APPLICATION OF MAJORIZED GROUP OF TRANSFORMATIONS TO FUNCTIONAL EQUATIONS

By

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Preface.

In *S. Bochner and W. T. Martin, Several Complex Variables* (1948) p. 53, there has been proved the following theorem:

If the group $\mathfrak{G}$ *of transformations* $T(\alpha)$ *of the following forms*

$$T(\alpha): \quad 'x^{\nu}=\varphi^{\nu}(x,\alpha)=a^{\nu}_{\mu}(\alpha)x^{\mu}+a^{\nu}_{\mu_1\mu_2}(\alpha)x^{\mu_1}x^{\mu_2}+\cdots\cdots \quad ^{(1)}$$

is majorized, namely there exists a set of regular functions $\Phi^{\nu}(x)$ *such that*

$$\varphi^{\nu}(x,\alpha)\ll\Phi^{\nu}(x)$$

for all α*, then there exists a regular transformation* S *of the form*

$$S: \qquad 'x^{\nu}=f^{\nu}(x)=x^{\nu}+\cdots\cdots$$

such that

$$T(\alpha)=S^{-1}L\{T(\alpha)\}S$$

for all α*, where* $L\{T(\alpha)\}$ *denotes the linear parts of* $T(\alpha)$*. Here* S *is given as follows:*

$$S=\int_{\mathfrak{G}}L\{T(\alpha^{-1})\}\cdot T(\alpha)d\mu(\alpha),$$

where $\mu(\alpha)$ *is an invariant measure introduced in* $\mathfrak{G}$.

In this paper, we apply this theorem to the finite transformation and the one-parameter group of transformations. Thus we shall obtain the solutions of the equations of Schröder[2] of the certain kind and the solutions of the differential equations in the neighbourhood of the singularity of the certain kind.

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1) $a^{\nu}_{\mu}(\alpha)x^{\mu}$, $a^{\nu}_{\mu_1\mu_2}(\alpha)x^{\mu_1}x^{\mu_2}$ etc. mean $\sum_{\mu} a^{\nu}_{\mu}(\alpha)x^{\mu}$, $\sum_{\mu_1,\mu_2} a^{\nu}_{\mu_1\mu_2}(\alpha)x^{\mu_1}x^{\mu_2}$ etc. respectively. The unwritten terms are those of higher orders than those written explicitly. In the following, we use these conventions.

2) M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 113-131.

Chapter I. Equations of Schröder.

§ 1. Conditions for majorizedness.

We consider the finite transformation of the form as follows:

$$(1.1) \qquad T: \qquad 'x^\nu = \varphi^\nu(x) = a^\nu_\mu x^\mu + a^\nu_{\mu_1\mu_2} x^{\mu_1} x^{\mu_2} + \cdots\cdots .$$

When the group of $\{T^k\}$ ($k=0, \pm 1, \pm 2, \ldots\ldots$) is majorized, we say that *T itself is majorized.*

We can easily prove

Lemma. *If $\{T(\alpha)\}$ is majorized, then, for any regular transformation S of the form as follows:*

$$(1.2) \qquad S: \qquad \bar{x}^\nu = f^\nu(x) = s^\nu_\mu x^\mu + s^\nu_{\mu_1\mu_2} x^{\mu_1} x^{\mu_2} + \ldots\ldots ,$$

where det. $|s^\nu_\mu| \neq 0$, $\{\bar{T}(\alpha)\} = \{ST(\alpha)S^{-1}\}$ *is also majorized.*

By means of a suitable linear transformation, we can transform $\| a^\nu_\mu \|$ of (1.1) to the matrix of Jordan's form. Then, by the above lemma, if T is majorized, the transformed transformation is also majorized, and conversely. Thus, for our purpose, without loss of generality, we may assume that $\| a^\nu_\mu \|$ is of Jordan's form. In the following, we assume this.

Put

$$(1.3) \qquad T^k: \qquad 'x^\nu = \varphi^\nu(x, k) = a^\nu_\mu(k) x^\mu + a^\nu_{\mu_1\mu_2}(k) x^{\mu_1} x^{\mu_2} + \cdots\cdots ,$$

then $\| a^\nu_\mu(k) \| = \| a^\nu_\mu \|^k$. Therefore, if T is majorized, the absolute values of all the eigen values λ_ν of $\| a^\nu_\mu \|$ must be unity and moreover the matrix $\| a^\nu_\mu \|$ must be of diagonal form.

We shall prove

Theorem 1. *The necessary and sufficient condition that T is majorized, is that the absolute values of all the eigen values λ_ν of $\| a^\nu_\mu \|$ are unity and $\{T^k\}$ ($k=0, 1, 2, \ldots$) is majorized.*[1]

The necessity is evident. We consider the sufficiency. From majorizedness of $\{T^k\}$ ($k=0, 1, 2, \ldots$), it follows that $\| a^\nu_\mu \|$ is of diagonal form. For positive k, there exists a set of regular functions

$$\Phi^\nu(x) = x^\mu + A^\nu_{\mu_1\mu_2} x^{\mu_1} x^{\mu_2} + \cdots\cdots + A^\nu_{\mu_1\ldots\mu_N} x^{\mu_1} \cdots\cdots x^{\mu_N} + \cdots\cdots$$

such that

$$|a^\nu_{\mu_1\ldots\mu_N}(k)| \leq A^\nu_{\mu_1\ldots\mu_N} .$$

1) When there exists a set of regular functions $\Phi^\nu(x)$ such that $\varphi^\nu(x, k) \ll \Phi^\nu(x)$ for $k = 0, 1$ $2, \ldots$, we say that $\{T^k\}$ ($k = 0, 1, 2, \ldots$) is majorized.

Put

$$T^{-k}: \quad x^\nu = \psi^\nu('x, k) = b^\nu_\mu(k)\,'x^\mu + b^\nu_{\mu_1\mu_2}(k)'x^{\mu_1}{}'x^{\mu_2} + \cdots\cdots,$$

then, substituting this into (1.3), we have:

$$\begin{aligned}'x^\nu =\lambda_\nu^k\,&[b^\nu_\mu(k)\,'x^\mu + b^\nu_{\mu_1\mu_2}(k)'x^{\mu_1}{}'x^{\mu_2} + \cdots\cdots + b^\nu_{\mu_1\cdots\mu_N}(k)'x^{\mu_1}\cdots{}'x^{\mu_N} + \cdots\cdots]\\ &+ a^\nu_{\mu_1\mu_2}(k)\,[b^{\mu_1}_{\nu_1}(k)\,'x^{\nu_1} + b^{\mu_1}_{\omega_1\omega_2}(k)'x^{\omega_1}{}'x^{\omega_2} + \cdots\cdots]\\ &\times[b^{\mu_2}_{\nu_2}(k)'x^{\nu_2} + b^{\mu_2}_{\omega_1\omega_2}(k)'x^{\omega_1}{}'x^{\omega_2} + \cdots\cdots]\\ &+ \cdots\cdots\cdots\cdots\\ &+ a^\nu_{\mu_1\cdots\mu_N}(k)\,[b^{\mu_1}_{\nu_1}(k)'x^{\nu_1} + \cdots\cdots]\cdots\cdots[b^{\mu_N}_{\nu_N}(k)'x^{\nu_N} + \cdots\cdots]\\ &+ \cdots\cdots\cdots\cdots. \qquad\qquad (\text{not summed by } \nu)\end{aligned}$$

