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System of Differential Equations which are Equivalent to Dirac's Equation for Hydrogen Atom

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§ 1. Introduction.

Dirac's wave equation is

$$\{E+e\varphi+\beta E_0+\sum_{k=1}^{3}\alpha_k(cp_k+eA_k)\}\psi=0, \tag{1.1}$$

where

$A_k\,(k=1,2,3)$: vector potential, φ: scalar potential, E: energy, $E_0=m_0c^2$, $p_k=-i\hbar\dfrac{\partial}{\partial x^k}\,(k=1,2,3)$, $\hbar=\dfrac{h}{2\pi}$, $\alpha_k\,(k=1,2,3)$ and β are 4—4 matrices such as

$$\alpha_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\qquad \alpha_2=\begin{bmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{bmatrix}$$

$$\alpha_3=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}\qquad \beta=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

which satisfy the relations:

$$\alpha_{(k}\alpha_{l)}=\delta_{kl},\qquad \beta\beta=1,\qquad \alpha_k\beta+\beta\alpha_k=0,\qquad (k,l=1,2,3)$$

ψ: 1—4 matrix having components $\psi_1,\psi_2,\psi_3,\psi_4$.

Denoting the space and time coordinates x,y,z and t by x^1,x^2,x^3 and x^4, the equation (1.1) can be expressed in the form which is symmetrical with respect to space and time coordinates as follows:

$$\gamma^i\left(\frac{\partial}{\partial x^i}-\varphi_i\right)\psi=\mu\psi\qquad (i=1,\cdots,4). \tag{1.2}$$

In this expression, according to usual convention which will be used throughout, the term of the left hand side stands for the sum of 4 terms as i take the values

1 to 4. Here $\gamma^i\,(i=1,2,3,4)$ are defined by

$$\gamma^k=-\beta\alpha_k\,(k=1,2,3)\,,\qquad \gamma^4=\frac{1}{c}\beta\,, \tag{1.3}$$

and satisfy the following relations:

$$\gamma^{(i}\gamma^{j)}=g^{ij}\qquad (i,j=1,2,3,4)$$

g^{ij} being the fundamental tensor of the space such as

$$g^{11}=g^{22}=g^{33}=-1\,,\qquad g^{44}=\frac{1}{c^2}\,,$$

$$g^{ij}=0\qquad \text{if}\qquad i\neq j\,,$$

and

$$\mu=\frac{i}{c\hbar}E_0=\frac{im_0c}{\hbar}$$

$$\varphi_k=-\frac{i}{c\hbar}eA_k\,(k=1,2,3)\,,\qquad \varphi_4=\frac{i}{\hbar}e\varphi\,.$$

Since the equation (1.2) is expressed in a vector form we can consider the equation (1.2) by polar coordinates as well as by rectangular coordinates. Hereafter we take polar coordinates r,θ,φ instead of x,y,z.

The equation (1.2) gives the system of 4 equations for $\psi_1,\psi_2,\psi_3,\psi_4$. Now we consider differential equations for ψ which are not multiplied by γ^i such that[1]

$$\left(\frac{\partial}{\partial x^i}-\Gamma_i\right)\psi=\Sigma_i\psi\qquad (i=1,\cdots,4)\,, \tag{1.4}$$

which give a system of 16 equations for $\psi_1,\psi_2,\psi_3,\psi_4$. Here $\Gamma_i\,(i=1,\cdots,4)$ are certain 4—4 matrices determined from the following relations:

$$\frac{\partial\gamma^h}{\partial x^i}+\left\{ {h \atop ji} \right\}\gamma^j-\Gamma_i\gamma^h+\gamma^h\Gamma_i=0\,,\qquad (h,i,j=1,\cdots,4)\,, \tag{1.5}$$

$\left\{ {h \atop ji} \right\}$ being christoffel symbol formed with respect to g^{ij}, and $\Sigma_i\,(i=1,\cdots,4)$ are any 4—4 matrices.

