From (1), for every $h\in M$, we have

$$\bar{h}\leqq\bar{a}\vee\bigvee(\bar{k}\,;\,k\in N).$$

Hence by Lemma 2·7, h depends on N with respect to a, that is M depends on N with respect to a. Interchanging M and N, we conclude that M and N are equivalent with respect to a.

THEOREM 2·4. *A subset M of Φ is irreducible with respect to a, if and only if $\{\bar{a}\vee\bar{h}\,;\,h\in M\}$ is an independent system of points in $\bar{L}(\bar{a},1)$.*

PROOF. From Remark 1·3 and Remark 2·1, it is sufficient to consider the case where M is a finite subset $\{h_1,\dots,h_n\}$.

(i) Necessary. Since h_i does not depend on $\{h_1,\dots,h_{i-1},h_{i+1},\dots,h_n\}$ with respect to a, by Lemma 2·7 we have

$$\bar{h}_i\not\leqq\bar{a}\vee\bar{h}_1\vee\cdots\vee\bar{h}_{i-1}\vee\bar{h}_{i+1}\vee\cdots\vee\bar{h}_n. \tag{1}$$

Then $\bar{h}_i\not\leqq\bar{a}$, that is $\bar{a}<\bar{a}\vee\bar{h}_i$. Hence by Remark 1·3 and Remark 2·4, $\{\bar{a}\vee h_i\,;\,i=1,\dots,n\}$ is a set of points in $\bar{L}(\bar{a},1)$. Put $l_i=\bar{a}\vee\bar{h}_i$, by (1) we have, in $\bar{L}(\bar{a},1)$,

$$l_i\not\leqq l_1\vee\cdots\vee l_{i-1}\vee l_{i+1}\vee\cdots\vee l_n.$$

Therefore $(l_1\vee\cdots\vee l_{i-1})\frown l_i\leqq(l_1\vee\cdots\vee l_{i-1}\vee l_{i+1}\vee\cdots\vee l_n)\frown l_i=\bar{a}$. Hence by Remark 1·3 $\{l_i\,;\,i=1,\dots,n\}$ is an independent system in $\bar{L}(\bar{a},1)$.

(ii) Sufficient. Since $\{\bar{a}\vee\bar{h}_i\,;\, i=1,\dots,n\}$ is an independent system of points in $\bar{L}(\bar{a},1)$, we have

$$\bar{a}\vee\bar{h}_i \not\leqq (\bar{a}\vee\bar{h}_1)\vee\cdots\vee(\bar{a}\vee\bar{h}_{i-1})\vee(\bar{a}\vee\bar{h}_{i+1})\vee\cdots\vee(\bar{a}\vee\bar{h}_n).$$

Therefore $\quad \bar{h}_i \not\leqq \bar{a}\vee\bar{h}_1\vee\cdots\vee\bar{h}_{i-1}\vee\bar{h}_{i+1}\vee\cdots\vee\bar{h}_n \quad (i=1,\dots,n).$

Hence by Lemma 2·7, h_i does not depend on $\{h_1,\dots,h_{i-1},h_{i+1},\dots,h_n\}$ with respect to a. That is $\{h_1,\dots,h_n\}$ is irreducible with respect to a.

DEFINITION 2·8. Let $a\in L$, and $M\leqq\Phi$. We say that

$$b = a\smile\bigvee(h\,;\,h\in M)$$

is an *extension* of a by *adjunction* of M. Especially when M is a single element, we say that b is a *simple extension* of a. When M depends on a we say that b is a *dependent extension* of a, otherwise we say that b is a *transcendental extension* of a. When M is irreducible with respect to a, we say that b is a *purely transcendental extension* of a.

THEOREM 2·5. *When $a<c$ in L, then there exists a purely transcendental extension*

$$b = a\smile\bigvee(h\,;\,h\in N)$$

of a, and c is a dependent extension of b. In this case, the cardinal number of N is uniquely determined.