Comparing the coefficients of both sides, we have:

$$(1.4)\quad\left\{\begin{aligned}&\lambda_\nu^k b^\nu_\mu(k) = \delta^\nu_\mu,\\ &\lambda_\nu^k b^\nu_{\mu_1\mu_2}(k) + a^\nu_{\omega_1\omega_2}(k) b^{\omega_1}_{\mu_1}(k) b^{\omega_2}_{\mu_2}(k) = 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &\lambda_\nu^k b^\nu_{\mu_1\cdots\mu_N}(k) + a^\nu_{\omega_1\omega_2}(k)\,[b^{\omega_1}_{(\mu_1}(k) b^{\omega_2}_{\mu_2\cdots\mu_N)}(k) + b^{\omega_1}_{(\mu_1\mu_2}(k) b^{\omega_2}_{\mu_3\cdots\mu_N)}(k) + \cdots\cdots]^{(1)}\\ &\qquad + \cdots\cdots\cdots\cdots\\ &\qquad + a^\nu_{\omega_1\cdots\omega_N}(k) b^{\omega_1}_{\mu_1}(k)\cdots\cdots b^{\omega_N}_{\mu_N}(k) = 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots. \qquad\qquad (\text{not summed by } \nu)\end{aligned}\right.$$

From the first of these, we have $b^\nu_\mu(k) = \delta^\nu_\mu \lambda'^k_\nu$ (not summed by ν) where $\lambda'_\nu = 1/\lambda_\nu$. The coefficients $b^\nu_{\mu_1\mu_2}(k)$, ..., $b^\nu_{\mu_1\cdots\mu_N}(k)$, ... of $\psi^\nu(x, k)$ are successively determined from (1.4). We consider the transformation as follows:

$$A: \quad 'x^\nu = \Theta^\nu(x) = x^\nu - A^\nu_{\mu_1\mu_2} x^{\mu_1} x^{\mu_2} - \cdots - A^\nu_{\mu_1\cdots\mu_N} x^{\mu_1}\cdots x^{\mu_N} - \cdots\cdots,$$

and write the inverse transformation of this as follows:

$$B: \quad x^\nu = \Psi^\nu('x) = 'x^\nu + B^\nu_{\mu_1\mu_2}\,'x^{\mu_1}{}'x^{\mu_2} + \cdots + B^\nu_{\mu_1\cdots\mu_N}\,'x^{\mu_1}\cdots{}'x^{\mu_N} + \cdots\cdots,$$

then, among the coefficients of $\Theta^\nu(x)$ and $\Psi^\nu('x)$, we have the analogous relations as (1.4). Comparing these relations, we have

$$|b^\nu_{\mu_1\cdots\mu_N}(k)| \leqq B^\nu_{\mu_1\cdots\mu_N},$$

namely $\psi^\nu('x, k) \ll \Psi^\nu('x)$. Thus we see that $\{T^k\}$ $(k=0, \pm1, \pm2, \ldots\ldots)$ is majorized.

When the absolute values of all the eigen values λ_ν of $\|a^\nu_\mu\|$ are unity, we put $\lambda_\nu = e^{i\theta_\nu}$. We consider the case where all θ_ν's are commensurable

1) $b^{\omega_1}_{(\mu_1}(k) b^{\omega_2}_{\mu_2\cdots\mu_N)}(k)$ mean $\frac{1}{N!}\sum_{(i_1\cdots i_N)} b^{\omega_1}_{\mu_{i_1}}(k) b^{\omega_2}_{\mu_{i_2}\cdots\mu_{i_N}}$ where $(i_1 \ldots\ldots i_N)$ is an arbitrary permutation of $(123\ldots N)$.

with 2π. Put $\theta_\nu = 2\pi \frac{p_\nu}{q_\nu}$, where p_ν, q_ν are integers such that p_ν and q_ν are relatively prime. Let q be L.C.M. of all q_ν's. Then T^q is of the form as follows:

$$T^q: \qquad 'x^\nu = \varphi^\nu(x, q) = x^\nu + a^\nu_{\mu_1\mu_2}(q) x^{\mu_1} x^{\mu_2} + \cdots\cdots .$$

If T is majorized, then, by the uniqueness theorem of H. Cartan[1], T^q must be an identity. Conversely, when T^q becomes an identity for a suitable integer q, it is evident that T is majorized. Thus we have

Theorem 2. *When the absolute values of all the eigen values λ_ν of $\| a^\nu_\mu \|$ is unity and all the arguments of λ_ν are commensurable with 2π, the necessary and sufficient condition that T is majorized, is that, for a suitable integer q, T^q becomes an identity.*

§ 2. Equations of Schröder.

We consider the equations of Schröder for the transformation T. When the absolute values of all the eigen values of $\| a^\nu_\mu \|$ are either greater or less than unity, the equations of Schröder are already completely solved[2]. In this paper, we consider the case where the absolute values of all the eigen values are unity.

When T is majorized, by the general theorem on majorized group, there exists a regular transformation S of the form

$$(2.1) \qquad S: \qquad 'x^\nu = f^\nu(x) = x^\nu + \cdots\cdots$$

such that

$$(2.2) \qquad T^k = S^{-1} L(T^k) S$$

for $k=0, \pm 1, \pm 2, \ldots\ldots$. (2.2) can be written as follows:

$$(2.3) \qquad ST^k = L(T^k) S .$$

For $k=1$, we have

$$(2.4) \qquad f^\nu[\varphi(x)] = \lambda_\nu f^\nu(x) .$$

These are nothing but the equations of Schröder. Conversely, when the equations of Schröder (2.4) have solutions of the form (2.1), then $\bar{T} = STS^{-1}$ is of the form

$$\bar{T}: \qquad 'x^\nu = \lambda_\nu x^\nu ,$$

1) S. Bochner and W. T. Martin, Several Complex Variables (1948), p. 13.

2) M. Hukuhara, Kyūshū-Teikoku-Daigaku Rigaka-Hōkoku (1945), in Japanese. M. Urabe, ibid.

consequently, because $|\lambda_\nu|=1$, $\bar{T}$ is majorized, then, by the lemma in §1, we see that T is majorized. Making use of Theorem 1, we have

Theorem 3. *The necessary and sufficient condition that the equations of Schröder* (2.4) *for the transformation T such that $|\lambda_\nu|=1$ have regular solutions of the form* (2.1), *is that $\{T^k\}$ $(k=0,1,2,\ldots)$ is majorized.*

Though it is equivalent to the proof of the general theorem on majorized group, in order to seek for the concrete form of the solutions, making use of the idea of invariant integration, we shall directly prove the existence of the solutions of the equations of Schröder for the majorized transformation.