In this paper we shall show that if we determine Σ_i suitably under certain conditions the equations (1.4) give the same solutions as the solutions of Dirac's equation for hydrogen atom, namely, the solutions of (1.2) in the case of $\varphi_1=\varphi_2=\varphi_3=0$, $\varphi_4=\frac{i}{\hbar}\frac{Ze^2}{r}$.

§ 2. The actual form of the equations (1.4).

We take polar coordinates r,θ,φ and put $x^1=r, x^2=\theta, x^3=\varphi$. Then by the transformations: $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$, the matrices $\gamma^k(k=1,2,3)$ defined by (1.3) are transformed to $\gamma^r,\gamma^\theta,\gamma^\varphi$ as follows:

$$\left.\begin{aligned}\gamma^r &= \sin\theta\cos\varphi\gamma^x+\sin\theta\sin\varphi\gamma^y+\cos\theta\gamma^z,\\ \gamma^\theta &= \frac{1}{r}\cos\theta\cos\varphi\gamma^x+\frac{1}{r}\cos\theta\sin\varphi\gamma^y-\frac{1}{r}\sin\theta\gamma^z,\\ \gamma^\varphi &= -\frac{\sin\varphi}{r\sin\theta}\gamma^x+\frac{\cos\varphi}{r\sin\theta}\gamma^y,\end{aligned}\right\} \tag{2.1}$$

where $\gamma^x, \gamma^y, \gamma^z$ mean $\gamma^1, \gamma^2, \gamma^3$ defined by (1.3) respectively. Hereafter we write $\gamma^r, \gamma^\theta, \gamma^\varphi$ newly as $\gamma^1, \gamma^2, \gamma^3$ while leaving $\gamma^4=\frac{1}{c}\beta$ as it is. For these matrices $\gamma^r, \gamma^\theta, \gamma^\varphi, \gamma^4$, from the equations (1.5) we can show that $\Gamma_i=0$.

If we put
$$\gamma_5=cr^2\sin\theta\gamma^r\gamma^\theta\gamma^\varphi\gamma^4, \tag{2.2}$$

we know that $\gamma^i, \gamma^{[i}\gamma^{j]}, \gamma_5, \gamma^i\gamma_5\ (i,j=1,2,3,4)$ and unit matrix together form basis of 4—4 matrix, namely any 4—4 matrix can be expressed by linear combination of these 16 matrices. Hence now we express Σ_i of the equations (1.4) in the form:

$$\Sigma_i=A_i+A_i^5\gamma_5+A_{ijk}\gamma^j\gamma^k+A_{ij}\gamma^j+A_{ij}^5\gamma^j\gamma_5 \tag{2.3}$$

where $A_i, A_i^5, A_{ijk}, A_{ij}, A_{ij}^5$ are vectors and tensors with respect to the suffixes respectively and $A_{ijk}=-A_{ikj}$.

So long as $\Sigma_i\ (i=1,\cdots,4)$ are arbitrary 4—4 matrices these vectors and tensors are arbitrary. However, under certain consideration (which will not be described in this paper since we want to state briefly the result only) for hydrogen atom we can determine these vectors and tensors as follows:

$$\left.\begin{aligned}&A_1=-\frac{1}{r},\quad A_2=-\frac{1}{2}\cot\theta,\quad A_3=im,\quad A_4=-\frac{iE}{\hbar}\\ &A_i^5=0\ (i=1,2,3,4)\\ &\left\{\begin{aligned}&A_{114}=-A_{141}=-\frac{i}{2\hbar}\left(E+\frac{Ze^2}{r}\right),\\ &A_{212}=-A_{221}=\frac{1}{4}r,\quad A_{223}=-A_{232}=\frac{i}{2}mr^2,\\ &A_{313}=-A_{331}=\frac{1}{4}r\sin^2\theta,\quad A_{323}=-A_{332}=\frac{1}{4}r^2\sin\theta\cos\theta,\\ &\text{other } A_{ijk}=0\end{aligned}\right.\\ &A_{11}=-i\frac{m_0c}{\hbar},\quad A_{14}=\frac{ck}{r},\quad \text{other } A_{ij}=0,\\ &A_{23}^5=kr\sin\theta,\quad \text{other } A_{ij}^5=0,\end{aligned}\right\} \tag{2.4}$$

where k and m are any constants. In this manner the equations (1.4) are expressed as

$$\frac{\partial}{\partial x^i}\psi=(A_i+A_{ijk}\gamma^j\gamma^k+A_{ij}\gamma^j+A_{ij}^5\gamma^j\gamma_5)\psi. \tag{2.5}$$