PROOF. Since L is a Φ-relatively molecular lattice, by Theorem 1·1 there exists a subset M of Φ such that

$$c = a\smile\bigvee(h\,;\,h\in M).$$

By Lemma 2·5 we have, in $\bar{L}$,

$$\bar{c} = \bar{a}\vee\bigvee(\bar{h}\,;\,h\in M) = \bigvee(\bar{a}\vee\bar{h}\,;\,h\in M).$$

If we delete h from M such that $\bar{a}\vee\bar{h}=\bar{a}$, then we have a subset M_0 of M such that

$$\bar{c} = \bigvee(\bar{a}\vee\bar{h}\,;\,h\in M_0)$$

and $\bar{a}\vee\bar{h}>\bar{a}$ for every $h\in M_0$. By Remark 1·3 and Remark 2·4, $\bar{a}\vee\bar{h}(h\in M_0)$ is a point of $\bar{L}(\bar{a},1)$. Hence by Theorem 1·4, there exists a subset N of M_0 such that $\{\bar{a}\vee\bar{h}\,;\,h\in N\}$ is a basis of $\bar{c}$ in $\bar{L}(\bar{a},1)$, and by Remark 1·3, the cardinal number of N is uniquely determined. Thus

$$\bar{c} = \bigvee(\bar{a}\vee\bar{h}\,;\,h\in N)$$

and $\{\bar{a}\vee\bar{h}\,;\,h\in N\}$ is an independent system of points in $\bar{L}(\bar{a},1)$. Therefore by Theorem 2·4 N is irreducible with respect to a, and

$$b = a \smile \bigvee(h\,;\,h \in N)$$

is a purely transcendental extension of a.

Let $N' = M - N$, then

$$c = a \smile \bigvee(h\,;\,h \in M) = b \smile \bigvee(h\,;\,h \in N').$$

For $h \in N'$, we have $h \leqq c$, that is

$$\bar{h} \leqq \bar{c} = \mathsf{V}(\bar{a} \vee \bar{h}\,;\,h \in N) = \bar{a} \vee \mathsf{V}(\bar{h}\,;\,h \in N).$$

Hence by Lemma 2·7 h depends on $b = a \smile \bigvee(h\,;\,h \in N)$. That is N' depends on b. Consequently c is a dependent extension of b.

DEFINITION 2·9. The uniquely determined cardinal number of N, in Theorem 2·5, is called the *transcendence degree* of c over a.

§ 3. Applications to the Extensions of Fields.

Let $L(\Omega)$ be the set of all subfields of a field Ω. By Theorem 1·3, $L(\Omega)$ is a Φ-relatively molecular, upper continuous lattice partially ordered by set-inclusion, where the zero element of $L(\Omega)$ is the prime subfield π, and Φ is the set $\{\pi(\delta)\,;\,\delta \in \Omega - \pi\}$.[1]

We introduce a dependence relation in Φ from the algebraic dependency in Ω as follows. Let $\nu \in \Omega - \pi$, $\mathfrak{M} \leqq \Omega - \pi$ and put $S = \{\pi(\delta)\,;\,\delta \in \mathfrak{M}\}$. *$\pi(\nu)$ depends on S, if and only if ν depends algebraically on $\mathfrak{M}$ with respect to π*, that is ν is algebraic over $\pi(\mathfrak{M})$. Then this dependence relation satisfies Definition 2·1 (2°)—(5°), since the algebraic dependency satisfies (2°)—(5°).[2] With respect to (1°), if $\pi(\nu) \leqq \bigvee(\pi(\delta)\,;\,\delta \in \mathfrak{M})$, then $\nu \in \bigvee(\pi(\delta)\,;\,\delta \in \mathfrak{M}) = \pi(\mathfrak{M})$. Hence ν is algebraic over $\pi(\mathfrak{M})$ and $\pi(\nu)$ depends on $S = \{\pi(\delta)\,;\,\delta \in \mathfrak{M}\}$.