For brevity, we write $\varphi^\nu(x,k)$ as $\varphi^\nu(k)$. Put

$$F^\nu_{kn}=\frac{1}{n}\left[\frac{1}{\lambda_\nu^k}\varphi^\nu(k)+\frac{1}{\lambda_\nu^{k+1}}\varphi^\nu(k+1)+\cdots\cdots+\frac{1}{\lambda_\nu^{k+n-1}}\varphi^\nu(k+n-1)\right]. \tag{2.5}$$

Then, from our assumption that $\varphi^\nu(k)\ll\Phi^\nu$, it follows that $F^\nu_{kn}\ll\Phi^\nu$. Consequently there exists a sequence $\{n_i\}$ such that, for k fixed, $\{F^\nu_{kn_i}\}$ uniformly converges. Put these limit functions f^ν_k, then $f^\nu_k=x^\nu+\cdots\cdots$. Now

$$F^\nu_{k+1n}-F^\nu_{kn}=\frac{1}{n}\left[\frac{1}{\lambda_\nu^{k+n}}\varphi^\nu(k+n)-\frac{1}{\lambda_\nu^k}\varphi^\nu(k)\right],$$

consequently, $F^\nu_{k+1\,n}-F^\nu_{kn}\ll\dfrac{2}{n}\Phi^\nu$, therefore, when $n\to\infty$, $F^\nu_{k+1\,n}-F^\nu_{k\,n}$ uniformly tend to zero. Therefore, for any integers k and l, $f^\nu_k=f^\nu_l$, consequently we may write these f^ν_k as f^ν. Now

$$F^\nu_{kn}(\varphi)=\frac{1}{n}\left[\frac{1}{\lambda_\nu^k}\varphi^\nu(k+1)+\cdots\cdots+\frac{1}{\lambda_\nu^{k+n-1}}\varphi^\nu(k+n)\right]=\lambda_\nu F^\nu_{k+1n}(x).$$

Consequently, putting $n=n_i$ and making $i\to\infty$, we have

$$f^\nu(\varphi)=\lambda_\nu f^\nu(x).$$

Thus we see that *the equations of Schröder* (2.4) *have regular solutions f^ν, which are one of the sets of limit functions of $\{F^\nu_{kn}\}$.* Now, we can prove that the set of limit functions of $\{F^\nu_{kn}\}$ is unique. For an arbitrary small positive number ε, there exists a number G such that, for $i>G$,

$$|F^\nu_{k\,n_i}-f^\nu|<\varepsilon.$$

Then it follows that $|F^\nu_{kn_i}(\varphi)-f^\nu(\varphi)|<\varepsilon$, consequently $|\lambda_\nu F^\nu_{k+1\,n_i}-\lambda_\nu f^\nu|<\varepsilon$. Since $|\lambda_\nu|=1$, $|F^\nu_{k+1\,n_i}-f^\nu|<\varepsilon$. Thus, for any positive integers p, we have:

$$|F^{\nu}_{k+p\, n_i}-f^{\nu}|<\varepsilon .$$

Then, adding these inequalities, we have:

$$\left|\frac{1}{m_i}(F^{\nu}_{kn_i}+F^{\nu}_{k+1\, n_i}+\cdots\cdots+F^{\nu}_{k+m_i-1\, n_i})-f^{\nu}\right|<\varepsilon . \tag{2.6}$$

If there exists another set of limit functions g^{ν}, then, there exists a sequence $\{m_i\}$ and a positive number H such that, for $i>H$,

$$|F^{\nu}_{km_i}-g^{\nu}|<\varepsilon .$$

Then, similarly as above, we have

$$\left|\frac{1}{n_i}(F^{\nu}_{km_i}+F^{\nu}_{k+1\, m_i}+\cdots\cdots+F^{\nu}_{k+n_i-1\, m_i})-g^{\nu}\right|<\varepsilon . \tag{2.7}$$

Now it is evident that $\frac{1}{m_i}(F^{\nu}_{k\, n_i}+\cdots+F^{\nu}_{k+m_i-1\, n_i})=\frac{1}{n_i}(F^{\nu}_{km_i}+\cdots+F^{\nu}_{k+n_i-1\, m_i})$. Then, comparing (2.6) and (2.7) for $i>$ max. (G, H), we have

$$|f^{\nu}-g^{\nu}|<2\varepsilon .$$

Here ε is an arbitrary positive number, therefore it must be $f^{\nu}\equiv g^{\nu}$, namely the set of limit functions of $\{F^{\nu}_{kn}\}$ is unique. Now, from $F^{\nu}_{kn}\ll\Phi^{\nu}$, any sub-sequence of $\{F^{\nu}_{kn}\}$ has a limit, and from the above result, that limit is f^{ν}. Thus we see that the sequence $\{F^{\nu}_{kn}\}$ uniformly converges to f^{ν}, i.e.

$$f^{\nu}=\lim_{n\to\infty}F^{\nu}_{kn} . \tag{2.8}$$

Thus, we see that, *when T is majorized, the equations of Schröder* (2.4) *have regular solutions f^{ν} given by* (2.8).

When the equations of Schröder (2.4) have regular solutions f^{ν} of the from (2.1), we transform the variables x^{ν} as follows: $\bar{x}^{\nu}=f^{\nu}(x)$. Then the given transformation is expressed with regard to $\bar{x}^{\nu}$-system as follows: $'\bar{x}^{\nu}=\lambda_{\nu}\bar{x}^{\nu}$, namely $'\bar{x}^{\nu}=e^{i\theta_{\nu}}\bar{x}^{\nu}$. Therefore there exists a group $\mathfrak{G}$ containing the given transformation such that

$$\mathfrak{G}: \qquad '\bar{x}^{\nu}=e^{i\theta_{\nu}t}\bar{x}^{\nu} .$$

The operator functions $\bar{\xi}^{\nu}$ of $\mathfrak{G}$ become as follows: $\bar{\xi}^{\nu}=i\theta_{\nu}\bar{x}^{\nu}$, consequently, with regard to the initial coordinates, the operator functions ξ^{ν} are determined by the equations as follows:

$$\xi^{\mu}\frac{\partial f^{\nu}}{\partial x^{\mu}}=i\theta_{\nu}f^{\nu}.$$

Thus we see that, *for the transformation T such that $|\lambda_{\nu}|=1$, if $\{T^k\}$* ($k=0, 1, 2, \ldots$) *is majorized, then there exists a one-parameter group containing T and having regular operator functions.*

Next we consider the case where the arguments θ_ν of the eigen values λ_ν are all commensurable with 2π. Put $\theta_\nu = 2\pi\frac{p_\nu}{q_\nu}$. When T is majorized, by Theorem 2, it becomes $T^q = I$, where q is L.C.M. of all q_ν's. In this case, the regular solutions of the equations of Schröder (2.4) are readily sought as follows:

$$f^\nu(x) = \frac{1}{q}\left[x^\nu + \frac{1}{\lambda_\nu}\varphi^\nu(1) + \frac{1}{\lambda_\nu^2}\varphi^\nu(2) + \cdots\cdots + \frac{1}{\lambda_\nu^{q-1}}\varphi^\nu(q-1)\right]. \tag{2.9}$$

For

$$f^\nu(\varphi) = \frac{1}{q}\left[\varphi^\nu(1) + \frac{1}{\lambda_\nu}\varphi^\nu(2) + \cdots\cdots + \frac{1}{\lambda_\nu^{q-2}}\varphi^\nu(q-1) + \frac{1}{\lambda_\nu^{q-1}}\varphi^\nu(q)\right] = \lambda_\nu f^\nu(x),$$

because $\varphi^\nu(q) = x^\nu$ and $\lambda_\nu^q = 1$.