Here $x^1=r, x^2=\theta, x^3=\varphi$ and $\gamma^1=\gamma^r, \gamma^2=\gamma^\theta, \gamma^3=\gamma^\varphi$ are defined by (2.1) and $\gamma^4=\frac{1}{c}\beta$. Also γ_5 is defined by (2.2) and $A_i, A_{ijk}, A_{ij}, A^5_{ij}$ are given by (2.4). In the next section we shall show that the equations (2.5) have the same solutions as the solutions of Dirac's equation for hydrogen atom.

§ 3. Solutions of the equations (2.5).

Substituting (2.4) into (2.5), the equations (2.5) are expressed as

$$\left.\begin{aligned}
\frac{\partial}{\partial r}\psi&=\left\{-\frac{1}{r}-\frac{i}{\hbar}\left(E+\frac{Ze^2}{r}\right)\gamma^r\gamma^4-i\frac{m_0c}{\hbar}\gamma^r+\frac{ck}{r}\gamma^4\right\}\psi,\\
\frac{\partial}{\partial\theta}\psi&=\left\{-\frac{1}{2}\cot\theta+\frac{1}{2}r\gamma^r\gamma^\theta+imr^2\gamma^\theta\gamma^\varphi+kr\sin\theta\gamma^\varphi\gamma_5\right\}\psi,\\
\frac{\partial}{\partial\varphi}\psi&=\left\{im+\frac{1}{2}r\sin^2\theta\gamma^r\gamma^\varphi+\frac{1}{2}r^2\sin\theta\cos\theta\gamma^\theta\gamma^\varphi\right\}\psi,
\end{aligned}\right\} \tag{3.1}$$

$$\frac{\partial}{\partial t}\psi=-\frac{iE}{\hbar}\psi. \tag{3.1, t}$$

Calculating the actual forms of the matrices on the right hand side of the above, the second and the third equations of (3.1) are written down as follows:

$$\frac{\partial}{\partial\theta}\psi=\begin{pmatrix}
\left(m-\frac{1}{2}\right)\cot\theta, & \left(m+k-\frac{1}{2}\right)e^{-i\varphi} & 0 & 0\\
\left(m-k+\frac{1}{2}\right)e^{i\varphi}, & -\left(m+\frac{1}{2}\right)\cot\theta & 0 & 0\\
0 & 0 & \left(m-\frac{1}{2}\right)\cot\theta, & \left(m-k-\frac{1}{2}\right)e^{-i\varphi}\\
0 & 0 & \left(m+k+\frac{1}{2}\right)e^{i\varphi}, & -\left(m+\frac{1}{2}\right)\cot\theta
\end{pmatrix}\psi \tag{3.1,θ}$$

$$\frac{\partial}{\partial\varphi}\psi=i\begin{pmatrix}
m-\frac{1}{2} & 0 & 0 & 0\\
0 & m+\frac{1}{2} & 0 & 0\\
0 & 0 & m-\frac{1}{2} & 0\\
0 & 0 & 0 & m+\frac{1}{2}
\end{pmatrix}\psi \tag{3.1, φ}$$

The equations (3.1, t) and (3.1, φ) can be easily solved and the equation (3.1, θ) is solved by considering separately the case when the constant k is negative or positive. Hence we shall classify the solutions of (3.1) into two classes according to $k<0$ or $k>0$.