Let Δ be a subfield of Ω, and $\mathfrak{M}$ be a subset of $\Omega - \pi$. Then $\nu\,(\notin \pi)$ depends algebraically on $\mathfrak{M}$ with respect to Δ, if and only if ν is algebraic over $\Delta(\mathfrak{M})$. Since $\Delta(\mathfrak{M}) = \bigvee(\pi(\delta)\,;\,\delta \in (\Delta - \pi) \bigcup \mathfrak{M})$, this means that $\pi(\nu)$ depends on $\{\pi(\delta)\,;\,\delta \in (\Delta - \pi) \bigcup \mathfrak{M}\}$. Since

$$\{\pi(\delta)\,;\,\delta \in (\Delta - \pi) \bigcup \mathfrak{M}\} = \{\pi(\delta)\,;\,\delta \in \Delta - \pi\} \bigcup \{\pi(\delta)\,;\,\delta \in \mathfrak{M}\} = S(\Delta) \bigcup \{\pi(\delta)\,;\,\delta \in \mathfrak{M}\},$$

by Definition 2·3 this means that $\pi(\nu)$ depends on $M = \{\pi(\delta)\,;\,\delta \in \mathfrak{M}\}$ with respect to Δ. Hence *ν depends algebraically on $\mathfrak{M}$ with respect to Δ in Ω,*

1) When $\mathfrak{M}$ is a subset of Ω, $\pi(\mathfrak{M})$ means the subfield obtained by adjunction $\mathfrak{M}$ to π, that is, the least subfield which contains both π and $\mathfrak{M}$.

2) Cf. van der Waerden [1] 204.

if and only if $\pi(\nu)$ *depends on* $M=\{\pi(\delta);\delta\in\mathfrak{M}\}$ *with respect to* Δ *in* $\Phi=\{\pi(\delta);\delta\in\Omega-\pi\}$.

Thus we can apply the theory of §2 to the extensions of fields. In this case the algebraic and the transcendental extensions of subfields correspond to the dependent and the transcendental extensions in §2 respectively, and the closed element of $L(\Omega)$ is the relatively algebraically closed subfield $\mathfrak{K}$ of Ω, in the sense that every element of Ω algebraic over $\mathfrak{K}$ is contained in $\mathfrak{K}$.[1]

§4. Applications to the Extensions of Convex Sets.

We introduce the linear dependence in a relatively atomic, upper continuous lattice L, as follows. Let p be a point of L, and S a set of points in L. If for some disjoint subsets S_1 and S_2 of S,

$$\{p\smile\bigvee(q;q\in S_1)\}\frown\bigvee(q;q\in S_2)\neq 0,$$
$$\bigvee(q;q\in S_1)\frown\bigvee(q;q\in S_2)=0,$$

we say p *depends linearly on* S.[2]

This linear dependence satisfies Definition 2·1 (1°)—(4°), as follows.[3]

(1°) When $p\leqq\bigvee(q;q\in S)$, put $S_1=0$, $S_2=S$ in the above definition, then p depends linearly on S.

(2°) is evident.

(3°) When p depends linearly on S, by the definition, for some disjoint subsets S_1, S_2, there exists a point r such that

$$r\leqq\{p\smile\bigvee(q;q\in S_1)\}\frown\bigvee(q;q\in S_2)$$

and

$$\bigvee(q;q\in S_1)\frown\bigvee(q;q\in S_2)=0.$$

From Theorem 1·2, there exist points $q_i^{(1)}$, $q_j^{(2)}$ such that

$$r\leqq p\smile q_1^{(1)}\smile\cdots\smile q_m^{(1)}\quad (q_i^{(1)}\in S_1),$$
$$r\leqq q_1^{(2)}\smile\cdots\smile q_n^{(2)}\qquad (q_j^{(2)}\in S_2).$$

Then

$$(p\smile q_1^{(1)}\smile\cdots\smile q_m^{(1)})\frown(q_1^{(2)}\smile\cdots\smile q_n^{(2)})\neq 0,$$
$$(q_1^{(1)}\smile\cdots\smile q_m^{(1)})\frown(q_1^{(2)}\smile\cdots\smile q_n^{(2)})=0.$$

Therefore p depends linearly on the finite subset $\{q_1^{(1)},\dots,q_m^{(1)},q_1^{(2)},\dots,q_n^{(2)}\}$.