When the equations of Schröder (2.4) have formal solutions of the form (2.1), it follows formally that

$$\varphi^\nu(x) = g^\nu[e^{i\theta}f(x)], \tag{2.10}$$

where $g^\nu = (f^\nu)^{-1}$. Consequently, it follows formally that

$$\varphi^\nu(q) = g^\nu[e^{iq\theta}f(x)] = x^\nu,$$

namely $T^q = I$. Therefore T is majorized. Thus, by means of Theorem 2, we have

Theorem 4. *When the transformation T has the eigen values λ_ν, of which the absolute values are all unity and the argumets are all commensurable with 2π, the following three conditions are equivalent to one anther:*
(i) *T is majorized;*
(ii) *there exists an integer q such that T^q becomes an identity;*
(iii) *the equations of Schröder* (2.4) *have formal solutions f^ν of the form* (2.1).
When one of these three conditions is satisfied, the equations of Schröder have regular solutions given by (2.9) *which are of the form* (2.1).

Here we need to pay attention to the fact that we have not concluded the convergence of the formal solutions of the equations of Schröder if they exist. Our conclusion is that, if there exists at least one set of formal solutions, then there exists at least one set of regular solutions.

Chapter II. Differential equations.

§ 3. Conditions for majorizedness.

We consider the differential equation of the form as follows:

$$(3.1) \qquad Xf \equiv \xi^{\mu}\frac{\partial f}{\partial x^{\mu}} = 0,$$

where

$$(3.2) \qquad \xi^{\nu} = c^{\nu}_{\mu}x^{\mu} + c^{\nu}_{\mu_1\mu_2}x^{\mu_1}x^{\mu_2} + \cdots\cdots.$$

When, in the complex plane, the eigen values μ_{ν} of $\| c^{\nu}_{\mu} \|$ lie all in the same side of the straight line passing through the origin, the differential equation (3.1) is completely solved.[1] In this paper, we assume that all the eigen values lie on a straight line passing through the origin. Multiplying (3.1) by a suitable constant, without loss of generality, we may assume that all the eigen values are of the form as follows:

$$(3.3) \qquad \mu_{\nu} = i\theta_{\nu},$$

where θ_{ν} are real. Moreover, by effecting a suitable linear transformation of the variables, without loss of generality, we may assume that $\| c^{\nu}_{\mu} \|$ is of Jordan's form. In the following, we assume these.

We consider the one-parameter group $\mathfrak{G}$ of transformations with the operator functions ξ^{ν}. Then the transformations of $\mathfrak{G}$ are given by the integrals $'x^{\nu} = \varphi^{\nu}(x, t)$ of the differential equations as follows:

$$(3.4) \qquad \frac{d'x^{\nu}}{dt} = \xi^{\nu}('x).$$

Here t is real and $\varphi^{\nu}(x, t)$ are of the form as follows:

$$(3.5) \qquad 'x^{\nu} = \varphi^{\nu}(x, t) = a^{\nu}_{\mu}(t)x^{\mu} + a^{\nu}_{\mu_1\mu_2}(t)x^{\mu_1}x^{\mu_2} + \cdots\cdots,$$

and $a^{\nu}_{\mu}(0) = \delta^{\nu}_{\mu}$, $a^{\nu}_{\mu_1\mu_2}(0) = 0, \ldots\ldots, a^{\nu}_{\mu_1\cdots\mu_N}(0) = 0, \ldots\ldots$. Substituting (3.5) into (3.4), we have:

$$\frac{d}{dt}\| a^{\nu}_{\mu}(t) \| = \| c^{\nu}_{\mu} \| \cdot \| a^{\nu}_{\mu}(t) \|.$$

From the initial condition that $\| a^{\nu}_{\mu}(0) \|$ is a unit matrix, it follows that

$$(3.6) \qquad \| a^{\nu}_{\mu}(t) \| = e^{t \| c^{\nu}_{\mu} \|}.$$

If $\mathfrak{G}$ is majorized, there exists a set of functions $\Phi^{\nu}(x)$ such that $\varphi^{\nu}(x, t) \ll \Phi^{\nu}(x)$, consequently the origin is stable for the differential equations (3.4). Conversely, if the origin is stable, then, for any given ε, there exists δ such that, for $|x^{\nu}| \leq \delta$, $|\varphi^{\nu}(x, t)| \leq \varepsilon$, consequently $\mathfrak{G}$ is majorized. Thus we see that $\mathfrak{G}$ *is majorized if and only if the origin is stable for the differential equations* (3.4).

1) H. Dulac, Bull. Soc. Math. France (1912).
M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 25-37.

From (3.6), it is readily seen that, *if* $\mathfrak{G}$ *is majorized, then the matrix* $\| c_\mu^\nu \|$ *must be of diagonal form.*

If $\varphi^\nu(x, t)$ is majorized for positive t, namely there exist regular functions $\Phi^\nu(x)$ such that $\varphi^\nu(x, t) \ll \Phi^\nu(x)$ for positive t, then, in the same way as the proof of Theorem 1, we see that $\varphi^\nu(x, t)$ is majorized also for negative t, namely $\mathfrak{G}$ is majorized. Thus *we have the theorem corresponding to Theorem* 1.

We consider the case where all θ_ν's are mutually commensurable. In this case, there exists a real number ω such that

$$\omega\theta_\nu = 2\pi p_\nu \tag{3.7}$$

where p_ν are integers. In this case, we shall seek for the condition that $\mathfrak{G}$ is majorized.