Case I. When $k<0$, putting $k=-(l+1)$, the solution of the three equations (3.1, t), (3.1, φ) and (3.1, θ) is given by

$$\psi=e^{-\frac{iEt}{\hbar}}\begin{pmatrix} if(r)\sqrt{\frac{l-m+3/2}{2l+3}}\,Y_{l+1,m-\frac{1}{2}}(\theta,\varphi) \\ if(r)\sqrt{\frac{l+m+3/2}{2l+3}}\,Y_{l+1,m+\frac{1}{2}}(\theta,\varphi) \\ g(r)\sqrt{\frac{l+m+1/2}{2l+1}}\,Y_{l,m-\frac{1}{2}}(\theta,\varphi) \\ -g(r)\sqrt{\frac{l-m+1/2}{2l+1}}\,Y_{l,m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \tag{3.2}$$

where $Y_{l,m}(\theta,\varphi)$ is defined by

$$\left.\begin{aligned} Y_{l,m}(\theta,\varphi)&=\frac{1}{\sqrt{2\pi}}\,\mathfrak{p}_{lm}(\theta)e^{im} \\ \mathfrak{p}_{lm}(\theta)&=\sqrt{\frac{2l+1}{2}\,\frac{(l-m)\,!}{(l+m)\,!}}\,P_l^m(\cos\theta), \\ P_l^m(x)&=\frac{1}{2^l l\,!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l,\ (x=\cos\theta), \end{aligned}\right\} \tag{3.3}$$

and $f(r)$ and $g(r)$ are functions of r only. Substituting (3.2) into the first equation of (3.1) and using some identities among the functions $\sin\theta\mathfrak{p}_{lm}$, $\cos\theta\mathfrak{p}_{lm}$ and $\mathfrak{p}_{lm}$, we can reduce the first equation of (3.1) to the following two equations between $f(r)$ and $g(r)$:

$$\left.\begin{aligned} -\frac{df(r)}{dr}+(k-1)\frac{f(r)}{r}&=\frac{1}{c\hbar}\left(E+\frac{Ze^2}{r}-m_0c^2\right)g(r), \\ \frac{dg(r)}{dr}+(k+1)\frac{g(r)}{r}&=\frac{1}{c\hbar}\left(E+\frac{Ze^2}{r}+m_0c^2\right)f(r), \end{aligned}\right\} \tag{3.4}$$

Case II. When $k>0$, putting $k=l$, the solution of (3.1) is given by

$$\psi=e^{-\frac{iEt}{\hbar}}\begin{pmatrix} if(r)\sqrt{\frac{l+m-1/2}{2l-1}}\,Y_{l-1,m-\frac{1}{2}}(\theta,\varphi) \\ -if(r)\sqrt{\frac{l-m-1/2}{2l-1}}\,Y_{l-1,m+\frac{1}{2}}(\theta,\varphi) \\ g(r)\sqrt{\frac{l-m+1/2}{2l+1}}\,Y_{l,m-\frac{1}{2}}(\theta,\varphi) \\ g(r)\sqrt{\frac{l+m+1/2}{2l+1}}\,Y_{l,m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \tag{3.5}$$

Here $f(r)$ and $g(r)$ are the solutions of the same equations as (3.4).

We see that the solutions (3.2) and (3.5) accompanied by (3.4) coincide with the solutions of Dirac's equation for hydrogen atom,[2] regarding the constants l and m as azimuthal quantum number and magnetic quantum number respectively and determining energy E from the equations (3.4). Further the inner quantum number j is given by $j=l+\frac{1}{2}$ $(l=0, 1, 2, \cdots)$ or $j=l-\frac{1}{2}$ $(l=1, 2, \cdots)$ corresponding to the solutions (3.2) or (3.5). Therefore we can say that *the aggregate of the differential equations (2.5) corresponding to all possible values of constants k and m, is equivalent to Dirac's equation for hydrogen atom.*

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1) The equations of the form (1.4) have been treated in many papers on wave geometry published during 1935—1942, this journal Vol. 5 (1935)—Vol. 11 (1942).

2) Darwin, Proc. Roy. Soc. (A) Bd. 118 (1928) 654; Gordon, ZS. f. Phys. Bd. 48 (1928), 11; Bethe, Handbuch der physik, Bd. XXIV/1 (1933), Kap. 3, 311.