1) When Ω is an algebraically closed subfield, the closed element of $L(\Omega)$ is the algebraically closed subfield of Ω.

2) Prenowitz [1, p. 667] defined the linear dependence when S is a finite set of points.

3) By Theorem 1·1 and Theorem 1·2, Φ-relatively molecular, upper continuous lattices have the same properties as relatively atomic, upper continuous lattices, we can define as above the linear dependence for $h\in\Phi$ and $S\leqq\Phi$. Then this linear dependence satisfies Definition 2·1 (1°)—(4°).

(4°) Let p depend linearly on $S_0=\{q_1, \dots, q_n\}$ but not on any proper subset of S_0. Then there are points $q_1^{(1)}, \dots, q_i^{(1)}, q_1^{(2)}, \dots, q_m^{(2)}$ which belong to S_0, such that

$$(p\smile q_1^{(1)}\smile \cdots \smile q_i^{(1)})\frown(q_1^{(2)}\smile \cdots \smile q_m^{(2)}) \neq 0, \tag{1}$$
$$(q_1^{(1)}\smile \cdots \smile q_i^{(1)})\frown(q_1^{(2)}\smile \cdots \smile q_m^{(2)}) = 0.$$

Since p does not depend linearly on proper subsets of S_0, it must be that $S_0=\{q_1^{(1)}, \dots, q_i^{(1)}, q_1^{(2)}, \dots, q_m^{(2)}\}$. When $q_n=q_i^{(1)}$, since p does not depend on $\{q_1, \dots, q_{n-1}\}$, we have

$$(p\smile q_1^{(1)}\smile \cdots \smile q_{i-1}^{(1)}\smile q_{i+1}^{(1)}\smile \cdots \smile q_i^{(1)})\frown(q_1^{(2)}\smile \cdots \smile q_m^{(2)}) = 0. \tag{2}$$

By (1) and (2) $q_n=q_i^{(1)}$ depends linearly on $\{q_1, \dots, q_{n-1}, p\}$. Similarly when $q_n=q_j^{(2)}$, we have

$$(p\smile q_1^{(1)}\smile \cdots \smile q_i^{(1)})\frown(q_1^{(2)}\smile \cdots \smile q_{j-1}^{(2)}\frown q_{j+1}^{(2)}\smile \cdots \smile q_m^{(2)}) = 0. \tag{3}$$

By (1) and (3) $q_n=q_j^{(2)}$ depends linearly on $\{q_1, \dots, q_{n-1}, p\}$.

Thus the linear dependence satisfies the conditions of dependence relation except the transitiveness of dependence, i.e. (5°).

By Prenowitz [1, p. 660], the lattice L_G of convex sets of a descriptive geometry G is characterized by the following properties in a slightly modified but equivalent form.

I. Relative atomic.

II. Upper continuous.[1]

III. Transitiveness of dependence.[2]

IV. Quasi modularity. That is, let c be closed, then $a\leqq c$ implies

$$(a\smile b)\frown c = a\smile(b\frown c).$$

V. Quasi simplicity. That is, there exist only trivial complete congruence relations.

VI$_2$. Not every element is closed.

Thus in L_G, the linear dependence satisfies Definition 2·1 (1°)–(5°), hence we can apply the theory of §2 to the extension of convex sets.

1) By Theorem 1·2, the finiteness of dependence is equivalent to the upper continuity.

2) That is, Definition 2·1 (5°).

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