First we assume that $\mathfrak{G}$ is majorized. Then it must be

$$c_\mu^\nu = \mu_\nu \delta_\mu^\nu = i\theta_\nu \delta_\mu^\nu . \tag{3.8}$$

Substituting (3.5) into (3.4), and comparing the coefficients of the products of x^ν, we have:

$$\left\{\begin{aligned} &\frac{da_\mu^\nu}{dt} = i\theta_\nu a_\mu^\nu , \\ &\frac{da_{\mu_1\mu_2}^\nu}{dt} = i\theta_\nu a_{\mu_1\mu_2}^\nu + c_{\omega_1\omega_2}^\nu a_{\mu_1}^{\omega_1} a_{\mu_2}^{\omega_2} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{da_{\mu_1\ldots\mu_N}^\nu}{dt} = i\theta_\nu a_{\mu_1\ldots\mu_N}^\nu + c_{\omega_1\omega_2}^\nu \{ a_{(\mu_1}^{\omega_1} a_{\mu_2\ldots\mu_N)}^{\omega_2} + a_{(\mu_1\mu_2}^{\omega_1} a_{\mu_3\ldots\mu_N)}^{\omega_2} + \cdots + a_{(\mu_1\ldots\mu_{N-1}}^{\omega_1} a_{\mu_N)}^{\omega_2} \} \\ &\qquad + \cdots\cdots\cdots \\ &\qquad + c_{\omega_1\ldots\omega_N}^\nu a_{\mu_1}^{\omega_1} \cdots\cdots a_{\mu_N}^{\omega_N} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad\qquad (\text{not summed by } \nu). \end{aligned}\right. \tag{3.9}$$

From the first of these relations, we have:

$$a_\mu^\nu = \delta_\mu^\nu e^{it\theta_\nu} . \tag{3.10}$$

Substituting these into the second of (3.9), we have:

$$a_{\mu_1\mu_2}^\nu = c_{\mu_1\mu_2}^\nu e^{it\theta_\nu} \int_0^t e^{it(\theta_{\mu_1}+\theta_{\mu_2}-\theta_\nu)} dt .$$

Therefore we have:

when $\theta_\nu \neq \theta_{\mu_1} + \theta_{\mu_2}$, $a_{\mu_1\mu_2}^\nu = c_{\mu_1\mu_2}^\nu \dfrac{e^{it(\theta_{\mu_1}+\theta_{\mu_2})} - e^{it\theta_\nu}}{(\theta_{\mu_1} + \theta_{\mu_2} - \theta_\nu)i}$;

when $\theta_\nu = \theta_{\mu_1} + \theta_{\mu_2}$, $a_{\mu_1\mu_2}^\nu = c_{\mu_1\mu_2}^\nu t e^{it\theta_\nu}$.

Since $\mathfrak{G}$ is majorized, for $\theta_\nu=\theta_{\mu_1}+\theta_{\mu_2}$, it must be $c^\nu_{\mu_1\mu_2}=0$, consequently $a^\nu_{\mu_1\mu_2}=0$. Thus $a^\nu_{\mu_1\mu_2}(t)$ are linear combinations of $e^{it\theta_\nu}$ and $e^{it(\theta_{\mu_1}+\theta_{\mu_2})}$ (inclusive of zero). We assume that, for $M\leqq N-1$, $a^\nu_{\mu_1\ldots\mu_M}$ are linear combinations of $e^{it\theta_\nu}$, $e^{it(\theta_{\lambda_1}+\theta_{\lambda_2})}$, ..., $e^{it(\theta_{\mu_1}+\cdots+\theta_{\mu_M})}$. Then, from (3.9), we have:

$$\begin{aligned} a^\nu_{\mu_1\ldots\mu_N} &= e^{it\theta_\nu}\int_0^t L\left[e^{it(\theta_{\lambda_1}+\theta_{\lambda_2})}, \ldots\ldots, e^{it(\theta_{\mu_1}+\cdots+\theta_{\mu_N})}\right]\cdot e^{-it\theta_\nu}dt \\ &= \kappa t e^{it\theta_\nu}+L\left[e^{it\theta_\nu}, e^{it(\theta_{\lambda_1}+\theta_{\lambda_2})}, \ldots\ldots, e^{it(\theta_{\mu_1}+\cdots+\theta_{\mu_N})}\right], \end{aligned}$$

where $L[\ldots\ldots]$ denotes a linear combination of the arguments and κ is a constant. Since $\mathfrak{G}$ is majorized, it must be $\kappa=0$, namely $a^\nu_{\mu_1\ldots\mu_N}$ become linear combinations of $e^{it\theta_\nu}$, $e^{it(\theta_{\lambda_1}+\theta_{\lambda_2})}$, ..., $e^{it(\theta_{\mu_1}+\cdots+\theta_{\mu_N})}$. Thus, for any N, it is valid that

$$a^\nu_{\mu_1\ldots\mu_N}(t) = L\left[e^{it\theta_\nu}, e^{it(\theta_{\lambda_1}+\theta_{\lambda_2})}, \ldots\ldots, e^{it(\theta_{\mu_1}+\cdots+\theta_{\mu_N})}\right].$$

Now, for $N\geqq 2$, $a^\nu_{\mu_1\ldots\mu_N}(0)=0$, namely, in the right-hand side of the above formulae, $L(1,1,\ldots,1)=0$. From (3.7), $e^{i\omega\theta_\nu}=e^{i\omega(\theta_{\lambda_1}+\theta_{\lambda_2})}=\cdots=e^{i\omega(\theta_{\mu_1}+\cdots+\theta_{\mu_N})}=1$, consequently $a^\nu_{\mu_1\ldots\mu_N}(\omega)=0$. For $N=1$, from (3.10), $a^\nu_\mu(0)=a^\nu_\mu(\omega)=\delta^\nu_\mu$. Thus we see that, if $\mathfrak{G}$ is majorized, then $\varphi^\nu(x,\omega)=x^\nu$, namely $\mathfrak{G}$ has a period ω with respect to the parameter t.

Conversely we assume that $\mathfrak{G}$ has a period ω with respect to t[1]. We take any functions Ξ^ν such that $\Xi^\nu\gg\xi^\nu$. Put $X\equiv\xi^\mu\dfrac{\partial}{\partial x^\mu}$, and $X'\equiv\Xi^\mu\dfrac{\partial}{\partial x^\mu}$. Then, if $F\gg f$, then $X'F\gg Xf$. Therefore $X'^p(x^\nu)\gg X^p(x^\nu)$ for any positive integer p. Then, for any non-negative number t, $t^pX'^p(x^\nu)\gg t^pX^p(x^\nu)$. Now

$$'x^\nu = \varphi^\nu(x,t) = e^{tX}(x^\nu) = \sum_{p=0}^\infty \frac{1}{p!}t^pX^p(x^\nu) \quad .$$

Therefore, if we put

$$\Phi^\nu(x,t) = e^{tX'}(x^\nu) = A^\nu_\mu(t)x^\mu + A^\nu_{\mu_1\mu_2}(t)x^{\mu_1}x^{\mu_2} + \cdots\cdots,$$

then

$$|a^\nu_\mu(t)|\leqq A^\nu_\mu(t),\quad |a^\nu_{\mu_1\mu_2}(t)|\leqq A^\nu_{\mu_1\mu_2}(t),\ \ldots\ldots .$$

Here $A^\nu_\mu(t)$, $A^\nu_{\mu_1\mu_2}(t)$, ... are power series of positive coefficients, consequently, for t such that $0\leqq t\leqq\omega$, $A^\nu_\mu(t)\leqq A^\nu_\mu(\omega)$, $A^\nu_{\mu_1\mu_2}(t)\leqq A^\nu_{\mu_1\mu_2}(\omega)$, ..., namely $\varphi^\nu(x,t)\ll\Phi^\nu(x,\omega)$. Since ω is a period of $\varphi^\nu(x,t)$, we see that, for all t, $\varphi^\nu(x,t)\ll\Phi^\nu(x,\omega)$, namely that $\mathfrak{G}$ is majorized.

Thus we have

1) We may assume that ω is positive. For, if ω is negative, $-\omega$ being also period, it suffices to take $-\omega$ instead of ω.

Theorem 5. *When the eigen values μ_ν of $\| c^\nu_\mu \|$ are all pure imaginary and their absolute values are all mutually commensurable, the necessary and sufficient condition that the group $\mathfrak{G}$ is majorized, is that the group has a period with respect to the parameter.*

This theorem corresponds to Theorem 2 for finite transformation.

§ 4. Characteristic equations.

In order to solve the differential equation (3.1), as in the case where all the eigen values of $\| c^\nu_\mu \|$ lie in the same side of the straight line passing through the origin, we consider the characteristic equations for the group $\mathfrak{G}$[1].

When $\mathfrak{G}$ is majorized, by the general theorem on majorized group, there exists a regular transformation S of the form

$$(4.1) \qquad S: \qquad 'x^\nu = f^\nu(x) = x^\nu + \cdots\cdots$$

such that

$$(4.2) \qquad T(t) = S^{-1}L\{T(t)\}S$$

for any t. (4.2) can be written as follows:

$$(4.3) \qquad ST(t) = L\{T(t)\}S\,.$$

Now, from (3.10), (4.3) can be written as follows:

$$(4.4) \qquad f^\nu[\varphi(x, t)] = e^{it\theta_\nu} f^\nu(x)\,.$$

These are the equations of Schröder for $\mathfrak{G}$. Differentiating both sides of (4.4) with respect to t and making use of (3.4), we have:

$$\xi^\mu('x)\frac{\partial f^\nu('x)}{\partial 'x^\mu} = i\theta_\nu e^{it\theta_\nu} f^\nu(x)\,.$$

Putting $t=0$, we have:

$$(4.5) \qquad Xf^\nu \equiv \xi^\mu \frac{\partial f^\nu}{\partial x^\mu} = i\theta_\nu f^\nu\,.$$

These are the characteristic equations for $\mathfrak{G}$. Thus we see that, when $\mathfrak{G}$ is majorized, the characteristic equations for $\mathfrak{G}$ have regular solutions of the form (4.1).

Conversely, we assume that the characteristic equations (4.5) have regular solutions of the form (4.1). Substituting $'x^\nu$ for x^ν, and making use of (3.4), we have:

1) M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 25-37.

$$\frac{df^\nu('x)}{dt} = i\theta_\nu f^\nu('x).$$

Integrating these equations, we have $f^\nu('x) = e^{it\theta_\nu} f^\nu(x)$, namely (4.4). When we transform the variables x^ν by $\bar{x}^\nu = f^\nu(x)$, (4.4) can be written as follows:

$$'\bar{x}^\nu = e^{it\theta_\nu} \bar{x}^\nu.$$

These are formulae of the transformations of $\mathfrak{G}$ with regard to $\bar{x}^\nu$-system. From these formulae it is evident that $\mathfrak{G}$ is majorized with regard to $\bar{x}^\nu$-system. By the lemma in § 1, $\mathfrak{G}$ is majorized with regard to any coordinate system, consequently, of course with regard to x^ν-system. Thus we have

Theorem 6. *For the group $\mathfrak{G}$ where the eigen values of $\| c^\nu_\mu \|$ are all pure imaginary, the necessary and sufficient conition that the characteristic equations* (4.5) *for $\mathfrak{G}$ have regular solutions of the form* (4.1), *is that $\mathfrak{G}$ is majorized.*

As in § 2, in order to seek for the concrete form of the solutions, we shall directly prove the existence of the solutions of the characteristic equations for the majorized group $\mathfrak{G}$.

Since $\mathfrak{G}$ is majorized, there exists a set of functions $\Phi^\nu(x)$ such that $\varphi^\nu(x, t) \ll \Phi^\nu(x)$. Put

$$F^\nu_{tn}(x) = \frac{1}{n}\int_t^{t+n} e^{-i\tau\theta_\nu} \varphi^\nu(x, \tau) d\tau,$$

where n is an arbitrary positive integer. Since $\varphi^\nu(x, \tau) \ll \Phi^\nu(x)$, $F^\nu_{tn}(x) \ll \Phi^\nu(x)$. Consequently there exists a sequence $\{n_i\}$ such that, for fixed t, $\{F^\nu_{tn_i}\}$ uniformly converges. Put these limit functions f^ν_t, then $f^\nu_t = x^\nu + \cdots\cdots$.
Now

$$\begin{aligned} F^\nu_{t_2 n} - F^\nu_{t_1 n} &= \frac{1}{n}\left[\int_{t_2}^{t_2+n} - \int_{t_1}^{t_1+n}\right] \\ &= \frac{1}{n}\left[\int_{t_1+n}^{t_2+n} - \int_{t_1}^{t_2}\right] \\ &= \frac{1}{n}\left[\int_{t_1}^{t_2} e^{-i(n+\tau)\theta_\nu} \varphi^\nu(x, n+\tau) d\tau - \int_{t_1}^{t_2} e^{-i\tau\theta_\nu} \varphi^\nu(x, \tau) d\tau\right]. \end{aligned}$$

Since $\varphi^\nu(x, n+\tau) \ll \Phi^\nu(x)$, $\left|\int_{t_1}^{t_2} e^{-i(n+\tau)\theta_\nu} \varphi^\nu(x, n+\tau) d\tau\right| \leq |t_2 - t_1| \Phi^\nu(|x|)$. Consequently, when $n \to \infty$, $F^\nu_{t_2 n} - F^\nu_{t_1 n}$ tend uniformly to zero. Therefore $f^\nu_t(x)$ are independent of t, consequently we write these f^ν_t as f^ν. Now

$$\begin{aligned}F^{\nu}_{tn}[\varphi(x, t_0)] &= \frac{1}{n}\int_t^{t+n} e^{-i\tau\theta_\nu}\varphi^\nu[\varphi(x, t_0), \tau]d\tau \\ &= \frac{1}{n}\int_t^{t+n} e^{-i\tau\theta_\nu}\varphi^\nu(x, \tau+t_0)d\tau \\ &= \frac{1}{n}\int_{t+t_0}^{t+t_0+n} e^{-i(\tau-t_0)\theta_\nu}\varphi^\nu(x, \tau)d\tau \\ &= e^{it_0\theta_\nu}F^{\nu}_{t+t_0\, n}(x).\end{aligned}$$

Putting $n=n_i$ and making $n_i\rightarrow\infty$, we have

$$f^\nu[\varphi(x, t_0)] = e^{it_0\theta_\nu}f^\nu(x).$$

These are nothing but (4.4), consequently $f^\nu(x)$ satisfy the characteristic equations (4.5) for $\mathfrak{G}$. Namely we see that, when $\mathfrak{G}$ is majorized, the characteristic equations (4.5) for $\mathfrak{G}$ have regular solutions of the form (4.1), which are limit functions of $\{F^{\nu}_{tn}\}$. We shall show that the set of limit functions of $\{F^{\nu}_{tn}\}$ is unique. For an arbitrary small positive number ε, there exists a number G such that, for $i>G$,

$$|F^{\nu}_{tn_i} - f^\nu| < \varepsilon.$$

Then $|F^{\nu}_{tn_i}[\varphi(x, n_i)] - f^\nu[\varphi(x, n_i)]| < \varepsilon$, consequently

$$|e^{in_i\theta_\nu}F^{\nu}_{t+n_i\, n_i}(x) - e^{in_i\theta_\nu}f^\nu(x)| < \varepsilon,$$

namely

$$|F^{\nu}_{t+n_i\, n_i} - f^\nu| < \varepsilon.$$

Then, for any positive integer p, we have

$$|F^{\nu}_{t+pn_i\, n_i} - f^\nu| < \varepsilon.$$

Therefore it follows that

$$\left|\frac{1}{m_i}\left(F^{\nu}_{tn_i} + F^{\nu}_{t+n_i n_i} + \cdots\cdots + F^{\nu}_{t+(m_i-1)n_i\, n_i}\right) - f^\nu\right| < \varepsilon. \tag{4.6}$$

If there are another set of limit functions g^ν, then there exists a sequence $\{m_i\}$ such that, for $i>H$,

$$|F^{\nu}_{tm_i} - g^\nu| < \varepsilon.$$

Similarly as above, we have

$$\left|\frac{1}{n_i}\left(F^{\nu}_{tm_i} + F^{\nu}_{t+m_i m_i} + \cdots\cdots + F^{\nu}_{t+(n_i-1)m_i\, m_i}\right) - g^\nu\right| < \varepsilon. \tag{4.7}$$

Now

$$\frac{1}{m}\left(F^{\nu}_{tn}+F^{\nu}_{t+nn}+\cdots\cdots+F^{\nu}_{t+(m-1)nn}\right)=\frac{1}{mn}\int_t^{t+mn}e^{-i\tau\theta_\nu}\varphi^\nu(x,\tau)d\tau$$
$$=\frac{1}{n}\left(F^{\nu}_{tm}+F^{\nu}_{t+mm}+\cdots\cdots+F^{\nu}_{t+(n-1)mm}\right).$$

Then, comparing (4.6) and (4.7), for $i>$ max. (G, H), we have

$$|f^\nu-g^\nu|<2\varepsilon.$$

Here ε is an arbitrary positive number, therefore it must be $f^\nu\equiv g^\nu$, namely the set of limit functions of $\{F^{\nu}_{tn}\}$ is unique. Thus, as in §2, we have:

$$f^\nu=\lim_{n\to\infty}F^{\nu}_{tn}, \tag{4.8}$$

Thus we see that, *when* $\mathfrak{G}$ *is majorized, the characteristic equations* (4.5) *for* $\mathfrak{G}$, *have regular solutions* f^ν *given by* (4.8).

Next we consider the case where all θ_ν's are mutually commensurable. In this case, there exists a real number ω such that (3.7) hold. When $\mathfrak{G}$ is majorized, by the proof of Theorem 5, we see that $\varphi^\nu(x,\omega)=x^\nu$. In this case, the regular solutions of the characteristic equations (4.5) are readily sought as follows:

$$f^\nu(x)=\frac{1}{\omega}\int_0^\omega e^{-i\tau\theta_\nu}\varphi^\nu(x,\tau)d\tau. \tag{4.9}$$

For,

$$\begin{aligned}
f^\nu[\varphi(x,t)]&=\frac{1}{\omega}\int_0^\omega e^{-i\tau\theta_\nu}\varphi^\nu[\varphi(x,t),\tau]\,d\tau\\
&=\frac{1}{\omega}\int_0^\omega e^{-i\tau\theta_\nu}\varphi^\nu(x,t+\tau)d\tau\\
&=\frac{1}{\omega}\int_t^{t+\omega} e^{-i(\tau-t)\theta_\nu}\varphi^\nu(x,\tau)d\tau\\
&=\frac{1}{\omega}e^{it\theta_\nu}\int_t^{t+\omega} e^{-i\tau\theta_\nu}\varphi^\nu(x,\tau)d\tau\\
&=e^{it\theta_\nu}f^\nu(x).
\end{aligned}$$

Because $e^{-i\tau\theta_\nu}\varphi^\nu(x,\tau)$ have the period ω.

Next, we assume that the characteristic equations (4.5) have the formal solutions f^ν of the form (4.1). Put

$$f^\nu(x)=x^\nu+k^\nu_{\mu_1\mu_2}x^{\mu_1}x^{\mu_2}+\cdots\cdots+k^\nu_{\mu_1\cdots\mu_N}x^{\mu_1}\cdots x^{\mu_N}+\cdots\cdots.$$

Then, from (4.5), we have formally:

$$
\begin{aligned}
(4.10)\quad & i\theta_\nu[{}'x^\nu + k^\nu_{\mu_1\mu_2}{}'x^{\mu_1}{}'x^{\mu_2} + \cdots\cdots + k^\nu_{\mu_1\ldots\mu_N}{}'x^{\mu_1}\cdots{}'x^{\mu_N}\cdots\cdots] \\
&= \xi^\mu({}'x)[\delta^\nu_\mu + 2k^\nu_{\mu\mu_1}{}'x^{\mu_1} + \cdots\cdots + Nk^\nu_{\mu\mu_1\ldots\mu_{N-1}}{}'x^{\mu_1}\cdots{}'x^{\mu_{N-1}} + \cdots\cdots] \\
&= \xi^\nu({}'x) + 2k^\nu_{\mu\mu_1}\xi^\mu({}'x){}'x^{\mu_1} + \cdots + Nk^\nu_{\mu\mu_1\ldots\mu_{N-1}}\xi^\mu({}'x){}'x^{\mu_1}\cdots{}'x^{\mu_{N-1}} + \cdots\cdots.
\end{aligned}
$$

Substituting

$$
{}'x^\nu = \varphi^\nu(x, t) = a^\nu_\mu(t)x^\mu + \cdots\cdots + a^\nu_{\mu_1\ldots\mu_N}(t)x^{\mu_1}\cdots x^{\mu_N} + \cdots\cdots
$$

into both sides of (4.10), we have the equations which are valid formally with respect to x^μ. Put

$$
f^\nu(\varphi) = A^\nu_\mu(t)x^\mu + \cdots\cdots + A^\nu_{\mu_1\ldots\mu_N}(t)x^{\mu_1}\cdots x^{\mu_N} + \cdots\cdots,
$$

then the left-hand sides of (4.10) are $i\theta_\nu f^\nu(\varphi)$ and the right-hand sides of (4.10) are

$$
\begin{aligned}
&\frac{d\varphi^\nu}{dt} + 2k^\nu_{\mu\mu_1}\frac{d\varphi^\mu}{dt}\varphi^{\mu_1} + \cdots\cdots + Nk^\nu_{\mu\mu_1\ldots\mu_{N-1}}\frac{d\varphi^\mu}{dt}\varphi^{\mu_1}\cdots\varphi^{\mu_{N-1}} + \cdots\cdots \\
&= \frac{d}{dt}f^\nu(\varphi) \\
&= \dot{A}^\nu_\mu(t)x^\mu + \cdots\cdots + \dot{A}^\nu_{\mu_1\ldots\mu_N}(t)x^{\mu_1}\cdots x^{\mu_N} + \cdots\cdots.
\end{aligned}
$$

Thus we have

$$
\begin{cases}
\dot{A}^\nu_\mu(t) = i\theta_\nu A^\nu_\mu(t) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{A}^\nu_{\mu_1\ldots\mu_N}(t) = i\theta_\nu A^\nu_{\mu_1\ldots\mu_N}(t), \\
\cdots\cdots\cdots\cdots\cdots\cdots.
\end{cases}
$$

Since

$$
A^\nu_\mu(0) = \delta^\nu_\mu,\ A^\nu_{\mu_1\mu_2}(0) = k^\nu_{\mu_1\mu_2},\ \cdots\cdots,\ A^\nu_{\mu_1\ldots\mu_N}(0) = k^\nu_{\mu_1\ldots\mu_N},\ \cdots\cdots,
$$

integrating the above equations, we have

$$
A^\nu_\mu = e^{it\theta_\nu}\delta^\nu_\mu,\ A^\nu_{\mu_1\mu_2}(t) = e^{it\theta_\nu}k^\nu_{\mu_1\mu_2},\ \cdots\cdots,\ A^\nu_{\mu_1\ldots\mu_N}(t) = e^{it\theta_\nu}k^\nu_{\mu_1\ldots\mu_N},\ \cdots\cdots,
$$

namely, we have formally (4.4). Then, solving (4.4) formally with respect to $\varphi^\nu(x, t)$, we have

$$
\varphi^\nu(x, t) = g^\nu[e^{it\theta}f(x)],
$$

where $g^\nu = (f^\nu)^{-1}$. Then $\varphi^\nu(x, \omega) = g^\nu[e^{i\omega\theta}f(x)] = x^\nu$. Consequently, by Theorem 5, $\mathfrak{G}$ is majorized. Thus we have

Theorem 7. *When the eigenvalues μ_ν of $\| c^\nu_\mu \|$ of the group $\mathfrak{G}$ are pure imaginary and their absolute values are all mutually commensurable, the following three conditions are equivalent to one another:*

(i) $\mathfrak{G}$ *is majorized*;

(ii) $\mathfrak{G}$ *has a period with respect to the parameter*;

(iii) *the characteristic equations* (4.5) *for* $\mathfrak{G}$ *have formal solutions of the form* (4.1).

When one of these three conditions is satisfied, the characteristic equations have regular solutions given by (4.9) *which are of the form* (4.1).

As in Theorem 4, we remark that our conclusion is that, if there exists at least one set of formal solutions of the characteristic equations, then there exists at least one set of regular solutions.

§ 5. Solutions of the differential equations.

In § 4, we have seen that, when the eigen values μ_ν of $\| c_\mu^\nu \|$ are all all pure imaginary, if $\mathfrak{G}$ is majorized, the following characteristic equations have reglur solutions f^ν of the form $f^\nu = x^\nu + \cdots\cdots$:

$$(5.1) \qquad \chi f^\nu \equiv \xi^\mu \frac{\partial f^\nu}{\partial x^\mu} = i\theta_\nu f^\nu ,$$

where $\mu_\nu = i\theta_\nu$. Then it is easily seen that $(n-1)$ functions[1]

$$(5.2) \qquad (f^\nu)^{\frac{1}{\theta_\nu}} / (f^1)^{\frac{1}{\theta_1}} \qquad (\nu \neq 1)$$

furnish the independent solutions of the differential equation $\chi f = 0$.

For the differential equations

$$(5.3) \qquad \frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \cdots\cdots = \frac{dx^n}{\xi^n},$$

the integrals are given by putting the functions of (5.2) constants. If we put $(f^1)^{\frac{1}{\theta_1}} = t$, then the integrals are given as follows:

$$(5.4) \qquad \begin{cases} f^1(x) = t^{\theta_1}, \\ f^\nu(x) = C_\nu t^{\theta_\nu}, \quad (\nu \neq 1) \end{cases}$$

where C_ν are arbitrary constants. When all θ_ν's are zero, namely all the eigen values μ_ν are zero, ω in (3.7) is arbitrary, consequently, by Theorem 5, $\varphi^\nu(x, \omega) = x^\nu$ for any ω. Thus $\xi^\mu = 0$. We exclude this trivial case. Then, without loss of generality, we may assume that $\theta_1 \neq 0$. We assume that $\theta_\alpha > 0$ $(\alpha = 2, \dots, p)$, $\theta_\tau < 0$ $(\tau = p+1, \dots, q)$ and $\theta_\lambda = 0$ $(\lambda = q+1, \dots, n)$. We consider the integrals passing through the origin. If $\theta_1 > 0$, then, from the

1) n is the number of the variables x^μ.

first of (5.4), $t\to 0$ when $x^{\nu}\to 0$. Then, from the second of (5.4), it must be $C_{\tau}=C_{\lambda}=0$. Thus, solving (5.4) with respect to x^{ν}, we have $x^{\nu}=\mathfrak{P}^{\nu}(t^{\theta_1}, C_{\alpha}t^{\theta_{\alpha}})$, where $\mathfrak{P}^{\nu}$ denote the regular functions of the arguments. If $\theta_1<0$, then from the first of (5.4), $t\to\infty$ when $x^{\nu}\to 0$. Then, from the second of (5.4), it must be $C_{\alpha}=C_{\lambda}=0$. Thus, solving (5.4), we have $x^{\nu}=\mathfrak{P}^{\nu}(t^{\theta_1}, C_{\tau}t^{\theta_{\tau}})$. Summarizing the results, we see that *the integrals passing through the origin are given by* $x^{\nu}=\mathfrak{P}^{\nu}(C_{\kappa}t^{\theta_{\kappa}})$, *where one of* C_{κ}*'s is unity and all* θ_{κ}*'s are either positive or negative.*

When all θ_{ν}'s are mutually commensurable, if the characteristic equations have formal solutions, then, by Theorem 7, the above results are valid. This is an extension of the results already found by Dulac[1] in the case of two variables, though the methods are entirely different.

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1) M. H. Dulac, Jour. École Polytech. (1904).